

CERTAIN VARIETIES OF RESOLVING SETS OF A GRAPH

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Abstract. Let $G = (V, E)$ be a simple connected graph. For each ordered subset $S = \{s_1, s_2, \dots, s_k\}$ of V and a vertex $u \in V$, we associate a vector $\Gamma(u/S) = (d(u, s_1), d(u, s_2), \dots, d(u, s_k))$ with respect to S , where $d(u, v)$ denote the distance between u and v in G . A subset S is called a *resolving set* of G if $\Gamma(u/S) \neq \Gamma(v/S)$ for all $u, v \in V - S$. The purpose of this paper is to introduce various types of r -sets and compute minimum cardinality of such sets. Using the superhereditaryness of resolving sets, it is shown that, for most of the graphs, the powerful resolving and simple resolving numbers are identical. However, the foul resolving number is constant for cycles, paths and wheels, and the maximal resolving number is proportional to the order.

Key words and Phrases: Simple resolving sets, Metric dimensions, Landmarks, Powerful resolving sets, Maximal resolving sets, Foul resolving sets.

1. INTRODUCTION

All the graphs considered in this paper are connected, simple, undirected and finite. The distance between two vertices a and b of a graph G is the length of a shortest path between them and is denoted by $d_G(a, b)$ or simply $d(a, b)$. The number of edges incident with a vertex v of G is the degree of the vertex v in G and is denoted by $\deg_G(v)$ or simply $\deg(v)$. A subset S of a vertex set of G is called a *resolving set* (r -set) of G if for each pair u, v of vertices of G there exist a vertex w in S with the property that $d(v, w) \neq d(u, w)$. The vertex w is then called a resolving vertex for the pair u, v , or we say w resolves u and v . A resolving set S is said to be a minimal resolving set of G if none of its proper subsets are resolving set of G . The *metric dimension* of G , denoted by $\beta(G)$, is the minimum cardinality

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of a minimal resolving set of G . A resolving set with minimum cardinality is called a *metric basis*. The concept of metric dimension was introduced by F. Harary and R. A. Melter [5] and independently by P.J. Slater [10] under the term locating set. For more works on metric dimension, we refer [3, 4, 6, 8, 9, 11, 12, 7, 14]. The resolving property and dominating property of a network are super hereditary in nature and both are essential for unique identification of the nodes in the network for various purposes. The complement of a minimum dominating set is also a dominating set. Hence various classes of dominating sets are studied together with their complements. This is not the case of resolving sets, here the complement of a resolving set need not be a resolving set. For example every resolving set of a complete graph K_m should include at least $m - 1$ vertices of it. Hence, complement of a resolving set of K_m can not be a resolving set for all $m \geq 3$.

In 2018, an attempt is made by B.Soorayanarayana and Suma A.S in [13] to study a special class of dominating set namely, the neighborhood set along with its complements and resolving property. In this paper, we extend the results of [13] by excluding the restriction of neighborhood property, so as to hold for more classes of resolving sets.

Throughout this paper P_m denotes a path on m vertices with the vertex set $V = \{v_i : 1 \leq i \leq m\}$ and the edge set $E = \{v_i v_{i+1} : 1 \leq i \leq m - 1\}$, C_m denotes a cycle on m vertices with the vertex set $V = \{v_i : 1 \leq i \leq m\}$ and the edge set $E = \{v_i v_{i+1} : 1 \leq i \leq m - 1\} \cup \{v_1 v_m\}$, K_m denotes the complete graph on m vertices with the vertex set $V = \{v_i : 1 \leq i \leq m\}$ and the edge set $E = \{v_i v_j : i < j, 1 \leq i, j \leq m\}$ and $W_{1,m}$ denotes wheel graph on $m + 1$ vertices with vertex set $V = \{v_i : 1 \leq i \leq m\} \cup \{v_0\}$ and edge set $E = \{v_0 v_i : 1 \leq i \leq m\} \cup \{v_i v_{i+1} : 1 \leq i \leq m - 1\} \cup \{v_1 v_m\}$. The terms not defined here may be found in [1, 4].

2. RELATIONSHIP BETWEEN p -SETS AND p^* -SETS

A property \mathcal{P} of a graph $G(V, E)$ is a class of subsets of V and is called super-hereditary if $S \in \mathcal{P}$ and $S \subseteq H$, then $H \in \mathcal{P}$. Equivalently, if $S \notin \mathcal{P}$ and $H \subseteq S$ implies that $H \notin \mathcal{P}$. For each $S \in \mathcal{P}$, the complement of S in V is $\bar{S} = V - S$. A property \mathcal{P}^* is a subset of \mathcal{P} such that for each $S \in \mathcal{P}^*$ we have $\bar{S} \in \mathcal{P}$. Clearly $\bar{\bar{S}} \in \mathcal{P}^*$ and \mathcal{P}^* need not be super-hereditary. A property $\hat{\mathcal{P}}$ is a subset of \mathcal{P} such that for each $S \in \hat{\mathcal{P}}$ its complement $\bar{S} \notin \mathcal{P}$. Further, $\overline{\mathcal{P}}$ is a property of G such that for each $S \in \overline{\mathcal{P}}$ neither $S \in \mathcal{P}$ nor its complement $\bar{S} \in \mathcal{P}$. An element of a property \mathcal{P} is called a p -set if no proper subset of it is a member of \mathcal{P} . The minimum cardinality of a p -set are denoted by $l_p(G)$. Similarly, $l_{p^*}(G)$, $l_{\hat{p}}(G)$ and $l_{\overline{p}}(G)$ are defined.

Theorem 2.1. *Let \mathcal{P} be a super-hereditary property of a graph G . Then either $l_p(G) = l_{p^*}(G)$ or $\mathcal{P}^* = \emptyset$.*

Proof. Let $S \in \mathcal{P}$ be a p -set of minimum cardinality and $P^* \neq \emptyset$. Let $S^* \in \mathcal{P}^*$ be an p^* -set with minimum cardinality in G . Since $S^* \in \mathcal{P}^* \subseteq \mathcal{P}$, we have $l_p(G) = |S| \leq |S^*|$. On the other hand, if $|S^*| > |S|$, then for every p -set in \mathcal{P} its complement is not in \mathcal{P} . Therefore, complement of each p -set of G requires at least one more element from the p -set to make its complement a p -set, that is $|\overline{S^*}| \geq |\overline{S}| + 1$ for every p -set S . But then, as S^* is in \mathcal{P} and S is a p -set of minimum cardinality, $|S^*| > |S|$ and hence $|\overline{S^*}| < |\overline{S}| \leq |\overline{S^*}| - 1$, a contradiction. \square

3. RESOLVING PROPERTY OF A GRAPH

The resolving property \mathcal{R} of a graph G , denoted by $\mathcal{R}(G)$, is super-hereditary. Each element of $\mathcal{R}(G)$ is a resolving set of G and is also called *simple resolving set*. An r -set is a minimal element of $\mathcal{R}(G)$. The minimum cardinality of an r -set are the lower metric dimension of G and is denoted by $l_r(G)$. Similarly, each element of \mathcal{R}^* , $\hat{\mathcal{R}}$ and $\overline{\mathcal{R}}$ are respectively called *powerful resolving set*, *maximal resolving set* and *foul resolving set*.

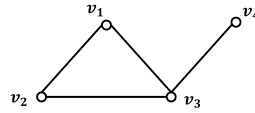


FIGURE 1. Illustration to resolving property of a graph

Let us consider a graph G of Figure 1, Let $\mathcal{R}(G) = \{ \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_2, v_3, v_4\}, \{v_1, v_3, v_4\}, V(G) \}$ among these elements of $\mathcal{R}(G)$ minimal r -sets of G are $\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}$ with cardinality 2. Hence, $l_r(G) = 2$. Let $\mathcal{R}^*(G) = \{ \{v_1, v_3\}, \{v_2, v_4\}, \{v_1, v_4\}, \{v_2, v_3\} \}$ and these sets are minimal sets with cardinality 2. Hence, $l_r^*(G) = 2$. Let $\hat{\mathcal{R}}(G) = \{ \{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_2, v_3, v_4\}, \{v_1, v_3, v_4\} \}$ among these elements the set $\{v_1, v_2\}$ is a minimal set with cardinality 2. Hence, $l_{\hat{r}}(G) = 2$. The set $\overline{\mathcal{R}}(G) = \emptyset$ due to the fact that either S or \overline{S} is an r -set.

Remark 3.1. *If G is a non-trivial graph of order m and S is any subset of vertices of G , then*

- i) $S \in \mathcal{R}(G)$, if $|S| \geq m - 1$.
- ii) $1 \leq l_r(G) \leq |V(G)| - 1$.
- iii) $|S| < m - 1$, if $S \notin \mathcal{R}(G)$.

Theorem 3.2 ([6]). *Let $l_r(G) = 2$ and $\{a, b\} \in \mathcal{R}(G)$. If P is a shortest path between a and b in G , then the following are true:*

1. P is unique.
2. $\deg_G(a) \leq 3$ and $\deg_G(b) \leq 3$.
3. $\deg_G(v) \leq 5$, for every $v \in V(P)$.

4. r -SETS AND DIMENSIONS OF A PATH

Remark 4.1. From the definition of an r -set, it is clear that every 2-element subset of vertices of a path P_m is in $\mathcal{R}(P_m)$ for every $m \geq 2$. In fact, if $S = \{a, b\}$ and u, v be arbitrary vertices of P_m such that $d(u, a) = d(v, a)$, then a is the central vertex of the uv -path in P_m , but then exactly one of the paths, ub -path or vb -path, in P_m contains the vertex a and hence $d(u, b) \neq d(v, b)$.

Remark 4.2. A singleton set $\{v\} \in \mathcal{R}(G)$ if and only if $G \cong P_m$ and v is an end vertex of P_m for some $m \in \mathbb{Z}^+$.

Observation 4.3. Every member of $\mathcal{R}(P_m)$ contains a pendent vertex of P_m whenever $1 \leq m \leq 3$.

Proposition 4.4. For any positive integer $m \geq 2$, $l_r(P_m) = 1 = l_{r^*}(P_m)$.

Proof. An immediate consequence of Theorem 2.1 and Remark 4.2. \square

Theorem 4.5. For any positive integer m , $l_{\hat{r}}(P_m) = \begin{cases} m & \text{for } m = 1, 2. \\ m - 1 & \text{for otherwise.} \end{cases}$

Proof. Let $S \in \hat{\mathcal{R}}(P_m)$ and $l_{\hat{r}}(P_m) = |S|$. Then $S \in \mathcal{R}(P_m)$ and $\bar{S} \notin \mathcal{R}(P_m)$. Hence by Remark 4.1 and Remark 4.2, $|\bar{S}| \not\geq 2$ and \bar{S} not contains any end vertex of P_m . Therefore, $|S| \geq m - 1$ if $m \geq 3$, and $|S| = m$ if $m = 1, 2$. But then, by Remark 3.1, $|S| = m - 1$ if $m \geq 3$, and $|S| = m$ if $m = 1, 2$ (Since S is a minimal r -set). \square

Remark 4.6. For each $m \in \mathbb{Z}^+$, $\bar{\mathcal{R}}(P_m) = \emptyset$ due to the fact that either $S \in \mathcal{R}(P_m)$ or $\bar{S} \in \mathcal{R}(P_m)$, for every $S \subseteq V(P_m)$.

5. r -SETS AND DIMENSIONS OF A CYCLE

Theorem 5.1. Let S be an i -element subset of vertices of a cycle C_m and $3 \leq i \leq m$. Then S is in $\mathcal{R}(C_m)$.

Proof. Let S be a subset of the vertices of C_m with cardinality at least 3. Let $a, b, c \in S$ and x, y be any two vertices of cycle C_m for $m \geq 3$. If possible, let $d(a, x) = d(a, y)$ and $d(b, x) = d(b, y)$. Then a and b lie in distinct xy -paths in C_m and m is even. In case if c lies between a and x , then $d(c, x) < d(c, y)$ and hence c resolves the pair x, y . Similarly, other cases follows by symmetry. \square

Remark 5.2. Let $m \geq 3$ and v_0, v_1, \dots, v_{m-1} be the vertices of C_m in order. Then the subset $\{v_i, v_{i \oplus_m 1}\}$ is in $\mathcal{R}(C_m)$ for every $i \in \mathbb{Z}_m$.

Theorem 5.3 ([6]). For every integer $m \geq 3$, $l_r(C_m) = 2$.

Theorem 5.4. For each integer $m \geq 3$, $l_r(C_m) = l_{r^*}(C_m) = 2$ if $m \neq 3$, $l_r(C_3) = 2$, and $\mathcal{R}^*(C_3) = \emptyset$.

Proof. The case $m \geq 4$ follows from Theorem 2.1, Remark 5.2 and Theorem 5.3. Further, for $m = 3$, by Theorem 5.3, $|S| \geq 2$ for each $S \in \mathcal{R}(C_3)$ and hence $|\bar{S}| \leq 1$. So, $\bar{S} \notin \mathcal{R}(C_3)$. Hence $l_r(C_3) = 2$ and $\mathcal{R}^*(C_3) = \emptyset$. \square

Lemma 5.5. *The set $\{a, b\}$ is in $\mathcal{R}(C_m)$ if and only if either (i) m is odd or (ii) m is even and $d_{C_m}(a, b) \neq \frac{m}{2}$.*

Proof. Let P be a shortest path between a and b in C_m .

Let $\{a, b\} \in \mathcal{R}(C_m)$. If m is odd then nothing to prove. We now suppose that m is even. If possible, let $d_{C_m}(a, b) = \frac{m}{2}$. Then $\bar{P} = \langle V(C_m) - V(P) \rangle$ is also a shortest path between a and b in C_m other than P , a contradiction to the fact that $\{a, b\}$ is in $\mathcal{R}(C_m)$ (by Theorem 3.2).

Conversely, let x and y be any two arbitrary vertices of C_m . If $d(a, x) = d(a, y)$ and $d(b, x) = d(b, y)$, then both x and y cannot be in a common short path between a and b . Let $x \in V(P)$ and $y \in V(\bar{P})$. But then, $m = |V(P)| + |V(\bar{P})| = d(a, x) + d(x, b) + d(a, y) + d(y, b) = 2d(a, x) + 2d(x, b) = 2|V(P)| = \text{even}$ and $|V(P)| = d(a, b) = \frac{m}{2}$, a contradiction. \square

Theorem 5.6. *For any integer $m \geq 3$,*

$$l_{\hat{r}}(C_m) = \begin{cases} m - 2 & \text{if } m \text{ is even and } m \neq 4. \\ m - 1 & \text{otherwise.} \end{cases}$$

Proof. Let $S \in \hat{\mathcal{R}}(C_m)$ be such that $l_{\hat{r}}(C_m) = |S|$. Then $S \in \mathcal{R}(C_m)$ and $\bar{S} \notin \mathcal{R}(C_m)$. If m is odd then by Remark 4.2, $|S| \geq 2$ (since $G \not\cong P_m$) and $|\bar{S}| < 2$ (since $\bar{S} \notin \mathcal{R}(C_m)$ and \mathcal{R} is super-hereditary). Thus, $|S| = m - 1$. If m is even, $|\bar{S}| \leq 2$ (since $\{a, b\} \notin \mathcal{R}(C_m)$ with $d(a, b) = \frac{m}{2}$ by Lemma 5.5). Hence $|S| = m - 2$ if $m \neq 4$ with two antipodal vertices in \bar{S} . Finally, when $m = 4$, S should contain two adjacent vertices to become an r -set but then \bar{S} resolves (by Lemma 5.5) and hence $|S| \geq 3$. Hence $|S| = 3$ (by Remark 3.1) if $m = 4$. \square

For $m = 3$ or $m \geq 5$ and a subset S of the vertices of the cycle C_m , by Remark 5.2 and the super-hereditary property of $\mathcal{R}(G)$, it follows that either $S \in \mathcal{R}(C_m)$ or $\bar{S} \in \mathcal{R}(C_m)$. Hence $\bar{\mathcal{R}}(C_m) = \emptyset$ for $m = 3$ or $m \geq 5$. However, for a subset S consisting of any two antipodal vertices of C_4 , neither $S \in \mathcal{R}(C_m)$ nor $\bar{S} \in \mathcal{R}(C_m)$. Thus we have the following theorem.

Theorem 5.7. *For an integer $m \geq 3$, $l_{\bar{r}}(C_4) = 2$ and $\bar{\mathcal{R}}(C_m) = \emptyset$ if $m \neq 4$.*

6. r -SETS AND DIMENSIONS OF A COMPLETE GRAPH

Theorem 6.1. *For any positive integer m , $l_r(K_m) = \begin{cases} 1 & \text{for } m = 1. \\ m - 1 & \text{otherwise.} \end{cases}$*

Proof. Let $S \in \mathcal{R}(K_m)$ such that $l_r(K_m) = |S|$. Then for each $x, y \in V(K_m) - S$, $d(x, w) = d(y, w) = 1$ for each $w \in S$. Hence $|\bar{S}| \leq 1$ implies that $|S| \geq m - 1$ if $m \geq 2$ and $|S| = 1$ if $m = 1$ (since S is non-empty). \square

If $S \in \mathcal{R}^*(K_m)$, then from the proof of the above theorem, we see that $|S| \geq m - 1$ and $|\bar{S}| \geq m - 1$. Hence such a set S exists (with $|S| = 1$) if and only if $m = 2$. We record this fact in the form of the following theorem.

Theorem 6.2. *For any positive integer m , $l_{r^*}(K_m) = 1$ if $m = 2$, and $\mathcal{R}^*(K_m) = \emptyset$ if $m \neq 2$.*

Theorem 6.3. *For any positive integer m , $l_{\hat{r}}(K_m) = \begin{cases} m & \text{for } m = 1, 2. \\ m - 1 & \text{for otherwise.} \end{cases}$*

Proof. Let $S \in \hat{\mathcal{R}}(K_m)$ be such that $|S| = l_{\hat{r}}(K_m)$. Then $S \in \mathcal{R}(K_m)$ and $\bar{S} \notin \mathcal{R}(K_m)$. For the case $m \leq 2$, the result follows by Theorem 4.5. Let $m \geq 3$. Then $|S| \geq m - 1$ (by Theorem 6.1) and $\{v\} \notin \mathcal{R}(K_m)$ for each $v \in V(K_m)$ (by Remark 4.2). Hence $|S| = m - 1$. \square

Theorem 6.4. *For any positive integer m , $l_{\bar{r}}(K_m) = 2$ if $m \geq 4$, and $\bar{\mathcal{R}}(K_m) = \emptyset$ if $m \leq 3$.*

Proof. Let $S \in \bar{\mathcal{R}}(K_m)$ be such that $|S| = l_{\bar{r}}(K_m)$. Then $S \notin \mathcal{R}(K_m)$ and $\bar{S} \notin \mathcal{R}(K_m)$. So, by Remark 3.1, $2 \leq |S| \leq m - 2$ and $2 \leq |\bar{S}| \leq m - 2$. Hence, $m \geq 4$ and $l_{\bar{r}}(K_m) = 2$. \square

7. r -SETS AND DIMENSIONS OF A WHEEL

Theorem 7.1 ([9]). *If $W_{1,m}$ is the wheel graph for $m \geq 3$, then*

$$\begin{aligned} (i) \quad & l_r(W_{1,3}) = l_r(W_{1,6}) = 3. \\ (ii) \quad & l_r(W_{1,4}) = l_r(W_{1,5}) = 2. \\ (iii) \quad & l_r(W_{1,x+5k}) = \begin{cases} 3 + 2k, & \text{for } x = 7 \text{ or } 8. \\ 4 + 2k, & \text{for } x = 9 \text{ or } 10 \text{ or } 11. \end{cases} \end{aligned}$$

Theorem 7.2. *For any positive integer $m \geq 3$,*

$$l_r(W_{1,m}) = \begin{cases} 3 & \text{for } m = 3, 6. \\ 2 & \text{for } m = 4, 5. \\ \lfloor \frac{2(m+1)}{5} \rfloor, & \text{for } m \geq 7. \end{cases}$$

Proof. The result follows from Theorem 7.1. \square

Definition 7.3. *A Chinese fan graph $F(1, m)$ is the sum graph $K_1 + P_m$. i.e it is a graph obtained by taking a vertex v adding it to every vertex of the path P_m by an edge.*

Theorem 7.4 ([2]). *For any integer m , $l_r(F(1, m)) = \begin{cases} 1 & \text{if } m = 1 \\ 2 & \text{if } m = 2, 3 \\ 3 & \text{if } m = 6 \\ \lfloor \frac{2m+2}{5} \rfloor & \text{if otherwise.} \end{cases}$*

Definition 7.5. Let $G(V, E)$ be a graph and $S \subseteq V$. Then for any two vertices $u, v \in V$, a path between u and v is called an \bar{S} path if each vertex of the path other than the end vertices are in \bar{S} .

Definition 7.6. If there exists an \bar{S} path between $u, v \in S$, then we define the gap between u and v in S , denoted by $g_S(u, v)$, as the number of vertices of \bar{S} in a shortest \bar{S} path between u and v .

For example, in the graph G of Figure 2, the shortest \bar{S} path between a and e is $a - c - d - e$ if $S = \{a, b, e\}$. Therefore $g_{\{a,b,e\}}(a, e) = 2$. Similarly, $g_{\{a,d,e\}}(a, e) = 1$, $g_{\{a,b,c\}}(a, b) = 0$, $g_{\{a,d,f\}}(a, d) = 1$, and $g_{\{a,b,d,e\}}(a, e) = 3$. Further, $g_{\{a,b\}}(a, b) = d(a, b) - 1 = 0$.

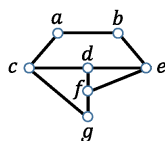


FIGURE 2. The graph G .

Lemma 7.7. For any integer $m \geq 3$ and a subset $S \subseteq V(W_{1,m})$, if $g_{S \cup \{c_0\}}(a, b) \geq 2$ and $g_{S \cup \{c_0\}}(a, c) \geq 2$, for some $a, b, c \in S$ with $b \neq c$, then $S \notin \mathcal{R}(W_{1,m})$.

Proof. Let $g_{S \cup \{c_0\}}(a, b) \geq 2$, for some $a, b \in S$. Then $a - v_i - v_{i+1} - \dots - b$ is a shortest $\overline{S \cup \{c_0\}}$ path between a and b for some i . Further, if $g_{S \cup \{c_0\}}(a, c) \geq 2$ and $b \neq c$ then, there are two rim vertices v_j, v_{j+1} in \bar{S} with $i \neq j$, such that $a - v_j - v_{j+1} - \dots - c$ is a shortest $\overline{S \cup \{c_0\}}$ path between a and c . But then $d(a, v_i) = d(a, v_j) = 1$ and $d(s, v_i) = d(s, v_j) = 2$, for all $s \in S - \{a\}$. Therefore, none of the vertices in S will resolve the pair v_i, v_j . Hence $S \notin \mathcal{R}(W_{1,m})$. \square

Lemma 7.8. For any integer $m \geq 3$ and a subset $S \subseteq V(W_{1,m})$, if $g_{S \cup \{c_0\}}(a, b) \geq 4$ for some $a, b \in S$, then $S \notin \mathcal{R}(W_{1,m})$.

Proof. Let $g_{S \cup \{c_0\}}(a, b) \geq 4$, for some $a, b \in S$. Then $a = v_i - v_{i+1} - v_{i+2} - \dots - v_{i+k} = b$ is a shortest $\overline{S \cup \{c_0\}}$ path between a and b for some i, k with $k \geq 6$. But then $d(s, v_{i+2}) = d(s, v_{i+3}) = 2$ for all $s \in S$. Therefore, none of the vertices in S will resolve the pair v_{i+2}, v_{i+3} . Hence $S \notin \mathcal{R}(W_{1,m})$. \square

Lemma 7.9. For any integer $m \geq 3$ and a subset $S \subseteq V(W_{1,m})$, if $g_{S \cup \{c_0\}}(a, b) = 3$ and $g_{S \cup \{c_0\}}(c, d) = 3$, for some $a, b, c, d \in S$ with $a \neq c$ then $S \notin \mathcal{R}(W_{1,m})$.

Proof. Let $g_{S \cup \{c_0\}}(a, b) = 3$ and $g_{S \cup \{c_0\}}(c, d) = 3$, for some $a, b, c, d \in S$. Then $a = v_i - v_{i+1} - v_{i+2} - v_{i+3} - v_{i+4} = b$ and $c = v_j - v_{j+1} - v_{j+2} - v_{j+3} - v_{j+4} = d$, for some i, j and $i \neq j$ are the shortest $\overline{S \cup \{c_0\}}$ path between the a, b and c, d respectively. But then, $d(s, v_{i+2}) = d(s, v_{j+2}) = 2$ for all $s \in S$. Therefore, none of the vertices in S will resolve the pair v_{i+2}, v_{j+2} . Hence $S \notin \mathcal{R}(W_{1,m})$. \square

Remark 7.10. For each i , $1 \leq i \leq m$, it is easy to see that $\{v_i, v_{i+1}\} \in \mathcal{R}(W_{1,m})$ for each $m = 4, 5$.

Theorem 7.11. A set S with of cardinality at least three is in $\mathcal{R}(W_{1,m})$ for any $m \geq 6$ if and only if the following hold.

- (i) $g_{S \cup \{c_0\}}(v_i, v_j) \leq 3$ for every $v_i, v_j \in S$, and $g_{S \cup \{c_0\}}(v_i, v_j) = 3$ for at most one pair $v_i, v_j \in S$.
- (ii) $g_{S \cup \{c_0\}}(v_i, v_{i+3}) = 2$ and $g_{S \cup \{c_0\}}(v_{i+k}, v_{i+k+3}) \geq 2$, for some $v_i, v_{i+3}, v_{i+k}, v_{i+k+3} \in S$ implies $4 \leq k \leq 5$.

Proof. Let $S \subseteq V(W_{1,m})$ be satisfying the conditions (i) and (ii). Since $|S| \geq 3$, S has a rim vertex. Without loss of generality we take rim vertex $v_1 \in S$. If possible, let us assume the contrary that $S \notin \mathcal{R}(W_{1,m})$. Then there exist $u, v \in V(W_{1,m}) - S$ such that

$$d(u, w) = d(v, w) \quad \forall w \in S. \quad (1)$$

We now prove the theorem by contradiction in different cases as follows.

Case 1: $d(u, w) = d(v, w) = 1$ for every $w \in S$.

Subcase 1: $c_0 \in \{u, v\}$.

In this case, from Equation 1, we get $1 = d(u, w) = d(v, w)$ for all $w \in S$. So, in particular $d(v, v_1) = 1$ implies that $v = v_2$ (or $v = v_m$) and hence $m = 3$, a contradiction.

Subcase 2: $c_0 \notin \{u, v\}$.

In this case, from Equation 1, we get $1 = d(u, w) = d(v, w)$ for all $w \in S$. So, in particular $d(u, v_1) = d(v, v_1) = 1$ implies that $u, v \in \{v_2, v_m\}$. But then, for any i , $1 < i < m$ if $v_i \in S$ then, $d(v_2, v_i) = d(v_m, v_i) = 1$ (from Equation 1 with $w = v_i$) implies that $i = 3$ and hence $m = 4$, a contradiction.

Case 2: $d(u, w) = d(v, w) = 2$, for some $w \in S$.

In this case, $c_0 \notin \{u, v\}$. Therefore, $u = v_i$ and $v = v_j$ for some $1 < i < j \leq m$. But then, $v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1} \notin S$. Let P be a path between u and v in C_m containing the vertex v_1 .

Subcase 1: $V(\bar{P}) \cap S = \emptyset$.

In this case, either $v_{i-1}, v_i, v_j, v_{j+1} \in V(\bar{P})$ or $v_i, v_{i+1}, v_{j-1}, v_j \in V(\bar{P})$. Hence, as there are no other vertices of \bar{P} in S , for the vertices $a, b \in S$ that are nearer to the end vertices of \bar{P} , we get $g_{\{a,b\}} = [|V(C_m)| - |V(P)|] + 2 \geq 4$, a contradiction to condition (i).

Subcase 2: $V(\bar{P}) \cap S \neq \emptyset$.

Let $v_k \in V(\bar{P}) \cap S$. Let a and c be the vertices in $S \cap V(\bar{P})$ nearer to v_j and v_i , respectively. Let b and d be the vertices in $S \cap V(P)$ be a vertex nearer to v_j and v_i , respectively. Then, $g_{\{a,b\}} = [\text{length of the path } P_1 \text{ between } a \text{ and } b \text{ containing } v_j] - 1 \geq 3$ (since $v_{j-1}, v_j, v_{j+1} \in V(P_1) \cap \bar{S}$) and $g_{\{c,d\}} = [\text{length of the path } P_2 \text{ between } c \text{ and } d \text{ containing } v_i] - 1 \geq 3$ (since $v_{i-1}, v_i, v_{i+1} \in V(P_s) \cap \bar{S}$), a contradiction to the condition (i).

Converse part follows by Lemma 7.7, Lemma 7.8 and Lemma 7.9. \square

Theorem 7.12. *For any positive integer $m \geq 4$*

$$l_{r^*}(W_{1,m}) = \begin{cases} 3, & \text{for } m = 6. \\ 2, & \text{for } m = 4, 5. \\ \lfloor \frac{2(m+1)}{5} \rfloor, & \text{for } m \geq 7. \end{cases}$$

Proof. In view of Theorem 2.1 and Theorem 7.2, it suffices to show that $\mathcal{R}^*(W_{1,m}) \neq \emptyset$, for all $m \geq 4$.

If $m = 4, 5$, then the set $S = \{v_1, v_2\} \in \mathcal{R}(W_{1,4})$ and $\bar{S} \in \mathcal{R}(W_{1,4})$, So Also it is easy to see that the set $S = \{v_1, v_2, v_3\} \in \mathcal{R}^*(W_{1,6})$. Let $m \geq 7$.

Case 1: $m \equiv 0 \pmod{5}$.

$$\text{Let } S = \{v_1, v_6, v_{11}, \dots, v_{m-4}\} \cup \{v_4, v_9, v_{14}, \dots, v_{m-1}\}.$$

Case 2: $m \equiv 1 \pmod{5}$.

$$\text{Let } S = \{v_1, v_6, v_{11}, \dots, v_{m-4}\} \cup \{v_4, v_9, v_{14}, \dots, v_{m-6}, v_m\}.$$

Case 3: $m \equiv 2 \pmod{5}$.

$$\text{Let } S = \{v_1, v_6, v_{11}, \dots, v_{m+1}\} \cup \{v_4, v_9, v_{14}, \dots, v_{m-1}\}.$$

Case 4: $m \equiv 3 \pmod{5}$.

$$\text{Let } S = \{v_1, v_6, v_{11}, \dots, v_{m-4}\} \cup \{v_4, v_9, v_{14}, \dots, v_{m-6}, v_m, v_{m+2}\}.$$

Case 5: $m \equiv 4 \pmod{5}$.

$$\text{Let } S = \{v_1, v_6, v_{11}, \dots, v_{m-4}\} \cup \{v_4, v_9, v_{14}, \dots, v_{m-6}, v_{m-1}\}.$$

In all the above cases the set S as well as \bar{S} satisfies all the conditions (i) and (ii) of Theorem 7.11. Hence $\bar{S} \in \mathcal{R}(W_{1,m})$. Therefore, $\mathcal{R}^*(W_{1,m}) \neq \emptyset$. \square

Remark 7.13. *If $m = 3$, then the graph $W_{1,2} \cong K_4$ and hence by Theorem 6.2, $\mathcal{R}^*(W_{1,m}) = \emptyset$.*

Theorem 7.14. *For any integer $m \geq 3$, $l_{\bar{r}}(W_{1,m}) = \begin{cases} 2 & \text{for } m = 3, 4. \\ 4 & \text{for } m \geq 8. \end{cases}$*

Proof. Let $S \in \bar{\mathcal{R}}(W_{1,m})$ be such that $|S| = l_{\bar{r}}(W_{1,m})$. Then $S \notin \mathcal{R}(W_{1,m})$ and $\bar{S} \notin \mathcal{R}(W_{1,m})$. If $m = 3$, then $W_{1,3} \cong K_4$ and hence the result is true by Theorem 6.4. If $m = 4$, then by Remark 3.1, we have $|\bar{S}| \leq 3$. So, $|S| \geq 2$. Further, the set $\{v_1, v_3\} \notin \mathcal{R}(W_{1,4})$ and its complement $\{v_2, v_4\} \notin \mathcal{R}(W_{1,4})$. Therefore, $l_{\bar{r}}(W_{1,4}) = |\{v_1, v_3\}| = 2$.

Let us now consider the case $m \geq 8$. If $|S| \leq 3$, then $g_{\bar{S} \cup \{c_0\}}(a, b) \leq 3$, for all $a, b \in \bar{S}$ and hence $\bar{S} \in \mathcal{R}(W_{1,m})$, a contradiction. Hence $l_{\bar{r}}(W_{1,m}) = |S| \geq 4$. Now, to prove the reverse inequality, consider the set $T = \{v_2, v_3, v_4, v_5\}$. For this set T , we see that $g_{T \cup \{c_0\}}(v_2, v_5) \geq 4$ and $g_{\bar{T} \cup \{c_0\}}(v_1, v_6) \geq 4$. So, by Theorem 7.11, $T \notin \mathcal{R}(W_{1,m})$ and $\bar{T} \notin \mathcal{R}(W_{1,m})$. Hence the set $T \in \bar{\mathcal{R}}(W_{1,m})$. Thus, $l_{\bar{r}}(W_{1,m}) \leq |T| = 4$. \square

Remark 7.15. *For $5 \leq m \leq 7$ and for any subset S of vertices of $W_{1,m}$ either the set S or the set \bar{S} satisfies the conditions (i) and (ii) of Theorem 7.11. Therefore, $\bar{\mathcal{R}}(W_{1,m}) = \emptyset$ for $5 \leq m \leq 7$.*

Lemma 7.16. *For any integer $m \geq 9$, if $S \in \mathcal{R}(W_{1,m})$ and $|S| = l_r(W_{1,m})$, then $g_{\bar{S} \cup \{c_0\}}(a, b) \leq 2$ for all $a, b \in \bar{S}$.*

Proof. Let $S \in \mathcal{R}(W_{1,m})$. If possible let $g_{\bar{S} \cup \{c_0\}}(a, b) \geq 3$ some $a, b \in \bar{S}$. Then $\{v_i, v_{i+1}, v_{i+2}\} \subset S$ for some $1 \leq i \leq m-2$. But then, the set $S' = S - \{v_{i+1}\}$ satisfies the condition (i) and (ii) of Theorem 7.11 (since, for all $c, d \in S'$, we have $g_{S' \cup \{c_0\}}(c, d) = 1$ if $\{c, d\} = \{v_i, v_{i+2}\}$, $g_{S' \cup \{c_0\}}(c, d) = g_{S \cup \{c_0\}}(c, d)$ otherwise, and S satisfies conditions (i) and (ii)). Hence $S' \in \mathcal{R}(W_{1,m})$ with $|S'| < |S|$, a contradiction to the fact that $|S| = l_r(W_{1,m})$. \square

Lemma 7.17. *For any integer $m \geq 9$, if $S \in \mathcal{R}(W_{1,m})$ and $|S| = l_r(W_{1,m})$, then $\bar{S} \in \mathcal{R}(W_{1,m})$.*

Proof. Let $S \in \mathcal{R}(W_{1,m})$ and $|S| = l_r(W_{1,m})$. If possible, let $\bar{S} \notin \mathcal{R}(W_{1,m})$. Then, by Lemma 7.16 and Theorem 7.11, without loss of generality we get $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\} \subset S$ for some $v_i \in \bar{S}$, $3 \leq i \leq m-2$. But then, the set $S' = [S \cup \{v_i\}] - \{v_{i-1}, v_{i+1}\}$ satisfies all the conditions of Theorem 7.11 (since, for all $c, d \in S'$, we have $g_{S' \cup \{c_0\}}(c, d) = 1$ if $\{c, d\} = \{v_{i-2}, v_i\}$ or $\{c, d\} = \{v_i, v_{i+2}\}$, $g_{S' \cup \{c_0\}}(c, d) = g_{S \cup \{c_0\}}(c, d)$ otherwise, and S satisfies conditions (i) and (ii)). Hence $S' \in \mathcal{R}(W_{1,m})$ with $|S'| < |S|$, a contradiction to the fact that $|S| = l_r(W_{1,m})$. \square

Lemma 7.18. *Let $S \in \mathcal{R}(W_{1,m})$ and $m \equiv 1 \pmod{5}$. Then if $g_{S \cup \{c_0\}}(v_{i-1}, v_{i+1}) = g_{S \cup \{c_0\}}(v_{i+1}, v_{i+3}) = 1$ for some $v_{i-1}, v_{i+1}, v_{i+3} \in S$, then $|S| > l_r(W_{1,m})$.*

Proof. Let $m = 5k + 1$ and $k \geq 2$. Then, by Theorem 7.2, we have $l_r(W_{1,m}) = 2k$. Let $S \in \mathcal{R}(W_{1,m})$ be arbitrary. If $\{v_{j+1}, v_{j+2}, v_{j+3}\} \not\subset \bar{S}$ for any $v_j, v_{j+4} \in S$, then by condition (ii) of Theorem 7.11, we get $|S| \geq \lceil \frac{2m}{5} \rceil = 2k + 1$ (since for every five element subset of $V(W_{1,m})$ has at least 2 elements in S). Hence the result holds in this case. Else if, $\{v_{j+1}, v_{j+2}, v_{j+3}\} \subset \bar{S}$ for some $v_j, v_{j+4} \in S$, then without loss of generality we can take $j = 1$ and hence $v_1, v_5 \in S$. But then, by condition (ii) of the Theorem 7.11, $|\{v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}\} \cap S| \geq 2$ for every $k \geq 3$. Now if $g_{S \cup \{c_0\}}(v_{i-1}, v_{i+1}) = g_{S \cup \{c_0\}}(v_{i+1}, v_{i+3}) = 1$, for some $v_{i-1}, v_{i+1}, v_{i+3} \in S$, then $|\{v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \cap S| = 3$. Hence, we get $|T| = \sum_{i=1}^{(m-1)/5} |T_i| > \frac{2(m-1)}{5} = 2k$, where $T_i = \{v_{5i-3}, v_{5i-2}, v_{5i-1}, v_{5i}, v_{5i+1}\}$. \square

Lemma 7.19. *For any integer $m \geq 9$, $l_{\hat{r}}(W_{1,m}) \geq l_r(W_{1,m}) + \xi$, where $\xi = 2$ if $m \equiv 1 \pmod{5}$ and $\xi = 1$ otherwise.*

Proof. Let $S \in \hat{\mathcal{R}}(W_{1,m})$ with $|S| = l_{\hat{r}}(W_{1,m})$. Then $S \in \mathcal{R}(W_{1,m})$ and $\bar{S} \notin \mathcal{R}(W_{1,m})$. So, by Lemma 7.17 it is suffices to prove the case $m \equiv 1$ under modulo 5. Let $m = 5k + 1$ and $k \geq 2$. If possible, let $|S| = l_r(W_{1,m}) + 1 = \lfloor \frac{2m+2}{5} \rfloor + 1 = \lfloor \frac{2(5k+1)+2}{5} \rfloor + 1 = 2k + 1$. Since $\bar{S} \notin \mathcal{R}(W_{1,m})$, by Theorem 7.11, we have (i) $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \subset S$ or (ii) $\{v_i, v_{i+1}, v_{i+3}, v_{i+4}\} \subset S$ for some $v_{i+2} \notin S$ and $1 \leq i \leq m-4$ or (iii) $\{v_i, v_{i+1}, v_{i+3}, v_{i+k}, v_{i+k+1}, v_{i+k+2}\} \subset S$ for some $k > 4$ and $5 < i+k \leq m-2$.

Case 1: $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \subset S$.

Consider the set $S' = [S \cup \{v_{i+4}\}] - \{v_{i+1}, v_{i+3}\}$ (or $S' = [S \cup \{v_{i-1}\}] - \{v_{i+1}, v_{i+3}\}$ if $i+4 > m$). The set $S' \in \mathcal{R}(W_{1,m})$ (since it $S \in \mathcal{R}(W_{1,m})$ and no two element of S' will create a gap more than 2). Further, $g_{S' \cup \{c_0\}}(v_i, v_{i+2}) = 1 = g_{S' \cup \{c_0\}}(v_{i+2}, v_{i+4})$ and hence by Lemma 7.18, we get $|S| - 1 \geq |S'| > l_r(W_{1,m}) \Rightarrow |S| \geq l_r(W_{1,m}) + 2$.

Case 2: $\{v_i, v_{i+1}, v_{i+3}, v_{i+4}\} \subset S$ for some $v_{i+2} \notin S$ and $1 \leq i \leq m-4$.

Consider the set $S' = [S \cup \{v_{i+2}\}] - \{v_{i+1}, v_{i+3}\}$. The set $S' \in \mathcal{R}(W_{1,m})$ (since $S \in \mathcal{R}(W_{1,m})$ and no two element of S' will create a gap more than 2). Further, $g_{S' \cup \{c_0\}}(v_i, v_{i+2}) = 1 = g_{S' \cup \{c_0\}}(v_{i+2}, v_{i+4})$ and hence by Lemma 7.18, we get $|S| - 1 = |S'| > l_r(W_{1,m}) \Rightarrow |S| \geq l_r(W_{1,m}) + 2$.

Case 3: $\{v_i, v_{i+1}, v_{i+3}, v_{i+k}, v_{i+k+1}, v_{i+k+2}\} \subset S$ for some $k > 4$ and $5 < i+k \leq m-2$.

Consider the set $S' = S - \{v_{i+1}, v_{i+k+1}\}$. The set $S' \in \mathcal{R}(W_{1,m})$ (since $S \in \mathcal{R}(W_{1,m})$ and no two element of S' will create a gap more than 2) and hence $|S| - 2 = |S'| \geq l_r(W_{1,m}) \Rightarrow |S| \geq l_r(W_{1,m}) + 2$. \square

Theorem 7.20. For any positive integer $m \geq 3$,

$$l_{\hat{r}}(W_{1,m}) = \begin{cases} m, & \text{for } m = 3. \\ \lfloor \frac{2(m+4)}{5} \rfloor, & \text{for } m \geq 4. \end{cases}$$

Proof. Let $S \in \hat{\mathcal{R}}(W_{1,m})$ be such that $|S| = l_{\hat{r}}(W_{1,m})$. Then $S \in \mathcal{R}(W_{1,m})$ and $\bar{S} \notin \mathcal{R}(W_{1,m})$. For $m = 3$, the result holds by Theorem 6.3. For the case $m = 4, 5$; if S has only two adjacent rim vertices, then \bar{S} contains two adjacent rim vertices. Hence, by Remark 7.10, $\bar{S} \in \mathcal{R}(W_{1,4})$, a contradiction. Or, if $m = 4, 5$ and S has only two non-adjacent rim vertices v_i, v_{i+2} , then S will not resolve the pair $c_0, v_{i+1} \in \bar{S}$, a contradiction to the fact that $S \in \mathcal{R}(W_{1,m})$. Thus, $|S| = 3$ for $m = 4, 5$ (Since $\{v_1, v_2, v_3\} \in \hat{\mathcal{R}}(W_{1,4})$ and $\{v_1, v_2, v_4\} \in \hat{\mathcal{R}}(W_{1,5})$). If $m = 6$ and $|S| = 2$, then only possible $S = \{v_i, v_{i+2}\}$ for some $1 \leq i \leq 4$ (follows by Theorem 7.12). But then, S will not resolve the pair $c_0, v_i \in \bar{S}$, again a contradiction to the fact that $S \in \mathcal{R}(W_{1,m})$. Further, any set S containing any three rim elements of $W_{1,6}$ satisfies the conditions (i) and (ii) of Theorem 7.14 and hence $l_{\hat{r}}(W_{1,6}) = 3$.

If $m = 7, 8$, then by Theorem 7.2 we have $|S| \geq 3$ (since $S \in \mathcal{R}(W_{1,m})$). Further if $|S| = 3$, then $\langle S \rangle$ is disconnected (else condition (i) of Theorem 7.8 fails) and hence $g_{\bar{S} \cup \{c_0\}}(a, b) \leq 2$ for all $a, b \in \bar{S}$. So, $\bar{S} \in \mathcal{R}(W_{1,m})$ (Theorem 7.8), a contradiction. Therefore, $l_{\hat{r}}(W_{1,m}) = |S| \geq 4$. Further, for the subset $T = \{v_i, v_{i+1}, v_{i+3}, v_{i+4}\}$ we see that $g_{T \cup \{c_0\}}(v_{i+1}, v_{i+3}) = 1$ and $g_{T \cup \{c_0\}}(v_i, v_{i+4}) \geq 2$. So, the set $T \in \mathcal{R}(W_{1,m})$ (by Theorem 7.11). Also, $g_{\bar{T} \cup \{c_0\}}(v_{i-1}, v_{i+2}) = 2 = g_{\bar{T} \cup \{c_0\}}(v_{i+2}, v_{i+5})$ and hence $\bar{T} \notin \mathcal{R}(W_{1,m})$. So, $T \in \hat{\mathcal{R}}(W_{1,m})$. So, $l_{\hat{r}}(W_{1,m}) \leq |T| = 4$. Thus, $l_{\hat{r}}(W_{1,m}) = 4$ if $m = 7, 8$.

When $m \geq 9$, let $S' = \{v_i : 8 \leq i \leq m\}$ and $W = \{v_2, v_3, v_5, v_6\}$. Then, W resolves all the pairs x, y whenever either $x, y \in \{v_j : 1 \leq j \leq 7\}$, or $x \in \{v_j : 1 \leq j \leq 7\}$ and $y \in W$. That is, it will not resolve any two vertices in $G' = \langle S' \rangle$. But then, as $G' \cong F(1, |S'|)$, we have $T = W \cup W' \in \mathcal{R}(W_{1,m})$ and

$\bar{T} \notin \mathcal{R}(W_{1,m})$, for every $W' \in \mathcal{R}(F(1, |S'|))$. Thus, $l_{\hat{r}}(W_{1,m}) \leq |T| = |W| + l_r(F(1, |S'|)) = 4 + \lfloor \frac{2((m-7)+1)}{5} \rfloor = \lfloor \frac{2(m+4)}{5} \rfloor$ (by Theorem 7.4). On the other hand, From Theorem 7.2 and Lemma 7.19 it follows, for all $m \geq 9$, that $l_{\hat{r}}(W_{1,m}) \geq l_r(W_{1,m}) + \xi = \lfloor \frac{2(m+2)}{5} \rfloor + \xi$, where $\xi = 2$ if $m \equiv 1 \pmod{5}$ and $\xi = 1$ otherwise. This is same as saying $l_{\hat{r}}(W_{1,m}) \geq \lfloor \frac{2(m+4)}{5} \rfloor$. \square

8. CONCLUSION

In this paper, it is shown that there are some graphs whose foul resolving number is constant but its metric dimension is not constant. However, for most of the graphs the powerful resolving number is constant whenever metric dimension is constant. Finally we found, for all the graphs considered here, that the maximal resolving number is not independent of its order. In conclusion, the new dimensions introduced and analyzed here may be a new network parameter in the study of network theory, and the networks with prescribed dimensions is the essence in network security issues.

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