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We consider F/M/Type IIA theory compactified to four, three, or two dimensions on a Calabi-Yau four-fold, and study the behavior near an isolated singularity in the presence of appropriate fluxes and branes. We analyze the vacuum and soliton structure of these models, and show that near an isolated singularity, one often generates massless chiral superfields and a superpotential, and in many instances in two or three dimensions one obtains nontrivial superconformal field theories. In the case of two dimensions, we identify some of these theories with certain Kazama-Suzuki coset models, such as the $\mathcal{N} = 2$ minimal models.

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1. Introduction

We have learned in recent years that it is fruitful to study singular limits of string compactifications. In this paper, we consider theories with four supercharges in four, three, and two dimensions, constructed by considering F-theory, M-theory, and Type IIA string theory on a Calabi-Yau four-fold with an isolated complex singularity. We can connect these theories to each other by circle compactifications from four to three to two dimensions. In addition to the choice of singularity, the description of these theories depends on certain additional data involving the four-form flux and membrane charge in M-theory (and related objects in the other theories).

We will analyze the vacuum structure of these theories and the domain walls connecting the possible vacua. We argue that in many cases, the nonperturbative physics near a singularity generates massless chiral superfields with a superpotential, leading in many instances, especially in two dimensions or in three dimensions with large membrane charge, to an infrared flow to a nontrivial conformal field theory. In some cases, we can identify the theories in question with known superconformal models; for example, Type IIA at a four-fold A_n singularity gives an $\mathcal{N} = 2$ Kazama-Suzuki model in two dimensions, as we argue using the vacuum and soliton structure. More generally, from the A-D-E four-fold hypersurface singularities with appropriate fluxes, we obtain all the $\mathcal{N} = 2$ Kazama-Suzuki models [1] at level one. This is a large list of exactly solvable conformal theories, which includes the $\mathcal{N} = 2$ unitary minimal models. It is quite satisfying that strings in the presence of singularities captures such a large class of known conformal theories in two dimensions, and suggests that maybe even in higher dimensions, strings propagating in singular geometries yield an equally large subspace of conformal theories. Moreover, since the Kazama-Suzuki models are exactly solvable conformal theories in two dimensions, it would be interesting to see to what extent its known spectrum and correlation functions can be extracted from string theory. This would be a natural testing grounds in view of potential application to higher dimensions where the conformal theories are less well understood.

Our result gives a relation between singularity theory, as in the Landau-Ginzburg approach to conformal theories in two dimensions [2-4], and the singularity of internal compactification geometry. This relation may well extend to non-supersymmetric examples; it would certainly be interesting to explore this.

In section 2, we analyze the fluxes, branes, and vacuum states near a four-fold singularity. In section 3, we show how to compute the spectrum of domain walls (which can

also be viewed as strings and kinks in the three and two-dimensional cases) for a special class of singularities. In section 4, we identify the models derived from A-D-E singularities with Kazama-Suzuki models at level 1. In section 5, we discuss some additional classes of singularities on four-folds, and in section 6, we discuss the reinterpretation of some of our results in terms of branes.

2. Classification Of Vacua

2.1. The G-Field And Domain Walls

For our starting point, we take M-theory on $\mathbf{R}^3 \times Y$, where Y is a compact eight-manifold. Soon, we will specialize to the case that Y is a Calabi-Yau four-fold, so as to achieve supersymmetry. We also will note in section 2.5 the generalization of our remarks to Type IIA or F-theory compactification on Y.

To fully specify a vacuum on Y, one must specify not just Y but also the topological class of the three-form potential C of M-theory, whose field strength is $G = dC$. Roughly speaking, C-fields are classified topologically by a characteristic class $\xi \in H^4(Y; \mathbf{Z})$. At the level of de Rham cohomology, ξ is measured by the differential form $G/2\pi$, and we sometimes write it informally as $\xi = [G/2\pi]$. [1]

Without breaking the three-dimensional Poincaré symmetries, this model can be generalized by picking N points $P_i \in Y$ and including N membranes with world-volumes of the form $\mathbf{R}^3 \times P_i$. More generally, we include both membranes and antimembranes and let N be the difference between the number of membranes and antimembranes; thus it can be a positive or negative integer. With Y being compact, the net source of the C-field must vanish; this gives a relation [6,5]

$$N = \frac{\chi}{24} - \frac{1}{2} \int_Y \frac{G \wedge G}{(2\pi)^2}. \tag{2.1}$$

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[1] The assertion that ξ takes values in $H^4(Y; \mathbf{Z})$ is a bit imprecise, since in general [5] the G-field is shifted from standard Dirac quantization and ξ is not an element of $H^4(Y; \mathbf{Z})$. But the difference between two C-fields is always measured by a difference $\xi - \xi' \in H^4(Y; \mathbf{Z})$. ξ itself takes values in a "principal homogeneous space" Λ for the group $H^4(Y; \mathbf{Z})$; the relation between $H^4(Y; \mathbf{Z})$ and Λ is just analogous to the relation between $H^2(Y; \mathbf{Z})$ and the set of Spin_c structures on Y. The shift in the quantization law of G arises precisely when the intersection form on $H^4(Y; \mathbf{Z})$ is not even. In our examples, this will occur only in section 5, and we will ignore this issue until that point.

If G obeys the shifted quantization condition mentioned in the last footnote, then the right hand side of (2.1) is always integral [5].

In this construction, models defined using the same Y but different ξ are actually different states of the same model.[2] To show this, it suffices to describe a domain wall interpolating between models with the same Y and with C-fields of arbitrary characteristic classes ξ_1 and ξ_2. By Poincaré duality, $H_4(Y; \mathbf{Z}) = H^4(Y; \mathbf{Z})$. Hence, there is a four-cycle $S \subset Y$, representing an element of $H_4(Y; \mathbf{Z})$, such that if $[S] \in H^4(Y, \mathbf{Z})$ is the class that is Poincaré dual to S, then $\xi_2 - \xi_1 = [S]$. Now, consider a fivebrane in $\mathbf{R}^3 \times Y$ whose worldvolume is $W = \mathbf{R}^2 \times S$, with $\mathbf{R}^2$ a linear subspace of $\mathbf{R}^3$. Being of codimension one in spacetime, this fivebrane looks macroscopically like a domain wall. Moreover, because the fivebrane is a magnetic source of G, the characteristic class $\xi = [G/2\pi]$ jumps by $[S]$ in crossing this domain wall. Hence if it equals ξ_1 on one side, then it equals $\xi_2 = \xi_1 + [S]$ on the other side.

Equation (2.1) implies that if ξ jumps in crossing a domain wall, then N must also jump. Let us see how this comes about. The key is that there is a self-dual three-form T on the fivebrane with a relation

$$dT = G|_W - 2\pi\delta(\partial M). \qquad (2.2)$$

where $G|_W$ is the restriction of the G-field to the world-volume W, ∂M is the union of all boundaries of membrane worldvolumes that terminate on W, and $\delta(\partial M)$ is a four-form with delta function support on ∂M. Because the G-field actually jumps in crossing the fivebrane, it is not completely obvious how to interpret the term $G|_W$. We will assume that this should be understood as the average of the G-field on the two sides: $G|_W = (G_1 + G_2)/2$. Since the left hand side of (2.2) is zero in cohomology, we get a relation in cohomology

$$[\partial M] = \frac{G_1 + G_2}{2(2\pi)} \qquad (2.3)$$

where $[\partial M]$ is the cohomology class dual to ∂M. We are interested in the case that the membrane worldvolumes are of the form $\mathbf{R}^3 \times P_i$, so that their boundaries on W are of the form $\mathbf{R}^2 \times P_i$. In evaluating (2.3), we can suppress the common $\mathbf{R}^2$ factor, and integrate over S to get a cohomology relation. The integration converts $[\partial M]$ into $N_1 - N_2$, the

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[2] This may also be true of models with different Y – as suggested by results of [7] in the threefold case – but this issue is much harder to explore.

change in N in crossing the domain wall or in other words the net number of membranes whose boundary is on the fivebrane. So

$$N_1 - N_2 = \frac{1}{2} \int_S \frac{G_1 + G_2}{2\pi} = -\frac{1}{2} \int_Y \frac{G_1^2 - G_2^2}{(2\pi)^2}. \tag{2.4}$$

Here we have used the fact that $[S] = [(G_2 - G_1)/2\pi]$ to convert an integral over S to one over Y. Clearly, (2.4) is compatible with the requirement that (2.1) should hold on both sides of the domain wall.

Incorporation Of Supersymmetry

Now we wish to specialize to the case that Y is a Calabi-Yau four-fold and to look for vacua with unbroken supersymmetry.

For this, several restrictions must be imposed. The requirements for G, assuming that one wants unbroken supersymmetry with zero cosmological constant, have been obtained in an elegant computation [8]. The result is that G must be of type $(2,2)$ and must be primitive, that is, it must obey

$$K \wedge G = 0, \tag{2.5}$$

where K is the Kähler form. We will analyze this condition in Appendix I, but for now we note that it implies that G is self-dual and hence in particular that

$$\int_Y \frac{G \wedge G}{(2\pi)^2} \geq 0, \tag{2.6}$$

with equality only if $G = 0$.

The second basic consequence of supersymmetry is that N must be positive. Only membranes and not anti-membranes on $\mathbf{R}^3 \times P_i$ preserve the same supersymmetry that is preserved by the compactification on the complex four-fold Y.

Given that N must be positive, the relation (2.1) implies that

$$\int_Y \frac{G \wedge G}{(2\pi)^2} \leq \frac{\chi}{12}. \tag{2.7}$$

This inequality together with self-duality implies that, for compact Y, there are only finitely many choices of G that are compatible with unbroken supersymmetry. For χ negative, there are none at all.

Energetic Considerations

There is another way to understand the result that $\Phi = N + \frac{1}{8\pi^2} \int G \wedge G$ should not change in crossing the domain wall. This is based on a finite energy condition. The condition that the domain wall be flat and have finite tension requires that the energy density on the two sides of the domain wall should be equal far away from the domain wall. The energy density in the bulk gets contribution from the G flux given by $\frac{1}{8\pi^2} \int G \wedge *G$ and from the membranes by N. For the supersymmetric situation we are considering, G is self-dual (as explained in Appendix I), i.e., $G = *G$ so the energy density is given by $N + \frac{1}{8\pi^2} \int G \wedge G$, and so its constancy across a domain wall is a consequence of the finite energy of the domain wall. This is also important in our applications later, as we will use a BPS formula for the mass of domain walls. Due to boundary terms at infinity, such formulas are generally not valid for objects of very low codimension in space (the codimension is one in our case). The cancellation of the boundary terms in question in our case is again precisely the condition of constancy of Φ across the domain wall.

2.2. Behavior Near A Singularity

So far we have considered the case of a compact smooth manifold Y. Our main interest in the present paper, however, is to study the behavior as Y develops a singularity. For practical purposes, it is convenient then to omit the part of Y that is far from the singularity and to consider a complete but not compact Calabi-Yau four-fold that is developing a singularity. In fact, some of the singularities we will study – like the A_n singularities of a complex surface for very large n – probably cannot be embedded in a compact Calabi-Yau manifold. Our discussion will apply directly to an isolated singularity of a non-compact variety.

Hypersurface singularities are an important example and will be our focus in this paper except in section 5. For example, one of our important applications will be to quasi-homogeneous hypersurface singularities. In this case, we begin with five complex variables z_a, $a = 1, \ldots, 5$ of degree $r_a > 0$ and a polynomial $F(z_1, \ldots, z_5)$ that is homogeneous of degree 1. We assume that F is such that the hypersurface $F = 0$ is smooth except for an isolated singularity at $z_1 = \ldots = z_5 = 0$. Then we let X be a smooth deformation of this singular hypersurface such as

$$F(z_1, \ldots, z_5) = \epsilon \tag{2.8}$$

with ϵ a constant, or more generally

$$F(z_1, \ldots, z_5) = \sum_i t_i A_i(z_1, \ldots, z_5), \tag{2.9}$$

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with complex parameters t_i and polynomials A_i that describe relevant perturbations of the singularity $F = 0$. The singular hypersurface $F = 0$ admits the $U(1)$ symmetry group

$$z_a \rightarrow e^{i\theta r_a} z_a. \tag{2.10}$$

Under this transformation, the holomorphic four-form

$$\Omega = \frac{dz_2 \wedge dz_3 \wedge dz_4 \wedge dz_5}{\partial F/\partial z_1} \tag{2.11}$$

has charge

$$r_\Omega = \sum_a r_a - 1. \tag{2.12}$$

The $U(1)$ symmetry in (2.10) is an R-symmetry group if $r_\Omega \neq 0$. If the model is to flow in the infrared to a superconformal field theory, an R-symmetry must appear in the superconformal algebra; we propose that it is the symmetry just identified. If the A_i have degrees r_i, then the dimensions of the corresponding operators are proportional to r_i/r_Ω (in other words, the R-charges normalized so that Ω has R-charge 1). Since the r_i are positive, requiring that the dimensions be positive gives a condition $r_\Omega > 0$:

$$\sum_a r_a > 1. \tag{2.13}$$

We will see the importance of this condition from several points of view.

In what sense is such an X a Calabi-Yau manifold? The holomorphic four-form Ω defined in (2.11) has no zeroes or poles on X, though it has in a certain sense (using the compactification described in the next paragraph) a pole at infinity. A theorem of Tian and Yau [9] asserts, assuming (2.13), that there is a Calabi-Yau metric on X with volume form

$$|\Omega \wedge \overline{\Omega}|, \tag{2.14}$$

and moreover (see the precise statement in eqn. (2.3) of [9]) this metric is asymptotically conical, that is it looks near infinity like

$$ds^2 = dr^2 + r^2 ds_\perp^2. \tag{2.15}$$

Here $ds_\perp$ is an "angular" metric, and r is a "radial" coordinate near infinity which scales under $z_a \rightarrow \lambda^{r_a} z_a$ as $r \rightarrow \lambda^{(\Sigma_a r_a - 1)/4} r$. This exponent ensures that the volume form derived from (2.15) scales like $|\Omega|^2$.

To apply the Tian-Yau theorem to the hypersurface X and deduce the statements in the last paragraph, one writes $r_a = b_a/d$ with b_a relatively prime integers and d a positive integer. Then one introduces another complex variable w of degree $1/d$, and one compactifies X to the compact variety Y' defined by the equation $F(z_i) - \epsilon w^d = 0$ in a weighted projective space. The discussion of [9] applies to this situation, with D the divisor $w = 0$, and identifies r with a fractional power of $|w|$.

It is very plausible that if a compact Calabi-Yau manifold Y develops an isolated hypersurface singularity that is at finite distance on the moduli space, then the Calabi-Yau metric on Y looks locally like the conelike metric that we have just described on the hypersurface X.[3] For our purposes, we do not strictly need to know that this is true, but the physical applications are certainly rather natural if it is.

Flux At Infinity

Noncompactness of X leads to several important novelties in the specification of the model. First of all, flux can escape to infinity, and hence (2.1) no longer holds. Rather, an extra term appears in (2.1), namely the flux Φ measured at infinity. This flux is a constant of the motion, invariant under the dynamics which occurs in the "interior" of X. If we absorb the constant $\chi/24$ in the definition of Φ,[4] then we can write the conserved quantity as

$$\Phi = N + \frac{1}{2} \int_X \frac{G \wedge G}{(2\pi)^2}. \tag{2.16}$$

We can think of Φ as a constant that must be specified (in addition to giving X) in order to determine the model. A model with given Φ has various vacuum states, determined by the values of N and G. For unbroken supersymmetry, both terms on the right hand side of (2.16) are positive (for the same reasons as in the case of compact X), so there are only finitely many possible choices of N and G for fixed Φ.

In addition to Φ, there is another quantity that characterizes the definition of the model – and commutes with the dynamics. For finiteness of the energy, it is reasonable to require that the flux G vanishes if restricted to ∂X, the region near infinity in X. This

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[3] There is no claim here that X can be globally embedded in Y, only that the behavior of Y near its singularity can be modeled by X. Note that the variety Y' used in the last paragraph in relation to the Tian-Yau theorem is not a Calabi-Yau manifold.

[4] When X is not compact, χ must in any event be defined by a curvature integral and need not coincide with the topological Euler characteristic.

does not imply that the cohomology class ξ vanishes if restricted to ∂X, but only that its restriction is a torsion class. Thus, the C-field at infinity is flat, but perhaps topologically non-trivial. Local dynamics cannot change the behavior at infinity, so the restriction of ξ to ∂X is another invariant of the local dynamics, which must be specified in defining a model.

There is another way to see more explicitly how this invariant comes about. For this, we have to look at precisely what Poincaré duality says in the case of a noncompact manifold X. Domain walls of the type introduced in section 2.1 are classified by $H_4(X; \mathbf{Z})$, which classifies the cycles S on which a fivebrane can wrap to make a domain wall. Poincaré duality says that this is the same as $H^4_{cpct}(X; \mathbf{Z})$ (where H^4_{cpct} denotes cohomology with compact support), [5] the rough idea being that if S is a four-cycle determining an element of $H_4(X; \mathbf{Z})$ (so in particular S is by definition compact), then the Poincaré dual class $[S]$ is represented by a delta function $\delta(S)$ that has compact support. C-fields on X are classified topologically by $\xi \in H^4(X; \mathbf{Z})$. The groups $H^4(X; \mathbf{Z})$ and $H^4_{cpct}(X; \mathbf{Z})$ that classify, respectively, topological classes of C-fields and of changes in C-fields in crossing domain walls are in general different for non-compact X. However, there is always a natural map

$$i : H^4_{cpct}(X; \mathbf{Z}) \to H^4(X; \mathbf{Z}) \tag{2.17}$$

(by "forgetting" that a class has compact support). Moreover, for hypersurface singularities, $H^4_{cpct}(X; \mathbf{Z})$ and $H^4(X; \mathbf{Z})$ are lattices, which we will call Γ and Γ^* respectively. Poincaré duality in the noncompact case says that Γ and Γ^* are dual lattices. When the intersection form on $H^4_{cpct}(X; \mathbf{Z})$ has no null vectors, the map i is an embedding, and Γ can be regarded as a sublattice of its dual lattice Γ^*. This makes things very simple.

The lattice Γ can actually be described rather simply. In fact, topologically, the hypersurface X is homotopic to a "bouquet" of four-spheres. [6] $H_4(X; \mathbf{Z})$ is a lattice Γ with one generator for every four-sphere in the bouquet.

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[5] The cohomology of X with compact support is generated by closed forms β on X with compact support, subject to the equivalence relation that $\beta \cong \beta + d\epsilon$ if ϵ has compact support.

[6] Such a bouquet is, by definition, associated with a tree diagram in which the vertices represent four-spheres and two vertices are connected by a line if and only if the corresponding four-spheres intersect. Such a diagram has the form of a simply-laced Dynkin diagram (with vertices for four-spheres and lines for intersections of them), except that the Cartan matrix may not be positive definite and thus one is not restricted to the A-D-E case.

In crossing a domain wall, ξ cannot change by an arbitrary amount, but only by something of the form $i([S])$ where $[S]$ is a class with compact support. The possible values of ξ modulo changes due to the dynamics, that is due to crossing domain walls, are thus classified by

$$H^4(X; \mathbf{Z})/i(H^4_{cpct}(X; \mathbf{Z})) = \Gamma^*/\Gamma. \tag{2.18}$$

This can be reinterpreted as follows. The exact sequence of the pair $(X, \partial X)$ reads in part

$$\ldots H^4(X, \partial X; \mathbf{Z}) \xrightarrow{i} H^4(X; \mathbf{Z}) \xrightarrow{j} H^4(\partial X; \mathbf{Z}) \to H^5(X, \partial X; \mathbf{Z}) \to \ldots. \tag{2.19}$$

Here $H^i(X, \partial X; \mathbf{Z})$ is the same as $H^i_{cpct}(X; \mathbf{Z})$. By Poincaré duality, $H^5(X, \partial X; \mathbf{Z}) = H_3(X; \mathbf{Z})$, and this is zero on dimensional grounds for a bouquet of four-spheres. So the exact sequence implies that

$$H^4(\partial X; \mathbf{Z}) = H^4(X; \mathbf{Z})/i(H^4_{cpct}(X; \mathbf{Z})) = \Gamma^*/\Gamma. \tag{2.20}$$

Thus the value of ξ modulo changes in crossing domain walls (the right hand side of (2.20)) can be identified with the restriction of ξ to ∂X (the left hand side).

If the intersection pairing on $H_4(X; \mathbf{Z})$ is degenerate, then we should define Γ to be the quotient of $H^4(X, \partial X; \mathbf{Z})$ by the group of null vectors (which can be shown to be precisely the image in $H^4(X, \partial X; \mathbf{Z})$ of $H^3(\partial X; \mathbf{Z})$). The dual Γ^* is the subgroup of $H^4(X; \mathbf{Z})$ consisting of elements whose restriction to ∂X is torsion. With these definitions of Γ and Γ^*, everything that we have described above carries over (i embeds Γ as a finite index sublattice of Γ^*; the G-fields, with appropriate boundary conditions, take values in Γ^*, and jump in crossing a domain wall by elements of Γ).

Examples

We will now illustrate these perhaps slightly abstract ideas with examples that will be important later.

Consider first the simple case that X is a deformation of a quadric singularity:

$$\sum_{a=1}^{5} z_a^2 = \epsilon. \tag{2.21}$$

If we assume that ϵ is real and write $z_a = x_a + iy_a$, we get $\vec{x}^2 - \vec{y}^2 = \epsilon$ and $\vec{x} \cdot \vec{y} = 0$. Setting $\vec{u} = \vec{x}/\sqrt{\epsilon + \vec{y}^2}$, we see that $\vec{u}$ is a unit vector. The subset of X with $\vec{y} = 0$ is a four-sphere

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S; since $\vec{y} \cdot \vec{u} = 0$, X is the cotangent bundle of S. In particular, X is homotopic to the four-sphere S. This is the case in which the bouquet of spheres is made from just a single sphere. The self-intersection number of S is $S \cdot S = 2$.[7] The lattice $\Gamma = H^4_{cpct}(X; \mathbf{Z})$ is generated by $[S]$, but the dual lattice $\Gamma^* = H^4(X; \mathbf{Z})$ is generated by $\frac{1}{2}[S]$ (whose scalar product with S is 1). So $H^4(\partial X; \mathbf{Z}) = H^4(X; \mathbf{Z})/H^4_{cpct}(X; \mathbf{Z}) = \frac{1}{2}\Gamma/\Gamma = \mathbf{Z}_2$.

A somewhat more sophisticated example is the A_{n-1} singularity in complex dimension four:

$$P_n(z_1) + \sum_{a=2}^{5} z_a^2 = 0. \tag{2.22}$$

Here $P_n(z_1)$ is a polynomial of degree n. For simplicity we take

$$P_n(z_1) = \prod_{i=1}^{n}(z_1 - b_i) \tag{2.23}$$

with real b_i, $b_1 < b_2 < \ldots < b_n$. For $i = 1, \ldots, n-1$, we define a four-sphere S_i by requiring that z_1 is real with $b_i < z_1 < b_{i+1}$, and that the z_j for $j > 1$ are all real or all imaginary depending on the value of i modulo two. The S_i generate the lattice $\Gamma = H^4_{cpct}(X; \mathbf{Z})$. The intersection numbers of the S_i are $S_i^2 = 2$, $S_i \cdot S_{i+1} = 1$, with others vanishing. (S_i intersects S_{i+1} at the single point $z_1 = b_{i+1}$, $z_j = 0$ for $j > 1$; S_i does not intersect S_j if $|j - i| > 1$.) Endowed with this intersection form, Γ is the root lattice of the Lie group $A_{n-1} = SU(n)$, while the dual lattice $\Gamma^* = H^4(X; \mathbf{Z})$ is the weight lattice. The quotient is $H^4(\partial X; \mathbf{Z}) = \Gamma^*/\Gamma = \mathbf{Z}_n$. It can be shown that X is homotopic to the union of the S_i, which form a "bouquet." In this case, the bouquet is associated with the Dynkin diagram of A_n.

More generally, if $H(z_1, z_2, z_3)$ is a polynomial in three complex variables that describes a deformation of an A-D-E surface singularity, we can consider the corresponding surface singularity in complex dimension four:

$$H(z_1, z_2, z_3) + z_4^2 + z_5^2 = 0. \tag{2.24}$$

The case just considered, with $H(z_1, z_2, z_3) = P_n(z_1) + z_2^2 + z_3^2$, corresponds to A_{n-1}. (The appropriate H's for the other cases are written at the end of section 2.5.) For any of the

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[7] To compute this, deform S to the four-sphere S' defined by $y_1 = u_2$, $y_2 = -u_1$, $y_3 = u_4$, $y_4 = -u_3$, $y_5 = 0$. Then S' intersects S at the two points $u_1 = \ldots = u_4 = 0$, $u_5 = \pm 1$, and each point contributes $+1$ to the intersection number. Hence $S \cdot S = S \cdot S' = 2$.

A-D-E examples, Γ is the root lattice of the appropriate simply-connected A-D-E group G, Γ^* is the weight lattice of G, and the quotient $H^4(\partial X; \mathbf{Z}) = \Gamma^*/\Gamma$ is isomorphic to the center of G. One approach to proving these assertions is to show that they are true for the middle-dimensional cohomology of the surface $H(z_1, z_2, z_3) = 0$, and are unaffected by "stabilizing" the singularity by adding two more variables with the quadratic terms $z_4^2 + z_5^2$.

2.3. Distance To Singularity And Hodge Structure Of Cohomology

In the present subsection, we return to the case of a compact Calabi-Yau four-fold Y. We suppose that, when some complex parameters t_i are varied, Y develops a singularity that looks like a quasihomogeneous hypersurface singularity $F(z_1, \ldots, z_5) = 0$, where the z_a have degrees $r_a > 0$ and F is of degree 1. Upon varying the complex structure of Y, the hypersurface is deformed to a smooth one which looks locally like

$$F(z_1, \ldots, z_5) + \sum_i t_i A_i(z_1, \ldots, z_5) = 0. \tag{2.25}$$

Here the t_i are complex parameters, and the A_i are perturbations of the equation.

The first question to examine is whether the singularity at $t_i = 0$ can arise at finite distance in Calabi-Yau moduli space. The Kähler form on the parameter space is

$$\omega = dt^i d\bar{t}^j \frac{\partial^2}{\partial t^i \partial \bar{t}^j} K, \tag{2.26}$$

where K is the Kähler potential K. On the parameter space of a compact Calabi-Yau manifold, the Kähler potential of the Weil-Peterson metric is[8]

$$K = -\ln \int_Y \Omega \wedge \overline{\Omega}. \tag{2.27}$$

We want to analyze a possible singularity of this integral near $z_a = 0$ in the limit that the t_i go to zero. If and only if there is such a singularity, the distance to $t_i = 0$ will be infinite in the metric (2.26). For analyzing this question, the large z_a behavior, which depends on how the singularity is embedded in a compact variety Y, is immaterial (as long as there is some cutoff to avoid a divergence at large z_a); we can, for instance, replace Y by the hypersurface in (2.25) and restrict the integral to the region $|z_a| < 1$.

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[8] The derivation of this formula is just as in the three-fold case; see [10] for an exposition.

To determine the small z_a behavior of the integral, we use a simple scaling. Under $z_a \to \lambda^{r_a} z_a$, Ω scales like $\lambda^{\Sigma_a r_a - 1}$ and so the integral in (2.27) scales like $|\lambda|^{2\Sigma_a r_a - 2}$. Small z_a corresponds to small λ. Hence the condition that the integral converges at small z_a is

$$\sum_a r_a - 1 > 0. \tag{2.28}$$

This is a satisfying result, in that this is the same condition that was needed to get an R-symmetry with positive charges and to apply the Tian-Yau theorem on existence of asymptotically cone-like Calabi-Yau metrics.

We will now apply this kind of reasoning to address the following question, whose importance will become clear: As Y becomes singular, what is the Hodge type of the "vanishing cohomology," that is, of the part of the cohomology that "disappears" at the singularity? We only have to look at middle dimensional cohomology, because the deformation of a hypersurface singularity has cohomology only in the middle dimension.

First let us ask if there is vanishing cohomology of type $(4, 0)$. For this, we normalize the holomorphic $(4, 0)$-form Ω of Y in such a way that far from $z_a = 0$ it has a limit as $t_i \to 0$. Then we ask if the integral

$$\int_Y \Omega \wedge \overline{\Omega} \tag{2.29}$$

converges as $t_i \to 0$. If the answer is *no*, then to make the integral converge as $t_i \to 0$, we would have to rescale Ω so that in the limit it vanishes pointwise away from the singularity. Then in the limit $t_i \to 0$, Ω would be a closed four-form that is non-zero but vanishes away from the singularity. There would thus be vanishing cohomology of type $(4, 0)$. If the answer is *yes*, there is no vanishing cohomology of type $(4, 0)$.

We have already seen that convergence of the integral in (2.29) is the condition that the singularity is at finite distance in moduli space. Hence, singularities that can arise in the dynamics of a compact Calabi-Yau four-fold have no vanishing cohomology of type $(4, 0)$.

Now let us look for vanishing cohomology of type $(3, 1)$. The $(3, 1)$ cohomology is generated by $\Omega_i = D\Omega/Dt_i$, where D/Dt_i is the covariant derivative computed using the Gauss-Manin connection. To determine if Ω_i is a vanishing cycle, we need to examine the integral

$$\int_Y \Omega_i \wedge \overline{\Omega}_i, \tag{2.30}$$

and ask if it is finite as all $t_j \to 0$. If not, then to make the integral converge, we would have to rescale Ω_i by a function of the t_j, and in the limit $t_j \to 0$, Ω_i would represent a

nonzero $(3,1)$ cohomology class that vanishes away from the singularity, or in other words a piece of the vanishing cohomology of type $(3,1)$. The integral (2.30) is more conveniently written as

$$\frac{\partial^2}{\partial t_i \partial \bar{t}_i} \int_Y \Omega \wedge \overline{\Omega}. \tag{2.31}$$

Whether this integral converges can, again, be determined by scaling. If the function A_i in (2.25) scales under $z_a \to \lambda^{r_a} z_a$ as λ^{s_i}, then t_i scales like λ^{1-s_i} and (2.31) scales like $|\lambda|^{w_i}$ with

$$w_i = 2\left(\sum_a r_a + s_i - 2\right). \tag{2.32}$$

Vanishing $(3,1)$ cohomology arises when $w_i \leq 0$, so that the integral in (2.31) diverges near $z = 0$. The most dangerous case is for $A_i = 1$, $s_i = 0$. The condition that $w_i > 0$ for all i, so that there is no vanishing $(3,1)$ cohomology, is thus

$$\sum_a r_a > 2. \tag{2.33}$$

We can classify the models that obey this condition. Consider a Landau-Ginzburg model with chiral superfields Φ_a, $a = 1, \ldots, 5$ and superpotential $F(\Phi_1, \ldots, \Phi_5)$. If Φ_a have degree r_a and F has degree one, then the central charge of this model is $\widehat{c} = \sum_{a=1}^{5}(1 - 2r_a) = 5 - 2\sum_a r_a$. The condition (2.33) thus amounts to[9]

$$\widehat{c} < 1. \tag{2.34}$$

The singularities that obey this condition are the A-D-E singularities. They are given, in a suitable set of coordinates, by

$$F(z_1, \ldots, z_5) = H(z_1, z_2, z_3) + z_4^2 + z_5^2, \tag{2.35}$$

where $H(z_1, z_2, z_3) = 0$ is the equation of an A-D-E surface singularity.

Application To Hypersurface

We have developed this discussion for the case of a compact Calabi-Yau manifold Y that develops a hypersurface singularity, but it is more in the spirit of the present paper to decompactify Y and focus on the hypersurface itself, that is to consider M-theory on

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[9] Note that in terms of $\widehat{c}$ the condition that the local singularity of the fourfold be at finite distance in moduli space (2.28) is that $\widehat{c} < 3$, which generalizes for an n-fold singularity to $\widehat{c} < n-1$.

$\mathbf{R}^3 \times X$, where X is a hypersurface that develops the given singularity. This is the natural framework for studying M-theory near a singularity, with extraneous degrees of freedom decoupled. Let us therefore now explain the significance of the above results for this case.

If we work on the noncompact hypersurface, the condition that $\sum_a r_a + s_i > 2$, which ensures that there is *not* a divergence of $\int |\Omega_i|^2$ near $z_a = 0$, also ensures that there *is* such a divergence near $z_a = \infty$. The large z_a divergence means that, in M-theory on $\mathbf{R}^3 \times X$, the modes that deform the singularity of X have divergent kinetic energy and are not dynamical. They correspond, instead, to coupling constants of the theory near the singularity; they can be specified externally as part of the definition of the problem.

In the A-D-E examples, the complex structure modes are all non-dynamical in this sense. For other examples, positivity of (2.32) does not hold for all i, and therefore some of the complex structure deformations of X are dynamical; they vary quantum mechanically in the theory at the singularity. Only those modes for which $w_i > 0$ can be specified externally and represent coupling constants.

Now let us consider the Hodge type of the G-field in the hypersurface case. For unbroken supersymmetry in flat spacetime, G must be a harmonic $\mathbf{L}^2$ form of type $(2,2)$ [8]. It must, as well, be integral and "primitive."

For hypersurface singularities with asymptotically conical metrics of the type predicted by the Tian-Yau theorem, the condition that G be a harmonic $\mathbf{L}^2$ form is a mild one in the following sense. For an asymptotically conical metric on a manifold X, one expects the space of $\mathbf{L}^2$ harmonic forms of degree i to be isomorphic to the image of $H^i_{cpct}(X; \mathbf{R})$ in $H^i(X; \mathbf{R})$. For hypersurface singularities of complex dimension four, there is only four-dimensional cohomology, so we expect $\mathbf{L}^2$ harmonic forms of degree four only. Assuming there are no null vectors in $H^4_{cpct}(X)$, the image of $H^i_{cpct}(X; \mathbf{R})$ in $H^i(X; \mathbf{R})$ is all of $H^i(X; \mathbf{R})$, so one expects that all of the four-dimensional cohomology is realized by $\mathbf{L}^2$ harmonic forms.

What about the requirement that G be primitive? Primitiveness means that $K \wedge G = 0$, where K is the Kähler form. If G is an $\mathbf{L}^2$ harmonic four-form on a manifold whose $\mathbf{L}^2$ harmonic forms are all four-forms, then $K \wedge G$ is automatically zero (if not zero, it would be an $\mathbf{L}^2$ harmonic six-form). Thus, for singularities of this type, the condition that G

should be primitive is automatically obeyed. [10] In section 5, we will examine a singularity of a different sort for which primitiveness of G is an important constraint.

The remaining constraint that we have not examined yet is a severe constraint in the case of hypersurface singularities. This is the condition that G should be of type $(2, 2)$. For A-D-E singularities, as we have seen above, the vanishing cohomology is all of type $(2,2)$, so the $\mathbf{L}^2$ harmonic forms have this property. For other singularities, with $\sum_a r_a < 2$, there is vanishing cohomology of types $(3,1)$, $(2,2)$, and $(1,3)$. Under such conditions, it is generically very hard to find a non-zero four-form that is of type $(2,2)$ and integral. Once an integral four-form G is picked, requiring that it be of type $(2, 2)$ will generally put restrictions on the complex structure of X. Since some of the complex structure modes are dynamical whenever there is vanishing $(3,1)$ cohomology, the restriction on complex structure that is entailed in making G be of type $(2,2)$ is likely to play an important role in the dynamics of these models. In this paper, to avoid having to deal with the dynamical complex structure modes and the Hodge structure of the singularity, we will study in detail only the A-D-E singularities. (For fourfold examples where moduli are dynamically frozen see [11].)

Here is another way to see the distinguished nature of the A-D-E singularities. As we explain in Appendix I, the intersection form on $H^4(X, \mathbf{Z})$ is positive definite on the primitive cohomology of type $(2,2)$, and negative definite on the primitive cohomology of types $(3,1)$ and $(1,3)$. Hence, in particular, having the primitive cohomology be entirely of type $(2,2)$ is equivalent to positive definiteness of the intersection form on $H^4(X; \mathbf{Z})$. For an intersection form specified by a bouquet of spheres to be positive definite is a condition that singles out the A-D-E Dynkin diagrams, so again we see that the A-D-E singularities are the ones with vanishing cohomology that is entirely of type $(2,2)$.

2.4. Interpretation Of Constraints On G

Since the constraints on G found in [8] have played an important role in this discussion, we will pause here to attempt to gain a better understanding of these constraints.

We consider compactification of M-theory on a compact four-fold Y. We first suppose that G is zero. Variations of the Calabi-Yau metric of Y arise either from variations of

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[10] A different explanation of this is as follows. In section 2.2, we compactified X to a complete but non-Calabi-Yau variety Y' by adding a divisor D at infinity. D is an ample divisor, and the "primitive" cohomology in this situation is the cohomology that vanishes when restricted to D. This is certainly so for the vanishing cohomology, whose support is far from D.

the complex structure or variations of the Kähler structure. The variations of the complex structure are parametrized classically by complex parameters t_i, which we promote to chiral superfields T_i. If $h^{p,q}$ is the dimension of the Hodge group $H^{p,q}(Y)$, then the number of T_i is $h^{3,1}$. The Kähler structure is parametrized classically by $h^{1,1}$ real parameters k_i. Compactification of the C-field on Y gives rise to $h^{1,1}$ $U(1)$ gauge fields a_i on $\mathbf{R}^3$ whose duals are scalars ϕ_i that combine with the k_i to make $h^{1,1}$ chiral superfields that we may call K_i.

If $G = 0$, the expectation values of the T_i and K_i are arbitrary, in the supergravity approximation to M-theory. (Instantons can lift this degeneracy [12].) For non-zero G, this is not so. After picking an integral four-form G (which must be such that $\int_X G \wedge G > 0$ or the equations we are about to write will have no solutions), we must adjust the complex structure of X so that

$$G_{0,4} = G_{1,3} = 0, \tag{2.36}$$

and the Kähler structure of X so that

$$G \wedge K = 0. \tag{2.37}$$

In (2.36), $G_{p,q}$ denotes the (p,q) part of G.

We want to describe an effective action for the T_i and K_j that accounts for (2.36) and (2.37). Since supersymmetric actions of the general kind $\int d^4\theta(\dots)$ do not lift vacuum degeneracies, we look for F-term interactions. Thus, we want a superpotential $W(T_i)$ that will account for (2.36), and an analog of a superpotential $\widetilde{W}(K_j)$ that will account for (2.37). In three dimensions, the fields K_j are in vector multiplets, and the function $\widetilde{W}(K_j)$ is related by supersymmetry to Chern-Simons couplings for those multiplets. We will not try to work out the full details of this here. Upon dimensional reduction to two dimensions, the K_j become twisted chiral multiplets and $\widetilde{W}(K_j)$ becomes the twisted chiral superpotential. Therefore, we will somewhen loosely call $\widetilde{W}$ a superpotential.

To obtain (2.36), we propose to let Ω be a holomorphic four-form on Y, and take

$$W(T_i) = \frac{1}{2\pi} \int_Y \Omega \wedge G. \tag{2.38}$$

This object is not, strictly speaking, a function of the T_i but a section of a line bundle over the moduli space $\mathcal{M}$ of complex structures on Y (on which the T_i are coordinates), since

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it is proportional to the choice of Ω. Let $\mathcal{L}$ be the line bundle over $\mathcal{M}$ whose fiber is the space of holomorphic four-forms on Y. The Kähler form of $\mathcal{M}$ can be written

$$\omega = -\partial\bar{\partial}\ln \int_Y \Omega \wedge \overline{\Omega}, \tag{2.39}$$

in other words it is $\partial\bar{\partial}\ln|\Omega|^2$ for Ω any section of $\mathcal{L}$, and this means [13] that W should be a section of $\mathcal{L}$. Thus, the linear dependence on Ω in (2.38) is the right behavior of a superpotential. In supergravity with four supercharges, the condition for unbroken supersymmetry in flat space is $W = dW = 0$. With W as in (2.38), the condition $W = 0$ is that $G_{0,4} = 0$. Also, since the objects $d\Omega/dt_i$ generate $H^{3,1}(Y)$, the condition $dW = 0$ is that $G_{1,3} = 0$. So we have found the supersymmetric interaction that accounts for (2.36).

Another way to justify (2.38) is to consider supersymmetric domain walls. The tension of a domain wall obtained by wrapping a brane on a four-cycle S is the absolute value of $\int_S \Omega$. If G changes from G_1 to G_2 in crossing the domain wall, then $G_2 - G_1 = 2\pi[S]$, so this integral is

$$\frac{1}{2\pi} \int_X \Omega \wedge (G_2 - G_1). \tag{2.40}$$

In a theory with four supercharges, the tension of a supersymmetric domain wall is the absolute value of the change in the superpotential W. So (2.40) should be the change in W in crossing the domain wall, a statement that is clearly compatible with (2.38).

In a similar spirit, one can readily guess the interaction responsible for (2.37):

$$\widetilde{W}(K_i) = \int_X \mathcal{K} \wedge \mathcal{K} \wedge G. \tag{2.41}$$

Here $\mathcal{K}$ is a complexified Kähler class whose real part is the ordinary Kähler class K. The condition $d\widetilde{W} = 0$ is $\mathcal{K} \wedge G = 0$, whose real part is (2.37). $\widetilde{W} = 0$ is a consequence of this, and imposes no further condition.

In M-theory on a *compact* Calabi-Yau four-fold Y, near a hypersurface singularity, the relation of the change in the superpotential in crossing a domain wall to (2.40) shows that W cannot vanish in all vacua. In the first version of the present paper it was conjectured that vacua with non-zero W correspond to supersymmetric AdS compactifications. However, a more careful analysis in the revised Appendix II (triggered by comments of J. Polchinski) shows that four-fold compactifications with $G_{0,4} \neq 0$ lead to

non-supersymmetric theories which are classically scale invariant[11], the so-called "no-scale models". In fact, since the supersymmetry conditions (2.36) and (2.37) are invariant under overall rescaling of the metric on Y, this conclusion is needed for the superpotential $W(T_i, K_j) = W(T_i) + \widetilde{W}(K_j)$ to describe correctly the effective dynamics of M-theory on a Calabi-Yau four-fold with a G-flux. Going to a non-compact manifold has the effect of decoupling gravity and allows us to avoid this problem in the present discussion.

Going back to supersymmetric compactifications to $\mathbf{R}^3$, it is interesting to compactify one of the directions in $\mathbf{R}^3$ on a circle and consider Type IIA on $\mathbf{R}^2 \times Y$. The above analysis carries over immediately for supersymmetric vacua with a nonzero value of the Ramond-Ramond four-form G. However, in Type IIA string theory, in view of mirror symmetry and other T-dualities, one naturally thinks that one should construct a more general effective superpotential to incorporate the possibility of turning on a full set of Ramond-Ramond fields, and not just the four-form. Indeed, the mirror of $G_{0,4}$ would be the RR zero-form (responsible [14] for the massive deformation of Type IIA supergravity), and the mirror of $G_{1,3}$ would be the RR two-form. This is under investigation [15].

Physical Interpretation

We will now discuss the physical interpretation of the superpotentials that we have computed.

We have computed the superpotential as a function of the superfields T_i and K_j with all other degrees of freedom integrated out. For Y a large, smooth Calabi-Yau four-fold, this is a very natural thing to do, since the superfields T_i and K_j are massless if $G = 0$, while other superfields are massive. However, we have argued that as one approaches a singularity, there are different vacuum states in the theory at the singularity that are specified by different choices of the G-field. We will interpret the theory near the singularity as a theory of dynamical chiral fields Φ_α such that the critical points of the superpotential as a function of Φ_α are given by the possible choices of G-field. Thus, a more complete description of the theory would involve a superpotential function $\widehat{W}(\Phi_\alpha; T_i, K_j)$, such that the function $W(T_i, K_j) = W(T_i) + \widetilde{W}(K_j)$ is obtained by extremizing $\widehat{W}$ with respect to the Φ_α. For fixed choices of T_i and K_j, the extremization with respect to Φ_α has

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[11] In general, quantum corrections presumably break the classical scale invariance which changes the normalization of the Lagrangian. For instance, five-brane instantons are expected to modify the effective superpotential and cause the model to roll down to a supersymmetric vacuum with negative cosmological constant.

different solutions, corresponding to the different choices of G. It is very difficult to see the superfields Φ_α explicitly, but for suitable examples we will identify the superpotential function $\widehat{W}(\Phi_\alpha; T_i, K_j)$ in section 3 by studying the soliton structure.

2.5. Analogs For Type IIA And F-Theory

We have formulated the discussion so far in terms of M-theory on $\mathbf{R}^3 \times Y$, with Y a Calabi-Yau four-fold, but there are close analogs for Type IIA on $\mathbf{R}^2 \times Y$ and (if Y is elliptically fibered) for F-theory on $\mathbf{R}^4 \times Y$.

The analysis of [8] carries over to Type IIA, with G now understood as the Ramond-Ramond four-form field. Our analysis of the vacuum structure also carries over readily to this case. One obvious change is that domain walls are now constructed from four-branes (with world-volume $\mathbf{R} \times S \subset \mathbf{R}^2 \times Y$). Another obvious change is that, in Type IIA, the space-filling membranes that contribute to the formula (2.16) for the flux at infinity are replaced by space-filling fundamental strings. Also, in the Type IIA case, alongside the Ramond-Ramond four-form, one would want to incorporate the Ramond-Ramond zero-form and two-form, as we have discussed briefly in section 2.4.

In going to F-theory, the space-filling membranes that contribute to the flux Φ at infinity are replaced by space-filling threebranes. Also we need to discuss the F-theory analog of the G-field. Let Y be a four-fold that is elliptically fibered over a base B. Let θ^i, $i = 1, 2$, be a basis of integral harmonic one-forms on the fibers, and let χ be an integral two-form generating the two-dimensional cohomology of the fibers. Then a four-form G on Y has at the level of cohomology an expansion

$$G = g + p \wedge \chi + \sum_i H_i \wedge \theta^i, \tag{2.42}$$

where g, p, and H_i are respectively forms of degree 4, 2, and 3 on B. (H_i is a three-form on B with values in the one-dimensional cohomology of the fibers, while g and p are ordinary four- and two-forms on B.) If G is primitive, then it is in particular self-dual (see Appendix I). For G to be integral, g and p must be integral. Self-duality of G gives a relation between g and p which, in the limit that the area of the fibers of $Y \to B$ is very small, is impossible to obey if g and p are non-zero and integral. Hence, the surviving part of G in the F-theory limit is contained in the H_i, which are interpreted physically as the Neveu-Schwarz and Ramond-Ramond three-form field strengths of Type IIB superstrings. With $g = p = 0$, G

is an element of the primitive cohomology of Y that is odd under the involution that acts as -1 on the elliptic fibers and trivially on the base.

In terms of a Type IIB description, we have the following structure. Let H^{NS}, H^R denote the NS and Ramond three-form field strengths. Let B denote the base of F-theory "visible" to type IIB. Consider

$$H^+ = H^R - \tau H^{NS}$$

$$H^- = H^R - \overline{\tau} H^{NS}$$

We view τ as varying over B with monodromies around the loci of seven-branes by $SL(2, \mathbf{Z})$ transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

Under such transformations

$$H^+ \to (c\tau + d)^{-1} H^+$$

$$H^- \to (c\overline{\tau} + d)^{-1} H^-$$

A supersymmetric configuration in this context is obtained by choosing an integral $(1, 2)$ form on the base, H, well defined modulo transformation by $(c\tau + d)^{-1}$ around the 7-branes. Alternatively, H is a section of $\Omega^{1,2} \otimes \mathcal{L}$ where $\mathcal{L}$ is a line bundle over B whose first chern class is $c_1(\mathcal{L}) = -12c_1(B)$. Then we identify

$$H^+ = H, \qquad H^- = \overline{H}$$

Moreover we require that $H \wedge k = 0$ where k denotes the Kähler class of B. In this case a given model is specified by fixing

$$\Phi = N + \frac{1}{4\pi^2} \int_B \frac{1}{\tau_2} H \wedge \overline{H}$$

where N denotes the number of $D3$ branes.

To describe the domain walls, recall that one can interpret F-theory on $\mathbf{R}^4 \times Y$ in terms of Type IIB on $\mathbf{R}^4 \times B$ with (p, q)-sevenbranes on a certain locus $L \subset B$. Domain walls across which the H_i jump are described by a five-brane wrapped on $\mathbf{R}^3 \times V \subset \mathbf{R}^4 \times B$ with V a three-cycle in B. The (p, q) type of the five-brane varies as V wraps around the discriminant locus in B.

This is a rather complicated structure in general, but to study the local behavior near a singularity, it simplifies considerably. One reason for this is that near an isolated singularity, one can replace B by $\mathbf{C}^3$. If we pick coordinates z_1, z_2, z_3 on $\mathbf{C}^3$, then the elliptic fibration over $\mathbf{C}^3$ can be described very explicitly by a Weierstrass equation for additional complex variables x, y:

$$y^2 = x^3 + f(z_1, z_2, z_3)x + g(z_1, z_2, z_3). \qquad (2.43)$$

The fibers degenerate over a singular locus L which is the discriminant of the cubic, given by $\Delta = 0$, where

$$\Delta = 4f^3(z_1, z_2, z_3) + 27g^2(z_1, z_2, z_3). \qquad (2.44)$$

A singular behavior of the elliptic fibration Y just corresponds in this language to a singularity of the hypersurface $L \subset \mathbf{C}^3$. We are interested in a singular point of L at which $4f^3 + 27g^2 = 0$ but f and g are not both zero.[12] Near such a singular point, the detailed construction of Δ in terms of f and g is irrelevant, and and one can regard L as a fairly generic deformation of a hypersurface singularity $\Delta = 0$.

Actually, the full structure of (p, q) sevenbranes also simplifies in this situation. The deformation of an isolated surface singularity is topologically a bouquet of two-spheres, and in particular simply-connected. Hence, there is no monodromy around which the type of brane can change; the (p, q) type of the sevenbrane is fixed, and one can think of it (for example) as a D7-brane. Thus, the F-theory analog of a Calabi-Yau fourfold singularity is a more elementary-sounding problem: the study of a D7-brane in $\mathbf{R}^{10} = \mathbf{R}^4 \times \mathbf{C}^3$ whose worldvolume is $\mathbf{R}^4 \times L$, where $L \subset \mathbf{C}^3$ is developing a singularity.

Now, let us describe the vacuum states and domain walls in this context. The D7-brane supports a $U(1)$ gauge field, whose first Chern class is an element of $H^2(L; \mathbf{Z})$. This group is a lattice Γ^*, whose rank is the number of two-spheres in the bouquet. A D5-brane can end on a D7-brane; its boundary couples magnetically to the gauge field on the D7-brane. Hence the domain walls across which the first Chern class jumps are built from fivebranes of topology $\mathbf{R}^3 \times V$, where V is a three-manifold in $\mathbf{C}^3$ whose boundary lies in L. In crossing such a domain wall, the first Chern class jumps by the cohomology class $[\partial V]$, which is an element of $\Gamma = H_2(L; \mathbf{Z}) = H^2_{cpct}(L; \mathbf{Z})$. Poincaré duality for noncompact manifolds

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[12] Singularities with f and g both zero are composite 7-branes of various types (the order of vanishing of discriminant would be bigger than 1). For such cases the simplifications described in the text do not arise and the full structure of (p, q) sevenbranes is relevant.

asserts that Γ and Γ^* are dual, and[13] the natural map $i : H^2_{cpct}(L; \mathbf{Z}) \to H^2(L; \mathbf{Z})$ gives an embedding of Γ in Γ^*. Thus we have a familiar situation: the vacuum is determined by a point in a lattice Γ^*, and in crossing a domain wall it can jump by an element of a sublattice Γ. Γ is endowed with an integral quadratic form (the intersection pairing), and as the notation suggests, Γ^* is the dual lattice of Γ with respect to this pairing.

The A-D-E singularities will furnish important examples in the present paper, for reasons that we have already explained. Thus, let us explain how they arise in the F-theory context. An example of an elliptic four-fold fibration Y with an isolated singularity is given by the following Weierstrass equation:

$$y^2 = x^3 - 3a^2 x + (H(z_1, z_2, z_3) + 2a^3). \tag{2.45}$$

Here a is an arbitrary non-zero constant, and H is a quasihomogeneous polynomial describing a singularity in three variables at $z_1 = z_2 = z_3 = 0$. If we shift x to $x + a$, the equation becomes

$$y^2 = x^3 + 3ax^2 + H(z_1, z_2, z_3), \tag{2.46}$$

and this makes it obvious that the singularity of the elliptic fibration is obtained by "stabilizing" the surface singularity $H = 0$ by adding the quadratic terms $3a^2 x^2 - y^2$ (the x^3 term is irrelevant near the singularity, which is at $x = y = 0$). The equation for the discriminant locus $L \subset \mathbf{C}^3$ reduces to $H = 0$ (plus higher order terms that are irrelevant near the singularity). So the singularity of the elliptic four-fold is just the "stabilization" of the singularity L. To obtain the A-D-E singularities, for both the surface L and the four-fold Y, we need only select the appropriate H:

$$
\begin{aligned}
H &= z_1^n + z_2^2 + z_3^2 & A_{n-1} \\
H &= z_1^n + z_1 z_2^2 + z_3^2 & D_{n+1} \\
H &= z_1^3 + z_2^4 + z_3^2 & E_6 \\
H &= z_1^3 + z_1 z_2^3 + z_3^2 & E_7 \\
H &= z_1^3 + z_2^5 + z_3^2 & E_8.
\end{aligned}
\tag{2.47}
$$

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[13] If there are null vectors in Γ, there is a slightly more elaborate story as mentioned in section 2.2.

2.6. Conformal Field Theory: First Results

Given Type IIA, M-theory, or F-theory on a singular geometry, one natural question is whether a non-trivial conformal field theory arises in the infrared.

In one situation, an affirmative answer to this question is strongly suggested by recent literature. This is the case of M-theory at a quasihomogeneous four-fold singularity (for the present discussion this need *not* be a hypersurface singularity) with a large value of the conserved quantity Φ that was introduced in section 2.2:

$$\Phi = N + \frac{1}{2} \int_X \frac{G \wedge G}{(2\pi)^2}. \tag{2.48}$$

We suppose that the four-fold X is a cone over a seven-manifold Q. Consider M-theory on $\mathbf{R}^3 \times X$, with a specified (flat) C-field at infinity that we call C_∞, and with a very large number of membranes near the singularity, such that the total membrane charge (including the contribution of the C-field) is Φ. This system is believed [16-18] to be described in the infrared by a conformal field theory that is dual to M-theory on $\mathrm{AdS}_4 \times Q$, with a constant curvature (but topologically trivial) C-field on AdS_4 that depends on Φ, and a flat but topologically nontrivial C-field on Q that is equal to C_∞. For a special case in which the role of C_∞ has been analyzed (for $Q = \mathbf{RP}^7$) see [19].

The AdS_4 dual of this CFT depends only on what one can measure on Q, that is C_∞ and Φ, and not the detailed way of decomposing Φ in terms of N and G as in (2.48). That decomposition arises if one makes a deformation of the theory, deforming X to a smooth hypersurface. M-theory on $\mathbf{R}^3 \times X$ with X such a smooth hypersurface has vacua corresponding to all choices of N and G obeying (2.48). When X develops a singularity, the G-field apparently "disappears" at the singularity, and the decomposition of Φ into membrane and G-field terms is lost.

Note that the vacua with $N \neq 0$ do not have a mass gap even after deforming to smooth X. There are at least massless modes associated with the motion of the membranes on X. To get a theory that after deformation of the parameters flows in the infrared to massive vacua only, one must set Φ to the smallest possible value for a given value of C_∞, so that after deforming to a smooth X, N will be zero for all vacua. We recall that C_∞ determines a coset in Γ^*/Γ. To get a massive theory, Φ must equal the minimum of

$$\tfrac{1}{2} \int_X \frac{G \wedge G}{(2\pi)^2}, \tag{2.49}$$

with G running over the elements of the coset of Γ^*/Γ determined by C_∞; the massive vacua are in correspondence with the choices of G that achieve the minimum.

Our goal in the next two sections will be to analyze, for the A-D-E singularities, the "massive" models just described. The analysis will be made by analyzing the domain wall structure, or, as it is usually called in two dimensions, the soliton structure. To justify the analysis, we need to know that there are no quantum corrections to the classical geometry (which we will use to find the solitons). Such corrections would come from appropriate instantons. For example, for Type IIA on a Calabi-Yau threefold near the conifold singularity, the Euclidean D2 brane instantons wrapped around the $\mathbf{S}^3$ in the conifold smooth out the singular classical geometry [20-22]. Likewise, in M-theory compactifications on suitable Calabi-Yau four-folds, a superpotential is generated by wrapped Euclidean fivebranes [12]. Such effects, however, are absent in the examples we are considering. For example, in the F-theory, we are really studying, as we have explained above, a sevenbrane on $L \subset \mathbf{C}^3$. Since the $\mathbf{C}^3$ has no non-trivial cycles, the relevant instantons will have to end on L, in order to have finite action. For the instanton to affect the quantum moduli space it has to be BPS, which in particular requires that the boundary of the instanton be a non-trivial compact cycle in L. In Type IIB string theory the only possible candidate instantons which could end on a sevenbrane are fivebranes and onebranes (of appropriate (p,q) type). Viewing them as instantons, their boundaries would be five- and one-dimensional respectively. So if L has no non-trivial compact five- or one-dimensional cycles, then there are no instantons, and quantum corrections do not modify the singular classical geometry. In our case, L, whose compact geometry consists of a bouquet of two-spheres, has only two-cycles, so there are no instantons. This is to be contrasted with the seemingly similar problem of F-theory on a Calabi-Yau threefold. In that case, L is a complex curve, with non-trivial one-cycles; instanton one-branes can and do modify the classical geometry. This is in fact the F-theory version of the description of the corrections to conifold geometry in Type IIA compactification (and reduces to it upon compactification on $\mathbf{T}^2$). For F-theory on a four-fold, if the singularity of the surface L is not isolated, then it would generically also have non-trivial one-cycles and would thus receive corrections.

For M-theory or Type IIA near a four-fold hypersurface singularity, a similar statement holds. In this case, the local geometry of the deformed singularity has non-trivial four-cycles only. Thus there is no room for instantons, i.e. wrapped Euclidean M2- or M5-branes, which would require non-trivial three or six-cycles on X. Thus the classical singularity survives quantum corrections.

Thus, to analyze the small Φ theories, we will look for supersymmetric domain walls using the classical geometry near the singularity. The domain walls are constructed from branes whose volumes vanish as the hypersurface X becomes singular, so their tensions go to zero. Thus one can reasonably hope to get a description in terms of an effective theory that contains only light degrees of freedom and generates these domain walls. In fact, for the massive models derived from A-D-E singularities, we will propose a description in terms of an effective superpotential for a certain set of chiral superfields that generate the same soliton structure. This description will make clear that one should expect flow to a non-trivial IR conformal field theory in the two-dimensional cases, and in a few cases in three dimensions.

The basic strategy for identifying a supersymmetric theory based on its BPS soliton structure is the classification approach of [23] to $\mathcal{N} = 2$ supersymmetric theories in two dimensions. Consider a theory with $\mathcal{N} = 2$ supersymmetry in two dimensions with k vacua, and consider the integral $k \times k$ matrix S given by

$$S = 1 - A \tag{2.50}$$

where 1 represents the identity matrix and A is a strictly upper triangular matrix whose A_{ij} entry for $i < j$ is the number of nearly massless BPS solitons interpolating between the i-th sector and the j-th sector weighted with the index $(-1)^F F$ [24], i.e.

$$A_{ij} = \text{Tr}_{ij-solitons}(-1)^F F.$$

It was argued that this massive deformation comes from a CFT in the UV limit with central charge $\widehat{c}$ and k chiral fields with R-charges q_i which satisfy

$$\text{Eigenvalues}(S^{-t}S) = \exp[2\pi i(q_i - \tfrac{1}{2}\widehat{c})]$$

(even the integral part of q_i can be determined from A_{ij} [23]). This is a strong restriction, and in case of deformations of minimal models, the solitons completely characterize the conformal theory. In other words, any theory which upon mass deformation has the same solitonic structure as that for a massive deformation of a minimal model is equivalent to it! For non-minimal models, the relation above between the spectrum of the solitons and the charges of chiral fields is still a very powerful connection and in particular fixes the central charge of the corresponding conformal theory.

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Above two dimensions, the domain wall or soliton analysis still identifies an effective superpotential, but it is less common for a theory with a given superpotential to flow to a nontrivial IR conformal field theory. For instance, a theory with a single chiral superfield Φ and superpotential $W = \Phi^n$ is believed to flow to a nontrivial CFT in two dimensions for all $n > 2$, while in three dimensions this is expected only for $n = 3$ [25], and in four dimensions, it is believed to flow to a trivial IR theory for all n. In any event, our analysis will identify the nonperturbative massless fields and superpotential near the singularity also in three and four dimensions. Also, even in four dimensions, a Φ^n superpotential can become relevant as a perturbation to certain fixed points [26], so with some modification of our construction, the superpotential we find may eventually be important in analyzing four dimensional CFT's that arise from string theory.

The soliton analysis will give detailed information about the behavior for small membrane charge, which is the opposite limit from the AdS description discussed above that governs the large charge behavior at least for the M-theory compactifications. For the Type IIA and F-theory compactifications, the description of the large charge behavior appears to be less simple.

3. Geometry of Domain Walls

As explained in section 2.6, our task now is to analyze the soliton structure for certain hypersurface singularities. In fact, we will consider the A_k singularities which were introduced in section 2.2.

Instead of specializing to four-folds, it proves insightful to consider the more general problem of identifying BPS states of wrapped n-branes in a Calabi-Yau n-fold near an isolated singularity. To study the behavior near an A_k singularity, we consider a local model for a Calabi-Yau n-fold given by

$$-P_m(z_1) + z_2^2 + ... + z_{n+1}^2 = 0$$

where $P_m(z_1)$ is a polynomial of degree $m = k + 1$ in z_1. When P_m has two equal roots, we get a singular geometry. The most singular gemetry arises when $P_m(z_1) = z_1^m$. For a generic polynomial $P_m(z_1)$, the geometry is not singular and the compact homology of this manifold has a basis made of $m - 1$ spheres of real dimension n intersecting each other according to the A_{m-1} Dynkin diagram. For a particular choice of P_m, we explained how to construct these spheres in section 2.2. The intersection form on the compact

homology is symmetric if n is even and antisymmetric if n is odd. We would like to consider minimal wrapped n-branes, i.e. minimal supersymmetric cycles, in this geometry. A supersymmetric cycle is a Lagrangian submanifold (that is, the Kähler form vanishes on it). Moreover, on a minimal supersymmetric n-cycle the holomorphic n-form Ω of Calabi-Yau is real (with a suitably chosen overall phase) and gives the volume of the n-brane. For a minimal supersymmetric cycle the quantity

$$V = \int_C |\Omega|$$

which is the volume of cycle C, is minimized and is given by

$$V = \alpha \int_C \Omega$$

for some choice of phase α. Or stated equivalently, the condition is that

$$\int_C |\Omega| = \left| \int_C \Omega \right|.$$

which is the condition for minimizing the volume of C among the Lagrangian submanifolds in a given homology class.

The holomorphic n-form Ω, up to an overall complex scale factor, is given by

$$\Omega = \frac{dz_1...dz_n}{z_{n+1}} = \frac{i dz_1...dz_n}{\sqrt{z_2^2 + ... + z_n^2 - P_m(z_1)}}$$

We would like to minimize the volume form given by $|\Omega|$. To count the minimal supersymmetric cycles, we follow the strategy in [27] and decompose the geometry to the "fiber and the base" as follows. Suppose C is a supersymmetric minimal cycle. Consider the image of C on z_1. This is a one-dimensional subspace, because for a fixed z_1, the manifold (being defined by $\sum_{j>1} z_j^2 = P_m(z_1)$) has for its only nontrivial cycle a sphere $\mathbf{S}_{z_1}^{n-1}$. Note that the radius of this sphere is $|P_m(z_1)|^{1/2}$, from which one can deduce by scaling that

$$\left| \int_{\mathbf{S}_{z_1}^{n-1}} \frac{dz_2 \wedge \ldots \wedge dz_{n-1}}{z_n} \right| = |P_m(z_1)|^{(n-2)/2} \tag{3.1}$$

up to an irrelevant multiplicative constant. The inverse image of C over a point in the z_1 plane must, if not empty, be a minimal cycle, and so must be $n-1$-dimensional; hence the image of C in the z_1 plane must be one-dimensional. The minimization of $|\Omega|$ will be done in two steps: We first consider Lagrangian submanifolds $C_f(z_1)$ for a fixed z_1 which

minimize the $\int_{C_f(z_1)} |\Omega|$ and next consider the minimization of the volume interval over an interval I in z_1. In this way we get using (3.1)

$$\int_{C_f(z_1) \times I} |\Omega| = \int_I \left| \int_{S^{n-1}(z_1)} \Omega \right| = \int_I |P_m(z_1)|^{\frac{n-2}{2}} dz_1. \tag{3.2}$$

We now minimize the volume of the supersymmetric n-cycle with respect to the choice of the one-dimensional line segment I representing the image of the supersymmetric cycle on the z_1 plane. One can allow the line segment to end at some special points on z_1 where $P_m(z_1) = 0$, and these are the only allowed boundaries. In fact, precisely if the line segment terminates at zeroes of P_m, the D-brane worldvolume is closed and smooth. Indeed the topology of the cycle is an $\mathbf{S}^n$ which can be viewed as an $\mathbf{S}^{n-1}$ sphere fibered over the interval, where at the boundaries of the interval the radius of $\mathbf{S}^{n-1}$ vanishes. The expression (3.2) is minimized if along the segment on z_1 plane the condition $|P_m(z)| = \alpha P_m(z)$ is satisfied for some z-independent phase α. Let us define a function W with

$$dW = P_m^{\frac{n-2}{2}} dz_1 \tag{3.3}$$

In terms of W, the condition for minimal volume is that the image of the curve in the W plane be a straight line along the direction specified by α^{-1}. Moreover, the end-points of the segment in the z_1 plane correspond to critical points in W, i.e., $dW = 0$ (for $n = 2$ the endpoints are defined by the condition that $P_m(z_1) = 0$). These conditions are identical [28] for finding solitons in an $\mathcal{N} = 2$ Landau-Ginzburg theory in two dimension (or more generally, BPS domain walls in theories with four supercharges in dimensions two, three, or four) with superpotential given by W! If n is even, (3.3) corresponds to a well defined function of z_1. If n is odd, it gives rise to a well defined (meromorphic) one-form on a hyperelliptic cover of the z_1 plane, branched over the zeroes of $P_m(z_1)$.

Strictly speaking we have constructed the supersymmetric cycle by assuming that the condition that the cycle C_f be Lagrangian is the same as being Lagrangian relative to the Kähler form induced on the fiber. This is not necessarily true. For example if the Kähler form has a piece of the form

$$k = ... + f_i dz_1 \wedge dz_i^* + ...$$

the condition of Lagrangian gets modified. In special cases, like when the polynomial P_m has real coefficient one can use a Z_2 antiholomorphic involution to argue that the cycles

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we constructed are both Lagrangian and supersymmetric. In the more general case we proceed as follows: Consider a generic

$$P_m(z_1) = \prod_i (z_1 - a_i)$$

Consider a one parameter family of Calabi-Yau metrics where

$$a_i(t) = ta_i.$$

Note that the BPS states will be the same for all t, because the condition of the BPS charges getting aligned does not change as we change t (the BPS charges only receive an overall rescaling). However to construct the Kähler metric as a function of t we note that it can be mapped to the previous metric by defining

$$\widetilde{z}_1 = tz_1$$

$$\widetilde{z}_i = t^{n/2} z_i \quad for \ i \neq 1$$

Thus we use the z variables but rescale the Kähler form accordingly. In this way as $t \to \infty$ the mixed terms in the Kähler form are dominated by the terms purely in the fiber direction (for $n > 2$ which is the case of main interest). Therefore in this limit the condition of Lagrangian submanifold in the fiber that we have used becomes accurate.

Let us consider some special cases. The cases for a K3 surface and for a Calabi-Yau threefold have already been considered in [27] (see also [29,30]), and will be reviewed below.

Solitons for K3

In the case $n = 2$, the above geometry is the complex deformation of the A_{m-1} singularity. For any choice of the polynomial $P_m(z_1)$, we expect $m(m-1)/2$ solitons (up to the choice of orientation) to complete the adjoint representation of $U(1)^{m-1}$ to the $SU(m)$ gauge multiplet. From (3.3), we see that in this case $W = z_1$. There are m roots for $P_m(z)$, and the solitons correspond to straight lines between the roots. Note that this gives $m(m-1)/2$ solitons up to the choice of the orientation, as was anticipated.

Solitons for CY3

For the case of Calabi-Yau threefolds, $n = 3$. In this case W is defined by

$$dW = P_m(z_1)^{\frac{1}{2}} dz_1.$$

Here dW can be viewed as a meromorphic one-form on a hyperelliptic Riemann surface over z_1 branched over the roots of $P_m(z_1)$. The geometry of these solitons for this class of conformal theories would correspond to straight lines on the Jacobian of this surface defined by the integrals dW and is presently under study [31].

Solitons for CY_4

For the case of four-folds, which are of course our main focus in the present paper, the definition of W in (3.3) shows that W is a polynomial of degree $m+1$ in the z_1 plane. We have already shown that the conditions for finding the solitons in this geometry are the same as those in an LG theory with the superpotential W. In this case, however, if we use our four-fold in Type IIA superstring theory, the analogy becomes more precise: compactification on the four-fold leads to a theory in two dimensions with $\mathcal{N} = 2$, and it is natural to identify the corresponding W with the superpotential of an $\mathcal{N} = 2$ Landau-Ginzburg theory. We will indeed argue that for a certain choice of the membrane charge, the Type IIA on a deformed A_1 singularity leads to an $\mathcal{N} = 2$ theory with the same W for its superpotential. For more general choices of the membrane charge, we find closely related Kazama-Suzuki coset models at level 1.

Before we discuss these subtleties, note that even though we have m critical points, it is no longer true in general that we have $m(m-1)/2$ solitons. In general the pre-image of a straight line connecting the images of critical points in the W plane will not connect the critical points in the z_1 space. In fact as we change the polynomial $P_m(z_1)$, it is known that the number of BPS states jumps [24]. For some choices of $P_m(z_1)$ we do have exactly the same number of solitons as in the K3 case. For example, it has been shown [28] that for

$$P_m(z_1) = z_1^m - \mu^n$$

for any constant μ, there is one soliton for each pair of m critical points $z_1 = \omega\mu$ with $\omega^m = 1$, though, unlike the K3 case the image in the z_1 plane is *not* a straight line.

It would be interesting to compare the formula for W that we have deduced from the soliton structure to the analysis of section 2.4. Although this is guaranteed to work, because both capture the mass of the BPS soliton, we have not attempted to check this correspondence explicitly.

4. Identifications With Kazama-Suzuki Models

Let us consider in more detail Type IIA strings propagating on a smooth hypersurface X obtained by deforming the A_{m-1} singularity:

$$-P_m(z_1) + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0.$$

As explained in section 2, in order to specify the problem fully, we must fix the value C_∞ of the C-field at infinity and also the flux

$$\Phi = N + \tfrac{1}{2}\xi^2, \tag{4.1}$$

where $\xi = [G/2\pi]$ is the characteristic class of the C-field.

As we explained in section 2.2, ξ is restricted to a fixed coset in Γ^*/Γ, where Γ and Γ^* are the root and weight lattices of the Lie group $SU(m)$. The coset in which ξ takes values is determined by C_∞. For the theory to have a mass gap, as discussed in section 2.6, we set Φ equal to the minimum value of $\frac{1}{2}\xi^2$ (for all ξ in the given coset), so that $N = 0$ for all vacua.

This can be made very explicit in the case of the A_{m-1} singularity. C_∞ takes values in Γ^*/Γ, which is isomorphic to the center of $SU(m)$, or $\mathbf{Z}_m$. For $k = 0, \ldots, m-1$, if $C_\infty = k$, then to minimize ξ^2, ξ must be a weight of the k-fold antisymmetric tensor product of the fundamental representation of $SU(m)$. We denote that representation as R_k. The number of choices of ξ is the dimension of R_k or $m!/k!(m-k)!$. This is the number of vacuum states of the kth model, if P_m is such that the hypersurface X is smooth.

For $k = 0$, there is only one vacuum ($\xi = 0$), and the theory is trivial and massive. Let us consider the next simplest case, where $k = 1$ and ξ is a weight of the fundamental representation of $SU(m)$. In this case, we have m vacua. To find the degeneracy of the solitons between these vacua, we use the analysis of section 3. We found that the solitons are exactly the same as those for an $\mathcal{N} = 2$ LG theory with a chiral field Φ and a superpotential W obeying $dW/d\Phi = P_m(\Phi)$. In fact we can identify the m vacua with the m critical points of P_m, and as we found in section 3, the condition for the existence of a soliton in the LG theory is exactly the same as the condition for a BPS wrapped four-brane in Type IIA near the A_{m-1} singularity. In this case the soliton data are enough to determine the theory [23] as discussed at the end of section 2; the two-dimensional theory with superpotential W is the A_m minimal model [3,4]. So the Type IIA theory near an

A_{m-1} four-fold singularity is governed by the A_m $\mathcal{N} = 2$ minimal model. This model can also be viewed as an $\mathcal{N} = 2$ Kazama-Suzuki coset model at level one, of the form

$$\frac{SU(m)}{SU(m-1) \times U(1)}$$

For the M-theory or F-theory near an A_{m-1} four-fold singularity, we still get a description in terms of a chiral field with the same superpotential, but in most instances (the exception being Φ^3 in three dimensions), a theory in three or four dimensions with a Φ^{m+1} superpotential is believed to flow to a free theory in the infrared.

We now consider the other choices of C_∞, so that ξ is a weight of the k-fold anti-symmetric product of the fundamental representation of $SU(m)$ with some $k > 1$. We argue that it has exactly the same solitonic spectrum as a deformation of the following LG theory, which we will call the k-fold symmetric combination of the $k = 1$ model. Consider the function of k variables

$$W(z_1, \ldots, z_k) = z_1^{m+1} + z_2^{m+1} + \ldots + z_k^{m+1}.$$

It is invariant under permutations of the z_i, and so can be expressed as a polynomial in the elementary symmetric functions

$$x_l = \sum_{i_1 < \ldots < i_l} z_{i_1} \ldots z_{i_l}.$$

The superpotential we consider is thus

$$W(x_1, \ldots, x_k) = W(z_1) + \ldots + W(z_k). \tag{4.2}$$

The LG model with superpotential (4.2) has been conjectured in [32] to be equivalent to the following Kazama-Suzuki coset model at level 1:

$$\frac{SU(m)}{SU(m-k) \times SU(k) \times U(1)}$$

For the deformed singularity, with $\partial W = P_m$, we claim that the deformed LG superpotential is given by

$$W(x_1, ..., x_k) = W(z_1) + ... + W(z_k)$$

where again what we mean by this expression is that the superpotential is $W(z_1) + \ldots + W(z_k)$ regarded as a polynomial in the elementary symmetric functions $x_1, \ldots, x_k$. Let us

see why this LG superpotential has exactly the same solitonic spectrum that we get for Type IIA at an A_{m-1} four-fold singularity with $C_\infty = k \mod m$. It is not too difficult to show [23,33] that the set of vacua of a LG theory made of a k-fold symmetric combination of a given LG theory (in the sense introduced above) can be identified with the k-fold *antisymmetric* tensor product of the space of vacua of the original LG theory.[14] As we already discussed, the $k = 1$ model has a one-variable superpotential $W(z)$, and its vacua correspond to the fundamental weights of the $SU(m)$ lattice. Thus we can identify the vacua of the k-fold symmetric combination of the $k = 1$ model with the weights of the k-fold anti-symmetric tensor representation R_k. As far as the allowed solitons, on the LG side, they can be constructed in the decoupled theory with superpotential $W(z_1, \ldots, z_k) = W(z_1) + \ldots + W(z_k)$ before re-expressing this in terms of the symmetric functions x_i. In this description, it is clear that soliton states are just the products of soliton states in the individual one-variable theories, and that irreducible solitons (which cannot break up into several widely separated mutually BPS solitons) are solitons in just one of the variables z_i. So if we label the vacua by $|i_1, \ldots, i_k\rangle$, with i_s denoting a vacuum in the sth one-particle theory, then the allowed solitons only change one vacuum index at a time. So the solitons of a LG theory that is constructed as a k-fold symmetric combination of a one-variable theory are in natural correspondence with the solitons of the one-variable theory. This is the same result that we get from Type IIA near the A_{m-1} singularity with $C_\infty = k \mod m$. Indeed, for this model, the solitons are constructed by finding supersymmetric four-cycles. The analysis of those cycles in section 3 depends only on the geometry of the hypersurface and makes no reference to C_∞. Hence the solitons for any k are in a natural sense the same as the solitons of the $k = 1$ model.

In other words the solitons are in 1-1 correspondence with those roots of $SU(m)$ that appear as solitons for the one-variable LG superpotential given by W. Whichever roots appear act in the natural way on the weights of the representation R_k. In the case $W = z^{m+1} - az$, all the roots appear with multiplicity 1. This structure for the solitons of the deformed Kazama-Suzuki model was suggested in [34] where it was argued to correspond (with a specific choice of Kähler potential) to an integrable model.

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[14] The main point that must be shown is that the vacua in the different factors must be distinct. To illustrate why, it suffices to consider the case that $k = 2$ and that each individual model has only one vacuum. So we start with $m = 2$: $W(z) = z^2$. Then we write $W(z_1, z_2) = z_1^2 + z_2^2$ in terms of the symmetric functions $x_1 = z_1 + z_2$, $x_2 = z_1 z_2$, getting $W(x_1, x_2) = x_1^2 - 2x_2$. This function has no critical points, so the combined model has no supersymmetric vacua, as expected.

4.1. Other A-D-E Singularities

So far we have mainly considered the local singularity to be

$$H(z_1, z_2, z_3) + z_4^2 + z_5^2 = 0$$

with H being an A_{m-1} singularity. Here we would like to generalize this to the case where H determines a D or E type singularity.

The general structure is quite like what we have seen for A_{m-1}. C_∞ takes values in Γ^*/Γ, where Γ is the root lattice of the appropriate simply-connected A-D-E group G, and Γ^* is the weight lattice of G. The quotient Γ^*/Γ is isomorphic to the center of G.

Just as in the $SU(m)$ case, to make possible a deformation to a massive theory, we need to pick Φ so that ξ ranges over the weights of the smallest representation with a given non-trivial action of the center of G. (If we pick the trivial action of the center, we will get the trivial representation and a massive free theory.) The appropriate representations are the representations with Dynkin label 1. In the D_n case, the relevant choices are the vector and spinor representations. For D_{2n}, there are two different spinor representations, but they differ by an outer automorphism of D_{2n} and give equivalent theories. So there are essentially two choices of C_∞ leading to massive theories based on the D_n singularity. For the E_6 theory, there is only the fundamental 27 dimensional representation (and its conjugate, which gives an equivalent theory); for E_7 there is the fundamental 56 dimensional representation. So E_6 and E_7 lead to one massive theory each. E_8 is simply-connected with trivial center, so we cannot use it to get a conformal theory with a massive deformation.

The distinguished representations that we have described are in one-to-one correspondence with nodes of index 1 on the A-D-E Dynkin diagram and thence with Hermitian symmetric spaces G/H (where H is obtained by omitting the given node from the Dynkin diagram). Apart from the Grassmannians $SU(m)/SU(k) \times SU(m-k) \times U(1)$ that we have already encountered, these Hermitian symmetric spaces are as follows. For the D_n case, there are two inequivalent choices, given by

$$\frac{SO(2n)}{SO(2n-2) \times SO(2)} \qquad \text{for fundamental rep.}$$

$$\frac{SO(2n)}{U(n)} \qquad \text{for spinor rep.,}$$

and for the E_6 and E_7 cases one has

$$\frac{E_6}{SO(10) \times U(1)} \qquad \text{for fundamental rep.}$$

$$\frac{E_7}{E_6 \times U(1)} \qquad \text{for fundamental rep.}$$

Such a Hermitian symmetric space determines a series of $\mathcal{N} = 2$ Kazama-Suzuki models (at level $1, 2, 3 \ldots$). It is natural to conjecture that, as we have found for A_{m-1}, the massive models obtained from Type IIA at an A-D-E singularity are the massive deformations of the corresponding level 1 Kazama-Suzuki (or KS) models. As a first check, it is known that for a level one G/H KS model, the dimension of the chiral ring is equal to the dimension of the corresponding representation of G. This in turn is equal to the dimension of the cohomology of G/H and it was conjectured in [32] that the chiral ring is isomorphic to the cohomology ring, which in turn was shown to arise from the ring of an LG theory. Thus the G/H theories at level 1 were identified with specific LG models.[15] Moreover, the structure of the solitons for a special (integrable) deformation of the KS model at level 1 was studied in [34] and it was conjectured that the solitons exist precisely for each allowed single root acting on the corresponding weight diagram. Though we have not analyzed the BPS spectrum of the D4-branes in this case to find the multiplicity of the solitons for each root, it is natural to expect that at least for specific deformations, just as in the A_{m-1} case, the solitons are given by the root lattice of the corresponding group with multiplicity 1. In this case we would reproduce the solitonic structure anticipated in [34]. It is quite satisfying that we apparently get all the Hermitian symmetric space KS models at level 1 in such a uniform way from considering singularities of CY four-folds.

5. Other Types Of Singularity

The only four-fold singularities that we have so far considered in any detail are hypersurface singularities. A Calabi-Yau four-fold can, however, develop singularities of many different types. We cannot offer any sort of overview of the possibilities, but will briefly analyze two cases in the present section.

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[15] The higher level KS models do not generally admit an LG description.

5.1. Hyper-Kähler Singularities

First we will consider what one might call hyper-Kähler singularities – singularities near which Y admits a hyper-Kähler structure, though Y may not be globally a hyper-Kähler manifold. An important fact here is that M-theory compactification on $\mathbf{R}^3 \times Y$ with Y hyper-Kähler has $\mathcal{N} = 3$ supersymmetry in three dimensions, because the space of covariantly constant spinors on a hyper-Kähler eight-manifold is three-dimensional.

To isolate the behavior near the singularity, we replace Y by an asymptotically conical hyper-Kähler manifold X that is developing a singularity. We will focus on a very concrete example, with $X = T^*\mathbf{CP}^2$, the cotangent bundle of complex projective two-space. This hyper-Kähler manifold is conveniently obtained by considering a $U(1)$ gauge theory with eight supercharges, and three hypermultiplets $\mathcal{A}^i$, $i = 1, 2, 3$, of charge 1. [16] There is an $SU(3)$ global symmetry group, with the $\mathcal{A}^i$ transforming as the **3**. There is also an $SU(2)$ R-symmetry group, and it is possible to add an $SU(2)$ triplet of Fayet-Iliopoulos terms $\vec{d}$ to the D-flatness equations. A manifestly $SU(2)$-invariant way to exhibit the D-flatness equations is as follows. The bosonic parts of the $\mathcal{A}^i$ can be regarded as a complex field A^i_α, $\alpha = 1, 2$, transforming as $(\mathbf{3}, \mathbf{2})$ under $SU(3) \times SU(2)_R$. The D-flatness condition is

$$\sum_i A^i_\alpha \overline{A}^\beta_i = \vec{d} \cdot \vec{\sigma}^\beta_\alpha, \tag{5.1}$$

with $\vec{\sigma}$ the traceless 2×2 Pauli matrices. The moduli space X of zero energy states of the classical gauge theory is the space of solutions of (5.1) divided by the action of the gauge theory. In this description, it is manifest that if $\vec{d} = 0$, then X has an $SU(3) \times SU(2)_R$ symmetry, broken if $\vec{d} \neq 0$ to $SU(3) \times U(1)_R$. The $SU(3)$ preserves the hyper-Kähler structure, and $SU(2)_R$ rotates the three complex structures on X. If $\vec{d} = 0$, X is a cone over a seven-manifold Q described by

$$\sum_i A^i_\alpha \overline{A}^\beta_i = \delta^\beta_\alpha. \tag{5.2}$$

It is fairly easy to see that this manifold is a copy of $SU(3)/U(1)$, where the $U(1)$ acts by right multiplication by

$$\mathrm{diag}(e^{i\theta}, e^{i\theta}, e^{-2i\theta}). \tag{5.3}$$

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[16] We will present this gauge theory as a formal device, but it may have a physical interpretation in terms of a membrane probe of the singularity.

$SU(3)$ acts on $SU(3)/U(1)$ by left multiplication, and $SU(2)_R$ acts by right multiplication by $SU(3)$ elements that commute with (5.3). Even if $\vec{d} \neq 0$, X is asymptotic to a cone over Q at big distances. The R-symmetry group that acts faithfully on X is actually $SO(3)_R = SU(2)_R/\mathbf{Z}_2$. That is because the center of $SU(2)_R$ is equivalent to a $U(1)$ gauge transformation. In M-theory on $\mathbf{R}^3 \times X$, the three spacetime supersymmetries transform as a vector of $SO(3)_R$.

Now let us explain why for $\vec{d} \neq 0$, X is equivalent to $T^*\mathbf{CP}^2$. In terms of a description that makes manifest only half the supersymmetry of the gauge theory, one can break up the bosonic part of the $\mathcal{A}_i$ into pairs of complex fields B^i, C_i, transforming as $\mathbf{3}$ and $\overline{\mathbf{3}}$ of an $SU(3)$ symmetry group, and with charges 1 and -1 under the $U(1)$ gauge group. (Compared to the previous description, $B^i = A^i_1$ and $C_i = \overline{A}_{i\,2}$.) This description breaks $SO(3)_R$ to $SO(2)_R = U(1)_R$, with $\vec{d}$ splitting as a real component $d_\mathbf{R}$ and complex component $d_\mathbf{C}$. The D-flatness equations of the $U(1)$ gauge theory are in this description

$$\sum_i |B^i|^2 - \sum_j |C_j|^2 = d_\mathbf{R}$$
$$\sum_i B^i C_i = d_\mathbf{C}. \tag{5.4}$$

One must also divide by the action of $U(1)$. By an $SO(3)_R$ rotation, one can set $d_\mathbf{C} = 0$ and $d_\mathbf{R} > 0$. The quantities $\widetilde{B}^i = B^i/\sqrt{d_\mathbf{R} + \sum_j |C_j|^2}$ obey $\sum_i |\widetilde{B}^i|^2 = 1$ and, after dividing by the gauge group, define a point in $\mathbf{CP}^2$. With $d_\mathbf{C} = 0$, the second equation in (5.4) can be interpreted to mean that C_i lies in the cotangent space to $\mathbf{CP}^2$, at the point determined by the $\widetilde{B}^i$. Thus X is isomorphic to $T^*\mathbf{CP}^2$. For any manifold W, regarded as the zero section of T^*W, the self-intersection number $W \cdot W$ is equal to the Euler characteristic of W. The Euler characteristic of $\mathbf{CP}^2$ is 3, so in our example

$$W \cdot W = 3 \tag{5.5}$$

with $W = [\mathbf{CP}^2]$.

Though turning on $\vec{d}$ breaks the $SO(3)_R$ symmetry of X to $SO(2)$, it preserves the hyper-Kähler structure and all of the supersymmetry of M-theory on $\mathbf{R}^3 \times X$.

The appearance of an $SO(3)_R$ symmetry at $\vec{d} = 0$ is a hint that M-theory on $\mathbf{R}^3 \times X$ flows to a superconformal field theory in the infrared as $\vec{d} \to 0$. Indeed, in three spacetime dimensions with $\mathcal{N}$ supercharges, the superconformal algebra contains an $SO(\mathcal{N})_R$ R-symmetry group.

To get more insight, let us now analyze the possible G-fields on the smooth manifold X with $\vec{d} \neq 0$. Since X is contractible to $\mathbf{CP}^2$, one has $H^4(X; \mathbf{R}) = H^4(\mathbf{CP}^2; \mathbf{R})$. The non-zero Betti numbers are $h^0 = h^2 = h^4 = 1$. The cohomology with compact support is, by Poincaré duality, the dual of this, so the non-zero Betti numbers with compact support are $h^4_{cpct} = h^6_{cpct} = h^8_{cpct}$. Hence, just on dimensional grounds, the natural map $i : H^k_{cpct}(X; \mathbf{R}) \to H^k(X; \mathbf{R})$ is zero except for $k = 4$. For $k = 4$, $H^4_{cpct}(X; \mathbf{R})$ is generated by the class $[W] = [\mathbf{CP}^2]$, and the nonzero intersection number (5.5) implies that $i \neq 0$. For an asymptotically conical manifold, one expects the space of $\mathbf{L}^2$ harmonic forms to coincide with the image of i, so in the present example we expect precisely one $\mathbf{L}^2$ harmonic form α, in dimension four. α is necessarily primitive with respect to all of the complex structures, since if K is any of the Kähler forms, then $\alpha \wedge K$, if not zero, would be an $\mathbf{L}^2$ harmonic six-form.

Hence, turning on a nonzero G-field, proportional to α, preserves all of the supersymmetries. In fact, we *must* turn on such a G-field, for the following reason. According to [5], on a spacetime X, the general flux quantization law for G is not that $G/2\pi$ has integral periods but that the periods of $G/2\pi$ coincide with the periods of $c_2(X)/2$ mod integers. (There is a slightly more general formulation if X is not a complex manifold.) In our situation, the integral of $c_2(X)/2$ over $\mathbf{CP}^2$ is a half-integer,[17] so we need

$$\int_{\mathbf{CP}^2} \frac{G}{2\pi} \in \mathbf{Z} + \frac{1}{2}, \tag{5.6}$$

and in particular G cannot be zero.

If we normalize the four-form α to represent the class $[\mathbf{CP}^2]$, then α generates $H^4_{cpct}(X; \mathbf{Z})$ (or rather its image in real cohomology). Also, $\alpha \cdot \alpha = 3$, so the dual lattice $H^4(X; \mathbf{Z})$ is generated by $\alpha/3$. Hence, we require

$$\left[\frac{G}{2\pi} \right] = \frac{\alpha}{3} \left(k + \tfrac{1}{2} \right) \quad \text{with } k \in \mathbf{Z}. \tag{5.7}$$

One also has $H^4(Q; \mathbf{Z}) = H^4(X; \mathbf{Z})/H^4_{cpct}(X; \mathbf{Z}) = \mathbf{Z}_3$. The different possibilities for the restriction of the C-field to $\partial X = Q$ are determined by the value of k modulo three.

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[17] Let the total Chern class of the tangent bundle of $\mathbf{CP}^2$ be $1 + c_1 + c_2$. The total Chern class of the cotangent bundle of $\mathbf{CP}^2$ is then $1 - c_1 + c_2$. The total Chern class of $T^*\mathbf{CP}^2$, restricted to $\mathbf{CP}^2 \subset T^*\mathbf{CP}^2$, is hence $(1 - c_1 + c_2)(1 + c_1 + c_2) = 1 - c_1^2 + 2c_2$, so $c_2(T^*\mathbf{CP}^2) = -c_1^2 + 2c_2$. Since $\int_{\mathbf{CP}^2} c_1^2 = 9$, which is odd, the claim follows.

In the presence of N membranes and a G-field, the membrane flux at infinity is

$$\Phi = N + \frac{1}{2} \int_X \frac{G \wedge G}{(2\pi)^2} = N + \frac{(k + \frac{1}{2})^2}{6}. \qquad (5.8)$$

In evaluating the integral, we used (5.7) and the fact that $\alpha \cdot \alpha = 3$. A check on (5.8) is that if k is shifted by an integer multiple of 3 (the 3 is needed so as to leave fixed the restriction of G to Q), Φ changes by an integer. According to the discussion in section 2, a model is specified by fixing the value of Φ and also by fixing the value of k modulo 3. A supersymmetric vacuum is then found by finding a nonnegative N and an integer k in the given mod 3 coset such that (5.8) is obeyed. There is precisely one case of a model having more than one vacuum, with all vacua having $N = 0$. This arises for $\Phi = 3/8$, with $k = 1$ and $k = -2$. We do not know a Landau-Ginzburg or other semiclassical description for this $\mathcal{N} = 3$ model with two vacua (but see below).

For sufficiently large Φ, this model (at $\vec{d} = 0$) is expected to flow to a nontrivial superconformal field theory in the infrared. Indeed, the standard conjectures would suggest that the SCFT in question is dual to M-theory on $AdS_3 \times Q$, with the C-field on Q being determined by the value of k modulo three. We have no good way at present to determine if the model flows to a nontrivial SCFT also for small Φ.

We expect that instead of $T^* \mathbf{CP}^2$, one could in a similar way analyze $T^* F$, where F is a two-dimensional Fano surface. One can also consider a collection of intersecting $\mathbf{CP}^2$'s (with a suitable normal bundle over it) and carry out a similar analysis.

Physical Interpretation Of Gauge Theory?

So far the $U(1)$ gauge theory with three charged hypermultiplets has been considered just as a mathematical device. It is natural to wonder whether, in fact, this gauge theory can be interpreted physically as the long wavelength theory of a membrane probe of the $\mathbf{R}^3 \times T^* \mathbf{CP}^2$ solution of M-theory. More generally, we would like to find an effective action for N membranes probing the $\mathbf{R}^3 \times T^* \mathbf{CP}^2$ singularity (in the limit that $\mathbf{CP}^2$ is "blown down") that will give a gauge theory dual of M-theory on $\mathbf{AdS}_4 \times Q$. In the spirit of [17,18], such a description might be roughly as follows. Consider an $\mathcal{N} = 4$ supersymmetric gauge theory in three dimensions with gauge group $S(U(N) \times U(N))$ and hypermultiplets consisting of three copies of $(\mathbf{N}, \overline{\mathbf{N}})$. Break $\mathcal{N} = 4$ to $\mathcal{N} = 3$ with some Chern-Simons interaction, determined by the C-field. (Gauge theories with Chern-Simons interactions are essentially the only known classical field theories in three spacetime dimensions without gravity with $\mathcal{N} = 3$ supersymmetry. For a study of their dynamics in the abelian case, see [35].) Such a model might have roughly the right properties.

5.2. Blowup Of Orbifold Singularity

The other kind of singularity that we will briefly examine is a simple orbifold singularity. We begin with $\mathbf{C}^4$, with complex coordinates $z_1, \ldots, z_4$, and consider the $\mathbf{Z}_4$ symmetry $z_a \to iz_a$. The quotient $\mathbf{C}^4/\mathbf{Z}_4$ is a Calabi-Yau orbifold.

If one analyzes this type of orbifold in Type IIA string theory, one finds that there is one blow-up mode and no complex structure deformation. The blow-up corresponds to a very simple resolution of the singularity, in which it is replaced by the total space W of a line bundle $\mathcal{L} = \mathcal{O}(-4)$ over $\mathbf{CP}^3$. Thus, $\mathbf{CP}^3$ is embedded in W as an exceptional divisor, the "zero section" of $\mathcal{L}$. W admits a Calabi-Yau metric, asymptotic in closed form to the flat metric on $\mathbf{C}^4/\mathbf{Z}_4$; because of the $SU(4)$ symmetry of W, it is actually possible to describe this metric by quadrature, though we will not do so here.

One might at first think that one could approach the $\mathbf{C}^4/\mathbf{Z}_4$ orbifold singularity in M-theory by a motion in Kähler moduli space, leading to a blow-down of the exceptional divisor $\mathbf{CP}^3 \subset W$. However, since the Hodge numbers $h^{i,0}(\mathbf{CP}^3)$ are zero for $i > 0$, fivebrane wrapping on $\mathbf{CP}^3$ will produce a superpotential [12], proportional roughly to e^{-V} with V the volume of $\mathbf{CP}^3$. Moreover, though the multiple cover formula for multiple fivebrane wrapping in M-theory is not known, analogy with other multiple cover formulas (such as the formula for multiple covers by fundamental strings [36,37]) suggests that the sum over multiple covers of $\mathbf{CP}^3$ will produce a pole at $V = 0$. If this is so, there will not be interesting long distance physics associated with the behavior of M-theory near a $\mathbf{C}^4/\mathbf{Z}_4$ singularity. At any rate, one certainly cannot expect to study M-theory on $\mathbf{C}^4/\mathbf{Z}_4$ while ignoring the superpotential.

Is it possible to include a G-field on W while preserving supersymmetry? If so, then since the G-field must vanish in cohomology on a fivebrane worldvolume (because of the existence of a field T on the fivebrane with $dT = G$), in the presence of the G-field the superpotential would be absent, and the question of the behavior near the $\mathbf{C}^4/\mathbf{Z}_4$ singularity would be restored.

The answer to the question of whether a supersymmetric G-field is possible turns out, however, to be "no," in the following interesting way. First of all, W is contractible to $\mathbf{CP}^3$, so its nonzero Betti numbers are $h^0 = h^2 = h^4 = h^6 = 1$. For cohomology with compact support, one has the dual Betti numbers $h^2_{cpct} = h^4_{cpct} = h^6_{cpct} = h^8_{cpct} = 1$. This suggests that the map $i : H^k_{cpct}(W; \mathbf{R}) \to H^k(W; \mathbf{R})$ may be nonzero for $k = 2, 4$, and 6. A topological analysis, using the fact that $c_1(\mathcal{L})^3|_W \neq 0$, shows that this is so.

Consequently, given the asymptotically conical nature of the Calabi-Yau metric on W, we expect the space of $\mathbf{L}^2$ harmonic forms on W to be three-dimensional, with one class each in dimension 2, 4, and 6. Given this, there is only one option for how the $SU(2)$ group R that acts on the cohomology of a Kähler manifold (see Appendix I) can act on the $\mathbf{L}^2$ harmonic forms on W: they transform with spin 1. Hence, though there is an $\mathbf{L}^2$ harmonic four-form on W, it is not primitive, and one cannot turn on a G-field without breaking supersymmetry.

6. Brane Perspective

We will conclude this paper by pointing out a reinterpretation of the problem in terms of singularities of branes. We already explained in section 2.6 that F-theory on a four-fold singularity can be reinterpreted as Type IIB with a D7-brane that has a world-volume $\mathbf{R}^4 \times L$, where $L \subset \mathbf{C}^3$ is a singular complex surface. By successive circle compactifications, it follows that M-theory or Type IIA at a four-fold singularity can be described by Type IIA with a singular sixbrane $\mathbf{R}^3 \times L$, or Type IIB with a singular fivebrane $\mathbf{R}^2 \times L$.

Analogous phenomena have been noted in the past in the context of $\mathcal{N} = 2$ conformal theories with NS or M5-branes worldvolumes with singular geometry $\mathbf{R}^4 \times \Sigma$ where Σ is a Riemann surface which develops a singularity, say of the form $y^2 = x^n$ locally (for $n > 2$). These give models for studying Type II strings at a Calabi-Yau threefold singularity capturing Argyres-Douglas points of $\mathcal{N} = 2$ conformal theories [38] and are presently under study [31].

As we discussed in section 2.6, it is important that in the cases that we have looked at, there are no corrections to the classical $\mathbf{R}^n \times L$ geometry. Let us raise the general question of this sort. Suppose we have a p-brane of some kind, with worldvolume

$$\mathbf{R}^n \times X^{p+1-n}$$

embedded in $\mathbf{R}^{10}$ or $\mathbf{R}^{11}$ depending on whether we are dealing with string theory or M-theory. We assume that this geometry preserves some number of supersymmetries in $\mathbf{R}^n$. Let us assume X develops a singularity. Is this singular geometry smoothed out in the quantum theory? A necessary condition for that is the existence of instantons which end on X. So if q-branes can end on this particular p-brane, the condition is the absence of compact q-cycles in X. So for Dp-branes in Type IIA or IIB string theory, since $D(p-2)$-branes and fundamental one-branes can end on them, the condition is the absence of topologically non-trivial compact one-cycles and $p - 2$-cycles in the geometry of X. For $M5$-branes, the condition is the absence of compact two-cycles in X.

Appendix I. Primitive Forms

The de Rham cohomology of a compact Kähler manifold X, or the space of $\mathbf{L}^2$ harmonic forms on any Kähler manifold, admits an $SU(2)$ action which is as follows. (See [39], pp. 122-6, for a mathematical introduction.) A diagonal generator J_3 of $SU(2)$ multiplies a p-form by $(n-p)/2$, where n is the complex dimension of X. The lowering operator J_- acts by wedge product with the Kähler form K:

$$G \rightarrow K \wedge G. \tag{I.1}$$

And the raising operator J_+ is the adjoint operation of contraction with K:

$$G_{i_1 i_2 \ldots i_n} \rightarrow K^{i_1 i_2} G_{i_1 i_2 \ldots i_n}. \tag{I.2}$$

Conceptually, this $SU(2)$ action arises as follows. Begin with the supersymmetric nonlinear sigma model in four dimensions with target space X, and dimensionally reduce it to $0+1$ dimensions. This gives a supersymmetric system in which the Hilbert space is the space of differential forms on X, the four supercharges are are the ∂ and $\overline{\partial}$ operators and their adjoints, and there is an $SU(2)$ symmetry that comes from rotations of the three extra dimensions. From this point of view, the $SU(2)$ arises as an R-symmetry group, so we denote it as R.

Since an $(n-p)$-form has J_3 eigenvalue $(n-p)/2$, it clearly transforms under R with spin at least $|n-p|/2$. For $n-p \geq 0$, we declare the primitive part of $H^{n-p}(X; \mathbf{R})$ to consist of the harmonic forms that transform with spin precisely $(n-p)/2$. For a middle-dimensional form, with $p = n$, this definition means that the primitive part of $H^n(X; \mathbf{R})$ consists precisely of the R-invariants.

For a noncompact Kähler manifold X, if all of the $\mathbf{L}^2$ harmonic forms are in the middle dimension, then they are all automatically primitive. For a middle-dimensional $\mathbf{L}^2$ harmonic form that is not R-invariant can be raised and lowered to make $\mathbf{L}^2$ harmonic forms of other dimensions.

For a middle-dimensional $\mathbf{L}^2$ harmonic form G, such as the G-field on a Calabi-Yau four-fold, primitiveness is equivalent to either $0 = J_- G$, which is the condition on G given in [8], or $0 = J_+ G = K \wedge G$.

An important illustrative case is that of a complex surface W. The middle-dimensional cohomology of W is two-dimensional and can be decomposed as follows. The space of self-dual forms at a given point is three-dimensional; the self-dual forms are the $(2,0)$ and

$(0, 2)$ forms and the multiples of the Kähler class K. The $(2, 0)$ and $(0, 2)$ forms are clearly primitive (the lowering operator would map them to $(3, 1)$ and $(1, 3)$-forms) but the Kähler class K is not (as $K \wedge K \neq 0$). The anti-self dual two-forms are of type $(1, 1)$ and are of the form $\alpha = a_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ where $a_{i\bar{j}}$ is traceless. Tracelessness of a means that α is annihilated by the raising and lowering operators and so is primitive. So for a complex surface, the middle-dimensional primitive cohomology is of type $(2, 0)$ or $(0, 2)$ and self-dual, or of type $(1, 1)$ and anti-self-dual.

Closer to our needs in this paper is the case of a complex four-fold X. At any point $P \in X$, the holonomy group $U(4)$ acts on the differential forms at P. Since the generator of the center of $U(4)$ simply multiplies a (p, q) form by $p - q$, we focus on the $SU(4)$ action. We look first at the (p, p) forms for $p = 0, 1, 2, \ldots$, since they are closed under the action of R. The $(0, 0)$ forms transform in the trivial representation $\mathbf{1}$ of $SU(4)$. The $(1, 1)$-forms $a_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ transform as $\mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{1} \oplus \mathbf{15}$, with $\mathbf{15}$ the adjoint representation. Since a $(2, 0)$ or $(0, 2)$-form $h_{ij} dz^i \wedge dz^j$ or $\widetilde{h}_{\bar{i}\bar{j}} d\bar{z}^{\bar{i}} \wedge d\bar{z}^{\bar{j}}$ transforms as the $\mathbf{6}$, the $(2, 2)$-forms transform as $\mathbf{6} \otimes \mathbf{6} = \mathbf{1} \oplus \mathbf{15} \oplus \mathbf{20}$. From this, it follows that $(2, 2)$-forms that transform as $\mathbf{20}$ of $SU(4)$ have $R = 0$, those that transform as $\mathbf{15}$ have $R = 1$, and those that transform as $\mathbf{1}$ have $R = 2$.

In particular, the primitive $(2, 2)$-forms transform in an irreducible representation of $SU(4)$. From this, it follows that they all transform with the same eigenvalue under the Hodge $*$ operator. To determine the sign, it suffices to consider the case that $X = W_1 \times W_2$, with the W_i complex surfaces, and to consider on X the primitive $(2, 2)$-form $G = \alpha_1 \wedge \alpha_2$, where for $i = 1, 2$, α_i is a primitive $(1, 1)$-form on W_i. Since the α_i are anti-self-dual, G is self-dual.

We can similarly analyze the primitive $(3, 1)$ cohomology. The $(2, 0)$-forms transform as $\mathbf{6}$ under $SU(4)$, while the $(3, 1)$-forms transform as $\mathbf{6} \oplus \mathbf{10}$. $(3, 1)$-forms that transform as $\mathbf{10}$ of $SU(4)$ have $R = 0$ and so are primitive, while those that transform as $\mathbf{6}$ have $R = 1$. Since the primitive $(3, 1)$-forms transform irreducibly under $SU(4)$, they again all have the same eigenvalue of $*$. Indeed, by considering the case $G = \alpha \wedge \beta$, with α a primitive $(1, 1)$-form on a surface W_1 and β a primitive $(2, 0)$-form on another surface W_2, we learn that the primitive $(3, 1)$ cohomology of a four-fold is anti-self-dual.

Finally, the $(4, 0)$ cohomology of a four-fold transforms trivially under $SU(4)$ and is primitive. By setting $G = \beta_1 \wedge \beta_2$ with β_i a $(2, 0)$ form on W_i for $i = 1, 2$, we learn that the $(4, 0)$ cohomology on a four-fold is self-dual.

In sum, the Hodge $*$ operator acts on the primitive $(p, 4-p)$ cohomology of a four-fold as $(-1)^p$. This type of argument can clearly be generalized to other dimensions.

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Appendix II. Supersymmetry Conditions in Eleven Dimensions

In this appendix we extend the analysis of [8] to compactifications of M-theory on Calabi-Yau four-folds with general G-flux, allowing for the possibility that the three-dimensional cosmological constant is non-zero. In the original version of this paper we showed that turning on holomorphic G-flux of type $(4,0)$ induces a mass for the gravitino fields, $m_\psi \sim \int G \wedge \Omega$. In a supersymmetric situation masses of the bosonic modes of the supergravity multiplet should be related to m_ψ in a supersymmetric fashion. In particular, one should expect a non-zero cosmological constant $\Lambda = -|m_\psi|^2$. However, it turns out that compactification with a $G_{4,0}$ flux leads to a solution with zero cosmological constant and, therefore, implies broken supersymmetry, see below and [40].

We follow the notations of [8] where capital letters M, N, ... run from 0 to 10 and denote eleven-dimensional indices; m, n, ... are real indices tangent to Y; and Greek letters μ, ν, ... stand for the three-dimensional Lorentzian indices 0,1,2. Finally, lower case letters a, b, ... from the beginning of the alphabet denote holomorphic indices tangent to Y.

The bosonic part of the eleven-dimensional effective action, corrected by the σ-model anomaly on the five-brane world-volume, has the following form:

$$S_{11} = \frac{1}{2} \int d^{11}x \sqrt{-g} R - \frac{1}{2} \int \left[\frac{1}{2} G \wedge *G + \frac{1}{6} C \wedge G \wedge G - (2\pi)^4 C \wedge I_8 \right] \tag{II.1}$$

The eight-form anomaly polynomial can be expressed in terms of the Riemann tensor [41]:

$$I_8 = \frac{1}{(2\pi)^4} \left(-\frac{1}{768} (\mathrm{tr}R^2)^2 + \frac{1}{192} \mathrm{tr}R^4 \right) \tag{II.2}$$

In these units the five-brane tension $T_6 = \frac{1}{(2\pi)^3}$. The field equation for G that follows from the action (II.1) looks like:

$$d * G = -\frac{1}{2} G \wedge G + (2\pi)^4 I_8 - \sum_{i=1}^{N} \delta^8(x^m - P_i) \tag{II.3}$$

This equation is a macroscopic analog of the anomaly equation (2.1). In fact, the right-hand side of (II.3) represents a local source for the field $G_{\mu\nu\rho m}$. For a compact Calabi-Yau space Y, the G-flux has nowhere to go. Hence, the integral of the right-hand side of (II.3) has to vanish, leading to the anomaly cancellation condition (2.1). In order to satisfy the equation of motion (II.3) we take the following ansatz for the three-dimensional components of G:

$$G_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho} \partial_m f(x^m) \tag{II.4}$$

We also allow for arbitrary internal components G_{mnpq}, the form of which will be fixed by the field equation and supersymmetry conditions. As we will see in a moment, a maximally symmetric compactification on Y with nontrivial G-flux typically leads to warped metric:

$$ds_{11}^2 = \Delta^{-1}\Big(ds_3^2(x^\mu) + ds_8^2(x^m)\Big) \tag{II.5}$$

where we introduced the warp factor $\Delta(x^m)$. For now, both $\Delta(x^m)$ and $f(x^m)$ are scalar functions of the coordinates on Y. Below we show that these two functions are related by the supersymmetry conditions which we are going to analyze now.

Assuming that the gravitino, ψ_M, vanishes in the background, supersymmetry variations of the bosonic fields are all identically zero. So, we only have to check that the variations of the gravitino also vanish for some Majorana spinor η:

$$\delta\psi_M \equiv \nabla_M \eta - \frac{1}{4}\Gamma_M{}^N \partial_N(\log \Delta)\eta - \tag{II.6}$$

$$-\frac{1}{288}\Delta^{3/2}(\Gamma_M{}^{PQRS} - 8\delta_M^P\Gamma^{QRS})G_{PQRS}\eta = 0$$

The first two terms in this expression come from the covariant derivative in the eleven-dimensional metric (II.5).

Following [8], we make the 11=3+8 split:

$$\Gamma_\mu = \gamma_\mu \otimes \gamma_9, \quad \Gamma_m = 1 \otimes \gamma_m$$

where the eleven-dimensional gamma-matrices Γ^M are hermitian for $M = 1, \ldots, 10$ and anti-hermitian for $M = 0$. They satisfy:

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN} \tag{II.7}$$

We use the standard notation

$$\Gamma_{M_1 \ldots M_n} = \Gamma_{[M_1} \ldots \Gamma_{M_n]} \tag{II.8}$$

for the antisymmetrized product of gamma-matrices. We decompose the supersymmetry parameter as:

$$\eta = \epsilon \otimes \xi + \epsilon^* \otimes \xi^* \tag{II.9}$$

where ϵ is an anti-commuting Killing spinor in three dimensions:

$$\nabla_\mu \epsilon = m_\psi \gamma_\mu \epsilon^* \tag{II.10}$$

and ξ is a commuting eight-dimensional complex spinor of definite chirality. From the commutation relation of ∇_μ in anti-de Sitter space we find the usual relation between the complex gravitino mass m_ψ and the value of the three-dimensional cosmological constant:

$$\Lambda = -|m_\psi|^2 \tag{II.11}$$

Without loss of generality we can take:

$$\gamma_9 \xi = \xi \tag{II.12}$$

Here γ_9 is the eight-dimensional chirality operator that anti-commutes with all the γ_m's and satisfies $\gamma_9^2 = 1$. The sign in (II.12) determines whether it is space-filling membranes or space-filling antimembranes that can be included without breaking supersymmetry. If the sign is changed, the corresponding supersymmetric vacuum can be obtained from that with $\gamma_9 \xi = +\xi$ by changing the sign of the function f and the chirality of ϵ.

The μ-component of the supersymmetry condition (II.6) takes the form:

$$\begin{aligned} \delta \psi_\mu \equiv \nabla_\mu \eta - \frac{1}{4} \partial_n (\log \Delta)(\gamma_\mu \otimes \gamma_9 \gamma^n) \eta - \\ - \frac{1}{288} \Delta^{3/2} (\gamma_\mu \otimes \gamma_9 \gamma^{mnpq}) G_{mnpq} \eta + \frac{1}{6} \Delta^{3/2} (\partial_m f)(\gamma_\mu \otimes \gamma^m) \eta = 0 \end{aligned} \tag{II.13}$$

Substituting the decomposition (II.9) for η and using equations (II.10) and (II.12) we obtain our first supersymmetry condition. Projecting the result onto subspaces of positive and negative chirality, we actually get two conditions: one comes from the first and the third term in (II.13); and the other one comes from the second and the fourth term. Moreover, in a vacuum with $\mathcal{N} = 2$ supersymmetry variations proportional to ϵ and ϵ^* must vanish separately. Therefore, we find:

$$f = \Delta^{-3/2}, \qquad 288 m_\psi \Delta^{-3/2} \xi = \not{G} \xi^*. \tag{II.14}$$

Here we have written $\not{G}$ for the total contraction $G_{mnpq} \gamma^{mnpq}$. According to our ansatz (II.4), the three-dimensional components of the four-form field strength:

$$G_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho} \partial_m \Delta^{-3/2} \tag{II.15}$$

have the form similar to the membrane solution with local "effective" membrane charge density, as follows from the field equation (II.3) for the internal components. Note, that for

a compact Y there is no global membrane charge. Substituting (II.15) into (II.3), we obtain two additional equations. One equation uniquely determines G given its cohomology class:

$$\partial_m(\Delta^{-3/2}G^{mnpq}) = \frac{1}{4!}(\partial_m\Delta^{-3/2})\epsilon^{mnpqrstu}G_{rstu} \qquad (II.16)$$

It is identically obeyed if G is self-dual (as we will find) and closed. The second equation which follows from (II.3) determines the warp factor:

$$d * d\log\Delta^{3/2} = \frac{1}{2}G \wedge G - (2\pi)^4 I_8 + \sum_{i=1}^{N}\delta^8(x^m - P_i) \qquad (II.17)$$

Now we return to the original supersymmetry condition (II.6) and consider its m-component:

$$\delta\psi_m \equiv \epsilon \otimes \nabla_m\xi - m_\psi\epsilon \otimes \gamma_m\xi^* + \frac{1}{24}\Delta^{3/2}G_{mnpq}\epsilon \otimes \gamma^{npq}\xi+ \qquad (II.18)$$

$$+\frac{1}{4}\partial_m(\log\Delta)\epsilon \otimes \xi - \frac{3}{8}\partial_n(\log\Delta)\epsilon \otimes \gamma_m{}^n\xi + \text{c.c.} = 0$$

where we used the explicit form of the solution (II.14) and standard properties of gamma-matrices. Once again ϵ and ϵ^* components of this equation must vanish separately. By means of the rescaling transformations:

$$\begin{array}{ccc} g_{mn} & \rightarrow & \Delta^{3/2}g_{mn} \\ \xi & \rightarrow & \Delta^{-1/4}\xi \end{array}$$

the ϵ-component of (II.18) can be written in the following compact form:

$$\nabla_m\xi - m_\psi\Delta^{3/4}\gamma_m\xi^* + \frac{1}{24}\Delta^{-3/4}G_{mnpq}\gamma^{npq}\xi = 0 \qquad (II.19)$$

Then, following [8], we choose ξ to be a covariantly constant spinor of unit norm, and use it to define the complex structure $J_m{}^n = i\xi^\dagger\gamma_m{}^n\xi$ and the Kähler form $J_{a\bar{b}} = ig_{a\bar{b}}$. Since the metric on Y is of type (1,1), it is convenient to think of 'holomorphic' gamma-matrices γ^a and γ_a as creation and annihilation operators that satisfy the algebra:

$$\{\gamma^a, \gamma^b\} = \{\gamma^{\bar{a}}, \gamma^{\bar{b}}\} = 0, \quad \{\gamma^a, \gamma^{\bar{b}}\} = 2g^{a\bar{b}}$$

Namely, γ^a and $\gamma_{\bar{a}}$ act on the Fock "vacuum" ξ as annihilation operators:

$$\gamma^a\xi = 0, \quad \gamma_a\xi^* = 0, \quad \gamma^{\bar{a}}\xi^* = 0, \quad \gamma_{\bar{a}}\xi = 0 \qquad (II.20)$$

To obtain the algebraic constraints on the field G, we multiply the differential equation (II.19) by γ^a which kills the first term in that equation:

$$24m_\psi\Delta^{3/2}\gamma^a\gamma_m\xi^* - G_{mnpq}\gamma^a\gamma^{npq}\xi = 0 \qquad \text{(II.21)}$$

Components of this equation with different gamma-matrix structure must vanish separately. For example, if we choose m to be an anti-holomorphic index and use (II.20), we find that $G^{(1,3)}$, the (1,3) piece of the field G, must be zero:

$$G_{\overline{a}b\overline{c}d} = 0. \qquad \text{(II.22)}$$

Moreover, $G^{(2,2)}$ must be primitive:

$$G_{\overline{a}b\overline{c}d}J^{\overline{c}d} = 0 \qquad \text{(II.23)}$$

Finally, taking the trace over the holomorphic index 'a' in the main supersymmetry condition (II.21), we can demonstrate that the (4,0) part of the G-flux breaks supersymmetry, as we expected in section 2.4. Indeed, we get a relation between $G_{4,0}$ and m_ψ of the form (II.14), but with a different numerical coefficient:

$$96m_\psi\Delta^{3/2}\xi^* = G_{abcd}\gamma^{abcd}\xi$$

Therefore, compactifications of M-theory on Calabi-Yau four-folds with $G_{4,0} \neq 0$ lead to three-dimensional vacua with broken supersymmetry, in accordance with the proposed expressions (2.38) and (2.41) for the effective superpotential. Indeed, from (2.38) it follows that (4,0) part of G contributes to the vacuum value of W. Now, in order to see that three-dimensional vacua with $W \neq 0$ are not supersymmetric we must be careful and use the appropriate covariant derivatives D/Dt_i and D/Dk_j. Since the supersymmetry conditions (2.36) and (2.37) are scale invariant, there is at least one massless vector multiplet in the effective three-dimensional theory corresponding to the volume of Y. After this multiplet is dualized to a massless chiral multiplet K_0, one finds no-scale supergravity theory where effective superpotential is independent on K_0, see [40], for otherwise this multiplet had a mass. Since $\partial_{k_0}W = 0$ but $\partial_{k_0}K(t_i, k_j) \neq 0$, where $K(t_i, k_j)$ is the Kähler potential, it follows that in a vacuum with $W \neq 0$ the covariant derivative $DW/Dk_0 = \partial_{k_0}W + (\partial_{k_0}K)W$ does not vanish, and supersymmetry is broken.

Note, that both $G_{4,0}$ and primitive G-flux of type $(2,2)$ are self-dual and, therefore, obey equations of motion (II.16). As one can easily see, self-duality of G also implies

vanishing of the three-dimensional cosmological constant. Indeed, it follows from (2.1) that $\frac{1}{2} \int G \wedge^* G - \chi(X)/24 = 0$, so that the positive vacuum energy due to the G-flux is cancelled by the negative contribution of the R^4 terms in the effective action of M theory. In particular, it means that compactification with G-flux of type $(4,0)$ classically gives a flat space solution with $\Lambda = 0$, *cf.* [42].

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