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CHAIN COMPLEXES AND STABLE CATEGORIES

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Abstract. Under suitable assumptions, we extend the inclusion of an additive subcategory $\underline{\mathcal{X}} \subset \underline{\mathcal{A}}$ (= stable category of an exact category with enough injectives) to an S-functor [15] $\mathcal{H}_{0]}\,\underline{\mathcal{X}} \to \underline{\mathcal{A}}$, where $\mathcal{H}_{0]}\,\underline{\mathcal{X}}$ is the homotopy category of chain complexes concentrated in positive degrees. We thereby obtain a new proof for the key result of J. Rickard's 'Morita theory for Derived categories' [17] and a sharpening of a theorem of Happel [12, 10.10] on the 'module-theoretic description' of the derived category of a finite-dimensional algebra.

1. Notation and Results

1.1 Let $\mathcal{B}$ be an additive category. We denote by

- $\mathcal{CB}$ the category of chain complexes

$$K = (\ldots \to K_{n+1} \xrightarrow{d^K_{n+1}} K_n \xrightarrow{d^K_n} K_{n-1} \to \ldots)\ ,\ K_n \in \mathcal{B}\ ,\ n \in \mathbf{Z}\ ,$$

- $\mathcal{HB}$ the homotopy catgory $\mathcal{CB}/\mathcal{N}$, where $\mathcal{N}$ is the ideal of morphisms homotopic to 0, endowed with the suspension functor

$$S : \mathcal{HB} \to \mathcal{HB}\ ,\ K \mapsto SK\ ,\ (SK)_n = K_{n-1}\ ,\ d^{SK} = -d^K$$

  and with the triangles $X \to Y \to Z \to SX$ furnished by the pointwise split exact sequences of $\mathcal{CB}$ (cf. [19]),

- $\mathcal{C}_+\mathcal{B}$, $\mathcal{C}_b\mathcal{B}$, $\mathcal{C}_{0]}\,\mathcal{B}$ and $\mathcal{C}^b_{0]}\,\mathcal{B}$ (resp. $\mathcal{H}_+\mathcal{B}$, $\mathcal{H}_b\mathcal{B}$, $\mathcal{H}_{0]}\,\mathcal{B}$ and $\mathcal{H}^b_{0]}\,\mathcal{B}$) the full subcategories of $\mathcal{CB}$ (resp. $\mathcal{HB}$) consisting of the right bounded ($K_n = 0\ \forall\, n \ll 0$), the right and left bounded ($K_n = 0\ \forall\, n \ll 0$ and $\forall\, n \gg 0$), the positive ($K_n = 0\ \forall\, n < 0$) and the bounded positive ($K_n = 0\ \forall\, n < 0$ and $\forall\, n \gg 0$) chain complexes, respectively.

We denote the homotopy class of a morphism of complexes f by $\overline{f}$. We identify $\mathcal{B}$ with the full subcategory of $\mathcal{HB}$ consisting of the complexes K with $K_n = 0\ \forall\, n \neq 0$.

The category $\mathcal{B}$ is *svelte* iff it is equivalent to a small category. In this case, the contravariant additive functors from $\mathcal{B}$ to the category of abelian groups $\mathcal{A}b$ form the abelian *category* $\operatorname{Mod}\mathcal{B}$.

1.2 For the convenience of the reader we include the following list of definitions from [15]:

a) If $\mathcal{H}$ is an arbitrary category endowed with a functor $S : \mathcal{H} \to \mathcal{H}$, a sequence of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

will be called an *S-sequence.* A *morphism of S-sequences* is given by a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & SX \\
a \downarrow & & b \downarrow & & c \downarrow & & \downarrow Sa \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & SX'
\end{array}$$

The composition is the obvious one.

A *suspended category* consists of an additive category $\mathcal{C}$, an additive functor associating with each $X \in \mathcal{C}$ its *suspension* $SX \in \mathcal{C}$, and a class of S-sequences called *triangles* and subject to the following axioms:

SP0 Each S-sequence isomorphic to a triangle is itself a triangle.

SP1 For each $X \in \mathcal{C}$ the S-sequence $0 \to X \xrightarrow{1} X \to S0$ is a triangle.

SP2 If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ is a triangle, then so is $Y \xrightarrow{v} Z \xrightarrow{w} SX \xrightarrow{-Su} SY$.

SP3 If the rows of the following diagram are triangles and the leftmost square is commutative, there is a $c : Z \to Z'$ making the whole diagram commutative.

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & SX \\
a \downarrow & & b \downarrow & & & & \downarrow Sa \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & SX'
\end{array}$$

SP4 For any two morphisms $X \xrightarrow{u} Y$ and $Y \xrightarrow{v} Z$ there is a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \to & SX \\
\| & & v \downarrow & & \downarrow & & \| \\
X & \to & Z & \to & Y' & \to & SX \\
 & & \downarrow & & \downarrow & & \downarrow Su \\
 & & X' & \xrightarrow{1} & X' & \xrightarrow{j} & SY \\
 & & j \downarrow & & \downarrow & & \\
 & & SY & \xrightarrow{Si} & SZ' & &
\end{array}$$

whose first two rows and whose two central columns are triangles.

b) Let $(\mathcal{A},\mathcal{E})$ be an exact category (cf. appendix A). An object $I \in \mathcal{A}$ is *injective*, if the functor $\mathcal{A}(?,I)$ takes conflations to short exact sequences of abelian groups. Suppose that $(\mathcal{A},\mathcal{E})$ has *enough injectives*, i.e. that for each $X \in \mathcal{A}$ there is a conflation

$$X \stackrel{i_X}{\to} IX \stackrel{d_X}{\to} SX$$

with an injective IX. Let $\mathcal{I}$ be the ideal of $\mathcal{A}$ formed by the morphisms factoring through an injective. The assignment $X \mapsto SX$ defines 'the' suspension functor [14] $S : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ of the residue class category $\underline{\mathcal{A}} = \mathcal{A}/\mathcal{I}$. Any conflation $X \stackrel{i}{\to} Y \stackrel{d}{\to} Z$ of $\mathcal{A}$ provides us with an S-sequence

$$X \stackrel{\overline{i}}{\to} Y \stackrel{\overline{d}}{\to} Z \stackrel{\overline{e}}{\to} SX\ ,$$

where $\overline{m}$ denotes the residue class of a morphism m of $\mathcal{A}$ and where e is determined by the commutative diagram

$$\begin{array}{ccccc} X & \stackrel{i}{\to} & Y & \stackrel{d}{\to} & Z \\ \| & & \downarrow & & \downarrow e \\ X & \stackrel{i_X}{\to} & IX & \stackrel{d_X}{\to} & SX \end{array} .$$

The *stable category* is the residue class category $\underline{\mathcal{A}}$ endowed with S and with the S-sequences isomorphic to S-sequences of the form $(\overline{i},\overline{d},\overline{e})$. It is a suspended category (compare [12]). If $\mathcal{A}$ is a *Frobenius category* (i.e. $\mathcal{A}$ has enough projectives and enough injectives and an object of $\mathcal{A}$ is projective iff it is injective), then $\underline{\mathcal{A}}$ is a *triangulated category*, i.e. a suspended category whose suspension is an equivalence. For example the category $\mathcal{CB}$ of 1.1 endowed with the pointwise split exact sequences is a Frobenius category and the associated stable category is $\mathcal{HB}$.

c) If $\mathcal{C}$ and $\mathcal{C}'$ are two suspended categories, an S-functor from $\mathcal{C}$ to $\mathcal{C}'$ is formed by an additive functor $F : \mathcal{C} \to \mathcal{C}'$ and by a morphism $\varphi : FS \to SF$ such that if $X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} SX$ is a triangle of $\mathcal{C}$, then

$$FX \stackrel{Fu}{\to} FY \stackrel{Fv}{\to} FZ \stackrel{(\varphi X)(Fw)}{\longrightarrow} SFX$$

is a triangle of $\mathcal{C}'$ (we denote all suspension functors by the same character S). When applied to $Y = 0$ this condition yields that φ is invertible. If (F,φ) and (F',φ') are two S-functors from $\mathcal{C}$ to $\mathcal{C}'$, a *morphism* from (F,φ) to (F',φ') is determined by a morphism of functors $\mu : F \to F'$ such that $(S\mu)\varphi = \varphi'(\mu S)$.

An S-functor $(F,\varphi) : \mathcal{C} \to \mathcal{C}'$ is an *S-equivalence* iff there is an S-functor (G,γ) such that the *composed S-functors* $(GF,(\gamma F)(G\varphi))$ and $(FG,(\varphi G)(F\gamma))$ are isomorphic to the *identical S-functors* $(1_{\mathcal{C}},1_S)$ and $(1_{\mathcal{C}'},1_S)$. One proves that *(F,φ) is an S-equivalence iff F is an equivalence of categories.*

1.3 Let $\mathcal{A}$ be an exact category with enough injectives (1.2). Let $\underline{\mathcal{X}}$ be a full additive subcategory of the stable category $\underline{\mathcal{A}}$ such that

$$\underline{\mathcal{A}}(S^nX,Y) = 0\ \forall n > 0\ ,\ \forall X,Y \in \underline{\mathcal{X}}.$$

In section 4, *we shall construct an S-functor $F_b : \mathcal{H}^b_{0]}\,\underline{\mathcal{X}} \to \underline{\mathcal{A}}$ such that $F_b|\,\underline{\mathcal{X}}$ is isomorphic to the inclusion $\underline{\mathcal{X}} \subset \underline{\mathcal{A}}$. If the suspension functor $S : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ is fully faithful and*

$$\underline{\mathcal{A}}\,(X, S^n Y) = 0 \;\forall\, n > 0\,,\; \forall\, X, Y \in \underline{\mathcal{X}}\,,$$

then F_b is fully faithful.

1.4 In addition to the assumptions of 1.3, we now suppose that each sequence

$$A^0 \xrightarrow{i^0} A^1 \to \ldots \to A^p \xrightarrow{i^p} A^{p+1} \to \ldots\,,\; p \in \mathbf{N}$$

of inflations (cf. Appendix A) of $\mathcal{A}$ has a direct limit $\varinjlim A^p$ in $\mathcal{A}$ and that $\coprod I_n$ is injective if $(I_n)_{n\in\mathrm{N}}$ is a family of injectives of $\mathcal{A}$.

In section 6, *we shall construct an S-functor $F : \mathcal{H}_{0]}\,\underline{\mathcal{X}} \to \underline{\mathcal{A}}$ whose restriction to $\mathcal{H}^b_{0]}\,\underline{\mathcal{X}}$ is isomorphic to F_b. In $\underline{\mathcal{A}}$, the image of a complex $X \in \mathcal{H}_{0]}\,\underline{\mathcal{X}}$ under F is isomorphic to $\varinjlim A^p$, where the limit is formed in $\mathcal{A}$ and*

$$A^0 \xrightarrow{i^0} A^1 \to \ldots \to A^p \xrightarrow{i^p} A^{p+1} \to \ldots\,,\; p \in \mathbf{N}$$

is an arbitrary sequence of inflations of $\mathcal{A}$ whose image in $\underline{\mathcal{A}}$ is isomorphic to the sequence

$$F_b X_{[0} \to F_b X_{[1} \to \ldots \to F_b X_{[p} \to F_b X_{[p+1} \to \ldots\,,\; p \in \mathbf{N}.$$

Here $X_{[p}$ denotes the 'subcomplex' of X with $(X_{[p})_n = 0$ for $n > p$ and $(X_{[p})_n = X_n$ for $n \le p$. *Moreover,*

a) F is fully faithful if the suspension functor $S : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ is fully faithful and

$$\underline{\mathcal{A}}\,(X, SFY) = 0$$

for all $X \in \underline{\mathcal{X}}$ and $Y \in \mathcal{H}_{0]}\,\underline{\mathcal{X}}$.

b) In case $\underline{\mathcal{X}}$ is svelte, the functor F has a right adjoint if, for each $Y \in \underline{\mathcal{A}}$ and all $n \in \mathbf{N}$, the restriction of $\underline{\mathcal{A}}\,(S^n\,?, Y)$ to $\underline{\mathcal{X}}$ is a resolvable (8.1) functor.

c) In case F is fully faithful, an object $A \in \underline{\mathcal{A}}$ lies in the image of F iff, in $\underline{\mathcal{A}}$, it is isomorphic to an object of the form $\varinjlim A^p$, where the limit is formed in $\mathcal{A}$ and

$$A^0 \xrightarrow{i^0} A^1 \to \ldots \to A^p \xrightarrow{i^p} A^{p+1} \to \ldots\,,\; p \in \mathbf{N}$$

is a sequence of inflations of $\mathcal{A}$ such that A^p lies in the image of F_b for all p and $\underline{\mathcal{A}}\,(S^n X, \overline{i^p})$ is invertible for each fixed $X \in \underline{\mathcal{X}}$ and $n \in \mathbf{N}$ for all $p \gg 0$.

We shall prove c) in 6.4, and a) and b) in sections 7 and 8, respectively.

1.5 The exact categories occurring in the applications frequently do not satisfy the assumptions of 1.4 because they have 'too few' direct limits. The first step in such cases is to replace $\mathcal{A}$ by a 'larger' category $\mathcal{A}'$ satisfying the assumptions of 1.4. It then remains to be shown in a second step that the image of F is essentially contained in $\underline{\mathcal{A}} \subset \underline{\mathcal{A}'}$ (cf. sections 2 and 3). Whereas the second step requires an investigation of the fine structure of the $\mathcal{A}$ at hand, the first one is possible in a general setting : For each exact category $\mathcal{A}$, one can construct the *countable envelope* $E : \mathcal{A} \to \mathcal{A}^\sim$, i.e. a 'universal' exact functor to an exact category $\mathcal{A}^\sim$ which has exact direct limits of sequences of inflations (cf. Appendix B). Countable sums of injectives of $\mathcal{A}^\sim$ are injective; if $\mathcal{A}$ has enough injectives then so does $\mathcal{A}^\sim$; the functor E preserves injectives and induces a fully faithful S-functor $\underline{\mathcal{A}} \to \underline{\mathcal{A}^\sim}$, which we also denote by E.

Thus, under the assumptions of 1.3, we obtain an S-functor $F^\sim : \mathcal{H}_{0]}\,\underline{\mathcal{X}} \to \underline{\mathcal{A}^\sim}$ from 1.4 such that the following square is commutative up to isomorphism

$$\begin{array}{ccc} \underline{\mathcal{X}} & \xrightarrow{\text{incl}} & \underline{\mathcal{A}} \\ \downarrow & & \downarrow E \\ \mathcal{H}_{0]}\,\underline{\mathcal{X}} & \xrightarrow{F^\sim} & \underline{\mathcal{A}^\sim} \,. \end{array}$$

If the suspension functor $S : \underline{\mathcal{A}^\sim} \to \underline{\mathcal{A}^\sim}$ *is fully faithful and*

$$\underline{\mathcal{A}}\,(X, S^n Y) = 0 \;\forall\, n > 0\,,\; \forall\, X, Y \in \underline{\mathcal{X}}\,,$$

then $F^\sim$ *is fully faithful* (proof in section 7).

2. Application : Homotopy categories

2.1 Let $\mathcal{B}$ be an additive category and $\underline{\mathcal{X}} \subset \mathcal{H}_+\mathcal{B}$ a full additive subcategory such that

- $\mathcal{H}\mathcal{B}\,(S^n X, Y) = 0 \;\forall\, n > 0,\; \forall\, X, Y \in \underline{\mathcal{X}}$ and
- $X_n = 0 \;\forall\, n < 0$ and $\forall\, n \gg 0$ for each complex $X \in \underline{\mathcal{X}}$.

We shall construct an S*-functor* $G : \mathcal{H}_+\underline{\mathcal{X}} \to \mathcal{H}_+\mathcal{B}$ *such that* $G|\,\underline{\mathcal{X}}$ *is isomorphic to the inclusion* $\underline{\mathcal{X}} \subset \mathcal{H}_+\mathcal{B}$ (cf. [17, 10.1]). *Moreover*

a) G *is fully faithful iff* $\mathcal{H}\mathcal{B}\,(S^n X, Y) = 0 \;\forall\, n \neq 0,\; \forall\, X, Y \in \underline{\mathcal{X}}$.

b) In case $\underline{\mathcal{X}}$ *is svelte,* G *has a right* S*-adjoint if, for each* $Y \in \mathcal{H}_+\mathcal{B}$*, the restriction of* $\mathcal{H}_+\mathcal{B}\,(\,?, Y)$ *to* $\underline{\mathcal{X}}$ *is a resolvable (8.1) functor.*

c) G *is an* S*-equivalence if* G *is fully faithful and* $\mathcal{H}_b\mathcal{B}$ *lies in the smallest full triangulated subcategory of* $\mathcal{H}_+\mathcal{B}$ *which contains* $\underline{\mathcal{X}}$ *and is closed under isomorphisms.*

The following sections contain preliminaries to the proof, which we give in 2.5.

2.2 We recall [18] that, for an inverse system of abelian groups

$$\ldots \to A^p \xrightarrow{a^p} A^{p-1} \to \ldots\,,\; p \in \mathbf{Z}\,,$$

the first right derived functor of $\varprojlim$ may be defined by the exact sequence

$$0 \to \varprojlim A^p \to \prod_{p\in\mathrm{Z}} A^p \xrightarrow{\alpha} \prod_{p\in\mathrm{Z}} A^p \to \varprojlim{}^1 A^p \to 0\,,$$

where α is given by the components

$$\prod_{p\in\mathrm{Z}} A^p \xrightarrow{\mathrm{can}} A^{q+1} \oplus A^q \xrightarrow{[-\theta\ 1]} A^q\,,\; \theta = a^{q+1}.$$

Now let

$$K^0 \xrightarrow{k^0} K^1 \to \ldots \to K^p \xrightarrow{k^p} K^{p+1} \to \ldots\,,\; p \in \mathbf{N}$$

be a sequence in $\mathcal{CB}$ such that

a) k^p_n admits a retraction in $\mathcal{B}$, $\forall\, n$, $\forall\, p$ and

b) $\varinjlim K^p =: K$ exists in $\mathcal{CB}$ (i.e. $\varinjlim K^p_n = K_n$ exists in $\mathcal{B}$, $\forall\, n$).

PROPOSITION. (compare [18]) *There is an exact sequence*

$$0 \to \varprojlim{}^1 \mathcal{HB}\,(SK^p, L) \xrightarrow{\delta} \mathcal{HB}\,(K, L) \xrightarrow{\mathrm{can}} \varprojlim \mathcal{HB}\,(K^p, L) \to 0,$$

which is functorial in $L \in \mathcal{HB}$. Here can *is induced by the canonical morphisms $K^p \to K$ (for δ see the remark below).*

PROOF. For $M \in \mathcal{CB}$, the complex of abelian groups $Hom\,(M, L)$ is defined by the components

$$Hom\,(M, L)_i = \prod_{n\in\mathrm{Z}} \mathcal{B}(M_{n-i}, L_n)\,,\; i \in \mathbf{Z}$$

and the differential

$$(f_n)_{n\in\mathrm{Z}} \mapsto (d_{n+1}\, f_{n+1} - (-1)^i f_n\, d_{n-i+1})_{n\in\mathrm{Z}}.$$

We have $\mathrm{H}_i\, Hom\,(M, L) \xrightarrow{\sim} \mathcal{HB}\,(S^i M, L)$. If we identify $Hom\,(K, L)$ with $\varprojlim Hom\,(K^p, L)$, we obtain an exact sequence of complexes

$$0 \to Hom\,(K, L) \to \prod_{p\in\mathrm{N}} Hom\,(K^p, L) \xrightarrow{\varphi} \prod_{p\in\mathrm{N}} Hom\,(K^p, L)\,,$$

where φ_i maps a family (f^p_n) to $(f^p_n - f^{p+1}_n k^p_n)$. Since, by assumption a), the maps

$$\mathcal{B}(K^p_{n-i}\,, L_n) \leftarrow \mathcal{B}(K^{p+1}_{n-i}\,, L_n)\,,\; p \in \mathbf{N}\,,\; n \in \mathbf{Z}$$

are surjective, we have

$$\operatorname{Cok} \varphi_i = \varprojlim{}^1 \prod_{n \in \mathbb{Z}} \mathcal{B}(K^p_{n-i}, L_n) = 0.$$

From the long exact homology sequence associated with

$$0 \to \mathcal{H}om\,(K, L) \to \prod_{p \in \mathbb{N}} \mathcal{H}om\,(K^p, L) \xrightarrow{\varphi} \prod_{p \in \mathbb{N}} \mathcal{H}om\,(K^p, L) \to 0\,,$$

we extract the sequence

$$0 \to \operatorname{Cok} \mathrm{H}_1\, \varphi \to \mathcal{HB}\,(K, L) \to \operatorname{Ker} \mathrm{H}_0\, \varphi \to 0,$$

which identifies with the sequence of the assumption.

EXAMPLE. For $K \in \mathcal{CB}$, the sequence of 'subcomplexes' $K_{[p}$, $p \in \mathbf{N}$ defined in 1.4 satisfies the assumptions of the proposition.

REMARK. In order to evaluate δ at the residue class g of

$$(\overline{g^p}) \in \prod_{p \in \mathbb{N}} \mathcal{HB}\,(SK^p, L)\,,$$

we first solve the system

$$g^p_n = f^p_n - f^{p+1}_n k^p_n\,,\ p \in \mathbf{N}\,,$$

for each fixed $n \in \mathbf{N}$, which is possible thanks to condition a). The morphisms $e^p : K^p \to L$ with the components

$$e^p_n = d_{n+1} f^p_{n+1} + f^p_n\, d_n$$

then form a compatible family (i.e. $e^{p+1} k^p = e^p\ \forall\, p$) of morphisms homotopic to 0. The corresponding morphism $e : K \to L$ equals δg. In general, it is not homotopic to 0.

2.3 In the following remarks, we collect some facts about $\mathcal{HB}$ and $\mathcal{H}_+\mathcal{B}$.

a) For a complex K, we denote by IK the complex given by

$$(IK)_n = K_n \oplus K_{n-1}\,,\ d^{IK} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and by $i_K : K \to IK$ the morphism of complexes with components

$$\begin{bmatrix} 1 \\ d_n \end{bmatrix} : K_n \to K_n \oplus K_{n-1}.$$

Obviously, a morphism of complexes $h : K \to L$ is homotopic to 0 iff h factors through i_K. This implies that $\overline{f} : K \to L$ *admits a retraction iff*

$$\begin{bmatrix} f \\ i_k \end{bmatrix} : K \to L \oplus IK$$

admits a retraction. In particular, K is homotopic to 0 iff i_K admits a retraction, that is to say iff K is a retract of a direct sum of complexes of the form

$$\ldots \to 0 \to B \xrightarrow{1} B \to 0 \to \ldots \quad .$$

b) *A morphism between right bounded complexes $\overline{f} : K \to L$ is invertible iff $H_n^\wedge \overline{f}$ is invertible $\forall\, n$.* Since this assertion involves only two complexes, we may assume for the proof that $\mathcal{B}$ is svelte. The canonical embedding

$$\mathcal{H}_+\mathcal{B} \to \mathcal{H}_+\mathrm{Mod}\,\mathcal{B}$$

then carries $\overline{f}$ to a quasi-isomorphism between right bounded complexes of free (hence projective) modules. It is well known that such a quasi-isomorphism is invertible in the homotopy category.

c) *A complex $K \in \mathcal{H}_+\mathcal{B}$ with $H_n^\wedge K = 0\ \forall\, n < 0$ is homotopy equivalent to a complex K' with $K'_n = 0$, $\forall\, n < 0$.* Using the technique of b), we can guarantee the existence of a family $r_n \in \mathcal{B}(K_n, K_{n+1})$, $n < 0$ such that

$$1_{K_n} = d_{n+1} r_n + r_{n-1} d_n \ ,$$

$\forall\, n < 0$. We set $L_n = 0$ for $n < 0$, $L_0 = K_0$ and $L_n = K_n \oplus K_{-n}$ for $n > 0$. With

$$d_1^L = [d_1^K \ \ r_{-1} d_0 r_{-1}] \ , \ \ d_n^L = d_n^K \oplus (r_{-n} d_{n+1} r_{-n}) \ , \ n > 1 \ ,$$

L becomes a differential complex and $f : L \to K$ with

$$f^0 = 1_{K_0} \text{ and } f^n = [1_{K_n}\ 0] \ , \ n > 0 \ ,$$

yields an isomorphism $\overline{f}$. (We do not assume that idempotents split in $\mathcal{B}$.)

d) Let $f : K \to L$ be a morphism of right bounded complexes. *If $H_n^\wedge \overline{f}$ is invertible for all $n < 0$, there is a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{\overline{f}} & L \\ \| & & \uparrow \overline{g} \\ K & \xrightarrow{\overline{f'}} & L' \end{array}$$

in $\mathcal{H}_+\mathcal{B}$, where $\overline{g}$ is invertible and the components f'_n admit retractions for all $n \in \mathbf{Z}$ and are invertible for all $n < 0$. We can replace $\overline{f}$ by

$$\begin{bmatrix} f \\ i_K \end{bmatrix} : K \to L \oplus IK$$

and hence may assume that the f_n admit retractions $\forall\, n \in \mathbf{Z}$. The assumptions then imply $\mathrm{H}_n^\wedge \operatorname{Cok} f = 0$, $\forall\, n < 0$; thus, by c), there is an isomorphism $\overline{h} : M \to \operatorname{Cok} f$, where $M_n = 0$, $\forall\, n < 0$. We define g and f' by base change

$$\begin{array}{ccccccc} 0 \to K & \xrightarrow{f} & L & \to & \operatorname{Cok} f & \to 0 \\ \| & & \uparrow g & & \uparrow h & \\ 0 \to K & \xrightarrow{f'} & L' & \to & M & \to 0. \end{array}$$

2.4 As we see from proposition 2.2, the direct limit $\varinjlim K^p$ formed in $\mathcal{CB}$ does not have a universal property in $\mathcal{HB}$, in general. For certain sequences K^p, we now characterize $\varinjlim K^p$ within $\mathcal{HB}$ in a different way. For short, we denote the restriction of the functor $\mathcal{HB}\,(S^n\,?,\ ,K)$ to $\mathcal{B}$ by $\mathrm{H}_n^\wedge K$. If $\mathcal{B}$ is svelte, we can interpret $\mathrm{H}_n^\wedge K$ as the n-th homology object of the complex

$$\ldots \to \mathcal{B}(?, K_n) \to \mathcal{B}(?, K_{n-1}) \to \ldots \quad .$$

Now let

$$C^0 \xrightarrow{\overline{c^0}} C^1 \to \ldots \to C^p \xrightarrow{\overline{c^p}} C^{p+1} \to \ldots\ ,\ p \in \mathbf{N}$$

be a sequence in $\mathcal{H}_+\mathcal{B}$. We assume that it is *admissible*, that is to say that

$$(*) \begin{cases} \text{there is an } n_0 \text{ such that } \mathrm{H}_n^\wedge C^p = 0\ \forall\, n < n_0,\ \forall\, p \text{ and} \\ \mathrm{H}_n^\wedge \overline{c^p} \text{ is invertible for each fixed } n \text{ and all } p \gg 0. \end{cases}$$

From 2.3 c) and d), it is easy to see that this is the case iff, in $\mathcal{HB}$, the given sequence is isomorphic to the image of a sequence

$$K^0 \xrightarrow{k^0} K^1 \to \ldots \to K^p \xrightarrow{k^p} K^{p+1} \to \ldots\ ,\ p \in \mathbf{N}$$

of $\mathcal{CB}$ satisfying

$$(**) \begin{cases} \text{there is an } n_0 \text{ such that } K_n^p = 0\ \forall\, n < n_0,\ \forall\, p \text{ and} \\ \text{for each fixed } n,\ k_n^p \text{ admits a retraction } \forall\, p \text{ and is invertible } \forall\, p \gg 0. \end{cases}$$

The *limit envelope* (cf. remark a) of the admissible sequence (C^p) consists in a complex C together with morphisms $\varphi^p \in \mathcal{HB}\,(C^p, C)$ such that $\varphi^{p+1}\overline{c^p} = \varphi^p$, $\forall\, p$ and that $\mathrm{H}_n^\wedge \varphi^p$ is invertible for each fixed n and all $p \gg 0$. We show that the limit envelope exists and is unique up to (non unique) isomorphism. For this, we first choose isomorphisms $\overline{f^p} : K^p \to C^p$ with $\overline{c^p}\,\overline{f^p} = \overline{f^{p+1}k^p}$, $\forall\, p$. From $(**)$, it is clear that the direct limit $\varinjlim K^p = K$ formed in $\mathcal{CB}$ together with the obvious morphisms $\psi^p \in \mathcal{HB}\,(C^p, K)$ forms a limit envelope. Now let C, $(\varphi^p)_{p\in\mathrm{N}}$ be the data of another limit envelope. By 2.2, there is a morphism $\overline{f} \in \mathcal{HB}\,(K, C)$ such that the diagrams

$$\begin{array}{ccc} K^p & \to & K \\ \overline{f^p}\downarrow & & \downarrow \overline{f} \\ C^p & \xrightarrow{\varphi^p} & C \end{array}$$

commute. In particular, $\overline{f}\psi^p = \varphi^p$ for all p and, since $\mathrm{H}_n^\wedge \psi^p$ and $\mathrm{H}_n^\wedge \varphi^p$ are invertible for $p \gg 0$, $\mathrm{H}_n^\wedge \overline{f}$ is invertible $\forall\, n$. By 2.3 b), $\overline{f}$ must be invertible.

REMARKS. a) In case $\mathcal{B}$ is svelte, we can interpret the morphism

$$\mathcal{H}_+\mathcal{B}\,(C, ?) \xrightarrow{\text{can}} \varprojlim \mathcal{H}_+\mathcal{B}\,(C^p, ?)$$

as a projective cover in $\mathrm{Mod}\,(\mathcal{H}_+\mathcal{B})^{op}$: Indeed, can is surjective by 2.2, and if an endomorphism $\overline{g}^\wedge = \mathcal{H}_+\mathcal{B}\,(\overline{g}, ?)$ of $\mathcal{H}_+\mathcal{B}\,(C, ?)$ satisfies $\mathrm{can}\,\overline{g}^\wedge = \mathrm{can}$, we conclude that $\overline{g}$ is invertible by the above argument.

b) We shall need later that for $L \in \mathcal{H}_b\mathcal{B}$ and $p \gg 0$ the maps

$$\begin{aligned} &\mathcal{HB}\,(L, \overline{c^p}) : \mathcal{HB}\,(L, C^p) \to \mathcal{HB}\,(L, C^{p+1}) \\ \text{and}\qquad &\mathcal{HB}\,(L, \varphi^p) : \mathcal{HB}\,(L, C^p) \to \mathcal{HB}\,(L, C) \end{aligned}$$

are invertible. Indeed this immediately follows from $(**)$.

2.5 We prove 2.1. Let $\mathcal{B}^+$ be the category whose objects are the sequences

$$B = (B_0, B_1, \dots, B_p, \dots)\,,\ p \in \mathbf{N}$$

of objects of $\mathcal{B}$ and whose morphisms $f : B \to C$ bijectively correspond to the 'matrices'

$$[f_{qp}] \in \prod_p \coprod_q \mathcal{B}\,(B_p, C_q).$$

The composition of morphisms is given by 'matrix multiplication'. By $B \mapsto (B, 0, \dots)$, we identify $\mathcal{B}$ with a full subcategory of $\mathcal{B}^+$. By [11, I, 3.2], the category $\mathcal{C}_+\mathcal{B}^+$ endowed with the pointwise split conflations is a Frobenius category, i.e. an exact category with enough projectives and enough injectives such that projectives and injectives coincide. The projective-injective objects of $\mathcal{C}_+\mathcal{B}^+$ are the complexes homotopic to 0. Thus $\underline{\mathcal{C}_+\mathcal{B}^+}$ coincides with the homotopy category $\mathcal{H}_+\mathcal{B}^+$. Now it is clear that $\mathcal{A} = \mathcal{C}_+\mathcal{B}^+$ and the subcategory $\underline{\mathcal{X}} \subset \underline{\mathcal{A}} = \mathcal{H}_+\mathcal{B}^+$ satisfy the assumptions of 1.4. We obtain an S-functor $F : \mathcal{H}_{0]}\,\underline{\mathcal{X}} \to \mathcal{H}_+\mathcal{B}^+$. Let $X \in \mathcal{H}_{0]}\,\underline{\mathcal{X}}$. We want to show that, up to isomorphism, FX lies in $\mathcal{H}_+\mathcal{B}$. The terms of the sequence

$$FX_{[0} \overset{\overline{a^0}}{\to} FX_{[1} \to \dots \to FX_{[p} \overset{\overline{a^p}}{\to} FX_{[p+1} \to \dots$$

are successive extensions of the objects S^pX_p, $p \in \mathbf{N}$. Hence, up to isomorphism, they lie in $\mathcal{H}_+\mathcal{B}$. For $p > 0$, the third corner of a triangle over $\overline{a^{p-1}}$ is isomorphic to S^pX_p, a complex whose components vanish in degrees $< p$. We conclude that the sequence of the $FX_{[p}$ is admissible (2.4). From 1.4, we see that its limit envelope is isomorphic to FX. Thus, up to isomorphism, FX lies in $\mathcal{H}_+\mathcal{B}$, since this is true of the $FX_{[p}$. Therefore, the functor F gives rise to an S-functor $\mathcal{H}_{0]}\,\underline{\mathcal{X}} \to \mathcal{H}_+\mathcal{B}$, which we extend to $G : \mathcal{H}_+\underline{\mathcal{X}} \to \mathcal{H}_+\mathcal{B}$ using [15, 2.2].

We want to prove 2.1 a). Let $X \in \underline{\mathcal{X}}$ and $Y \in \mathcal{H}_{0]}\,\underline{\mathcal{X}}$. The group $\mathcal{H}_+\mathcal{B}\,(X, SFY_{[p})$ vanishes because $SFY_{[p}$ is obtained from the $S^{n+1}Y_n$, $0 \le n \le p$ by successive extensions. Since X is bounded and the $SFY_{[p}$ form an admissible sequence with limit envelope FY, we have

$$\mathcal{H}_+\mathcal{B}\,(X, SFY_{[p}) \xrightarrow{\sim} \mathcal{H}_+\mathcal{B}\,(X, SFY)$$

for $p \gg 0$. The assertion follows from 1.4 a).

2.1 b) immediately follows from 1.4 b) and [15, 1.5].

We want to prove 2.1 c). A complex $Y \in \mathcal{C}_{0]}\underline{\mathcal{X}}$ is the limit of the sequence

$$Y_{[0} \xrightarrow{i^0} Y_{[1} \to \ldots \to Y_{[p} \xrightarrow{i^p} Y_{[p+1} \to \ldots .$$

By assumption, the $Y_{[p}$ all lie in the essential image of $F|\,\mathcal{H}^b_{0]}\,\underline{\mathcal{X}}$. Since the objects of $\underline{\mathcal{X}}$ are bounded complexes, the map $\mathcal{H}_+\mathcal{B}\,(S^nX, \overline{i^p})$ is invertible for each fixed $X \in \underline{\mathcal{X}}$ and $n \in \mathbf{N}$ for all $p \gg 0$. The assertion follows from 1.4 c).

3. Application : Finite-dimensional algebras

3.1 Let Λ be a finite-dimensional algebra over a field k, $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}_c\,\Lambda$) the category of finitely (resp. countably) generated right Λ-modules, $\nu :\ \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}_c\,\Lambda \to \mathrm{mod}_c\,\Lambda$) the Nakayama-functor $?\otimes_\Lambda \mathrm{Hom}\,(\Lambda, k)$, μ its right adjoint $\mathrm{Hom}_\Lambda(\mathrm{Hom}\,(\Lambda, k), ?)$ and $\mathcal{A}$ (resp. $\mathcal{A}_c$) the following Frobenius category : Its objects are the sequences $M = (M_n, m_n)_{n\in\mathbf{Z}}$ of Λ-modules $M_n \in \mathrm{mod}\,\Lambda$ (resp. $M_n \in \mathrm{mod}_c\,\Lambda$) and of morphisms $m_n \in \mathrm{Hom}_\Lambda(\nu M_n, M_{n-1})$ such that $m_n\nu m_{n+1} = 0$ for all n and $M_n = 0$ for all $n \ll 0$. A morphism $f : M \to M'$ is given by a sequence $f_n \in \mathrm{Hom}_\Lambda(M_n, M'_n)$ such that $m'_n\nu f_n = f_{n-1}m_n$ for all n.

We choose the suspension functor $S : \underline{\mathcal{A}_c} \to \underline{\mathcal{A}_c}$ as follows: For $M \in \mathcal{A}_c$, $n \in \mathbf{Z}$ let $i'_n : M_n \to I_n$ be an injective envelope in $\mathrm{mod}_c\,\Lambda$ and let $i''_n \in \mathrm{Hom}\,(M_n, \mu I_{n-1})$ be the morphism corresponding to $i'_{n-1}\,m_n \in \mathrm{Hom}\,(\nu M_n, I_{n-1})$. The projective-injective module $IM \in \mathcal{A}_c$ is given by

$$(IM)_n = I_n \oplus \mu I_{n-1}\,,\ \begin{bmatrix} 0 & \varphi \\ 0 & 0 \end{bmatrix} : \nu I_n \oplus \nu\mu I_{n-1} \to I_{n-1} \oplus \mu I_{n-2}\,,$$

where $\varphi : \mu\nu \to 1$ is the adjunction morphism. We put $SM = \mathrm{Cok}\, i_M$, where i_M has the components

$$\begin{bmatrix} i'_n \\ i''_n \end{bmatrix} : M_n \to I_n \oplus \mu I_{n-1}.$$

Let $\mathcal{P}$ and $\mathcal{P}_c$ be the full subcategories of the projectives of $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}_c\,\Lambda$, respectively.

THEOREM. (cf. [12, 10.10]) *The functor $\mathcal{P}_c \to \underline{\mathcal{A}_c}$ which associates with $P \in \mathcal{P}_c$ the sequence M with $M_0 = P$ and $M_n = 0\ \forall\, n \neq 0$ extends to a fully faithful S-functor $H : \mathcal{H}_+\mathcal{P}_c \to \underline{\mathcal{A}_c}$. If Λ has finite global dimension, H gives rise to an S-equivalence $\mathcal{H}_+\mathcal{P} \xrightarrow{\sim} \underline{\mathcal{A}}$.*

PROOF. The full subcategory $\mathcal{U}_c \subset \mathcal{A}_c$ consisting of the sequences M with $M_n = 0\ \forall\, n < 0$, is an abelian category with enough injectives (which are injective in $\mathcal{A}_c$ as well). It has countable unions and countable sums of injectives are injective. For $M \in \mathcal{U}_c$, the structure morphism $\nu(SM)_1 \to (SM)_0$ is surjective, which implies that $\mathcal{U}_c\,(SM, N)$ vanishes for each $N \in \mathrm{mod}_c\,\Lambda$ identified with the full subcategory of all $U \in \mathcal{U}_c$ with $U_n = 0 \,\forall\, n > 0$. In particular, $\underline{\mathcal{U}_c}\,(S^n P, Q)$ vanishes for $P, Q \in \mathcal{P}_c$, $n > 0$. Thus we obtain an S-functor $F : \mathcal{H}_{0]}\,\mathcal{P}_c \to \underline{\mathcal{U}_c}$ from 1.4. In order to show that F is fully faithful, we have to check that $\underline{\mathcal{U}_c}\,(P, SFY) = 0$ for all $P \in \mathcal{P}_c$, $Y \in \mathcal{H}_{0]}\,\mathcal{P}_c$. This is clear, since

$$\underline{\mathcal{U}_c}\,(P, SFY) \xrightarrow{\sim} \underline{\mathrm{mod}_c}\,\Lambda(P, S(FY)_0) \xrightarrow{\sim} \mathrm{Ext}^1_\Lambda\,(P, (FY)_0) = 0.$$

Using [15, 2.2], we infer the first part of the assertion.

Now suppose $\mathrm{gldim}\,\Lambda < N \in \mathbf{N}$. The construction of S shows that S^N moves the support of an $M \in \mathcal{A}$ to the left : We have $(S^N M)_n = 0$ for $n < 1$, if $M_n = 0$ holds for $n < 0$. By [15, 2.2] it is therefore enough to show that F gives rise to an S-equivalence $\mathcal{H}_{0]}\,\mathcal{P} \to \underline{\mathcal{U}}$, where $\mathcal{U} = \mathcal{U}_c \cap \mathcal{A}$. Let $X \in \mathcal{H}_{0]}\,\mathcal{P}$. We want to show that, up to isomorphism, FX lies in $\underline{\mathcal{U}}$ by 'constructing' FX using the procedure of 1.4 : The terms of the sequence

$$F_b X_{[0} \to F_b X_{[1} \to \ldots \to F_b X_{[p} \to F_b X_{[p+1} \to \ldots\,,\ p \in \mathbf{N}$$

are successive extensions of objects $S^n P$, $P \in \mathcal{P}$, $n \in \mathbf{N}$ and are therefore isomorphic to objects of $\underline{\mathcal{U}}$. Hence we can choose a sequence

$$Y^0 \xrightarrow{j^0} Y^1 \to \ldots \to Y^p \xrightarrow{j^p} Y^{p+1} \to \ldots\,,\ p \in \mathbf{N}$$

of morphisms of $\mathcal{U}$ whose image in $\underline{\mathcal{U}_c}$ is isomorphic to the sequence of the $F_b X_{[p}$. Moreover, we may assume the j^p to be injective (5.2). In $\underline{\mathcal{U}_c}$, the cokernel of j^{p-1} is isomorphic to

$$F\mathrm{Cok}\,(X_{[p-1} \to X_{[p}) \xrightarrow{\sim} S^p X_p$$

for $p \geq 1$. We choose isomorphisms $\overline{f^p} : S^p X_p \to \mathrm{Cok}\, j^{p-1}$. Because $S^p X_p$ has no injective summand, f^p is an isomorphism onto a direct summand of $\mathrm{Cok}\, j^{p-1}$ whose complement is projective-injective. Hence the pre-image $A^p \subset Y^p$ of $\mathrm{Im}\, f^p$ also has a projective-injective complement. We put $A^0 = Y^0$ and obtain the 'subsequence'

$$A^0 \xrightarrow{i^0} A^1 \to \ldots A^p \xrightarrow{i^p} A^{p+1} \to \ldots\,,\ p \in \mathbf{N}\,,$$

which, in $\underline{\mathcal{U}}$, is isomorphic to the sequence of the Y^p and which satisfies $\mathrm{Cok}\, i^{p-1} \cong S^p X_p$ in $\mathcal{U}$. Because $(S^p X_p)_n$ vanishes for $p \gg 0$, i^p_n is invertible for $p \gg 0$ and $\varinjlim A^p$ lies in $\mathcal{U}$. Hence, up to isomorphism, FX lies in $\underline{\mathcal{U}}$ and there is a fully faithful S-functor $G : \mathcal{H}_{0]}\,\mathcal{P} \to \underline{\mathcal{U}}$ whose composition with the inclusion $\underline{\mathcal{U}} \to \underline{\mathcal{U}_c}$ is isomorphic to F. Let $M \in \mathcal{U}$ be such that $M_n = 0$ for all $n \gg 0$. We want to show that M lies in the image of $G_b = G|\,\mathcal{H}^b_{0]}\,\mathcal{P}$. We use induction over the set of lexicographically ordered pairs (b, d), where b is the greatest index with $M_b \neq 0$ and d is the projective dimension of M_b. If $(b, d) = (0, 0)$, then M is in $\mathcal{P}$. If $(b, d) > (0, 0)$, we choose projective covers $q''_n : P_n \to M_n$, $n \in \mathbf{Z}$ in $\mathrm{mod}\,\Lambda$ and we define $X = (X_n, x_n)$ by

$$X_n = \nu P_{n+1} \oplus P_n \text{ for } n \geq 0,\ X_n = 0 \text{ for } n < 0 \text{ and } x_n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ for } n > 0.$$

Let $p : X \to M$ be the epimorphism with components $p_n = [q'_n \; q''_n]$, where $q'_n = m_{n+1}\, \nu q''_{n+1}$. For $M' = \operatorname{Ker} p$, we obviously have $(d', b') < (d, b)$ and, in $\underline{\mathcal{U}}$, X is isomorphic to an object of $\mathcal{P}$. By the triangle

$$M' \to X \xrightarrow{\overline{p}} M \to SM'$$

and the full faithfulness of G_b, it follows that M is in the essential image of G_b since M' and X are. In order to show that an arbitrary $M \in \mathcal{U}$ is in the image of G, we consider the sequence

$$M_{[0} \xrightarrow{i^0} M_{[1} \to \ldots \to M_{[p} \xrightarrow{i^p} M_{[p+1} \to \ldots\,,\; p \in \mathbf{N}$$

of submodules of M, where $(M_{[p})_n = M_n$ for $n \leq p$ and $(M_{[p})_n = 0$ for $n > p$. We have $\varinjlim M_{[p} \xrightarrow{\sim} M$ and we already know that $M_{[p}$ is in the image of G_b $\forall\, p$. Since the module $(S^n X)_k$ vanishes for almost all k for each fixed $X \in \mathcal{P}$ and $n \in \mathbf{N}$, it is clear that $\underline{\mathcal{U}}\,(S^n X, \overline{i^p})$ is invertible for all $p \gg 0$. The assertion follows from 1.4 c).

4. CONSTRUCTION OF F_b

4.1 Let $\mathcal{A}$ be an exact category with enough injectives. A complex A is *acyclic*, iff there are conflations

$$Z_n A \xrightarrow{j_n} A_n \xrightarrow{q_n} Z_{n-1} A$$

of $\mathcal{A}$ such that $d_n = j_{n-1}\, q_n$, $\forall\, n$. If A is acyclic and B is isomorphic to A in $\mathcal{HA}$, then B is a retract of the acyclic complex $A \oplus IB$ (2.3 a). If idempotents split in $\mathcal{A}$, it follows that B itself is acyclic.

LEMMA.

a) *We have $\mathcal{HA}\,(A, I) = 0$, if I has injective components and A is left bounded and acyclic.*

b) *For each left bounded complex K, there is a triangle*

$$aK \to K \to iK \to SaK$$

in $\mathcal{H}_-\mathcal{A}$ (= homotopy category of left bounded complexes), where aK is acyclic and iK has injective components.

c) *The inclusion of the full subcategory of acyclic complexes into $\mathcal{H}_-\mathcal{A}$ admits the right S-adjoint $K \mapsto aK$.*

PROOF. c) follows from a) and b) by [15, 1.6]. The proof of a) proceeds as in the case of an abelian category. With b) however, the argument of [13, I, 4.6] seems to fail. Suppose that $K_n = 0\ \forall\, n > 0$. First, we inductively construct a sequence of 'fibre summations' $(j'_n, j''_n, q'_n, q''_n)$, $n \in \mathbf{Z}$,

$$\begin{tikzcd}
K_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "q'_{n+2}"] & & K_n \arrow[rr, "d_n"] \arrow[dr, "q'_{n+1}"] & & K_{n-1} \arrow[dr, "q'_n"] & \\
& Z_{n+1} \arrow[ur, "j'_{n+1}"] \arrow[dr, "j''_{n+1}"] & & Z_n \arrow[ur, "j'_n"] \arrow[dr, "j''_n"] & & Z_{n-1} \\
I_{n+2} \arrow[ur, "q''_{n+2}"] & & I_{n+1} \arrow[ur, "q''_{n+1}"] & & I_n \arrow[ur, "q''_n"] &
\end{tikzcd}$$

in $\mathcal{A}$ such that I_n is injective, j''_n is an inflation, $j'_n\, q'_{n+1} = d_n$ and $j'_n\, q''_{n+1} = 0$. We put $I_n = Z_n = 0$ for $n > 0$ and $Z_0 = K_0$, $q'_1 = 1$. When the construction is completed up to Z_n, we choose $j''_n : Z_n \longrightarrow I_n$ as an inflation with injective I_n. From $(d_n\, j'_{n+1})\, q'_{n+2} = d_n\, d_{n+1} = 0$ and $(d_n\, j'_{n+1})\, q''_{n+2} = 0$, it follows that $d_n\, j'_{n+1} = 0$.

Hence there is a $j'_n \in \mathcal{A}(Z_n, K_{n-1})$ such that $j'_n\, q'_{n+1} = d_n$ and $j'_n\, q''_{n+1} = 0$. We define Z_{n-1} as the fibre sum of I_n and K_{n-1}. – By construction, $iK := (I_n, j''_{n-1}\, q''_n)$ is a chain complex and $f : K \to iK$ with $f_n = j''_n\, q'_{n+1}$ is a morphism of chain complexes. The mapping cone C over f has the differential $d^C_n = j_{n-1}\, q_n$, where $j_n = [j'_n\ j''_n]^t$ and $q_n = [q'_n\ q''_n]$. Since the sequences

$$Z_n \xrightarrow{j_n} K_{n-1} \oplus I_n \xrightarrow{q_n} Z_{n-1}$$

are conflations by construction, C is acyclic.

4.2 We use the notations and hypotheses of 1.3. In addition, we suppose for simplicity that *$\underline{\mathcal{X}}$ is closed under isomorphisms in $\underline{\mathcal{A}}$*, which obviously does not entail any restriction of generality. Let $\mathcal{X}$ be the full subcategory of $\mathcal{A}$ with the same objects as $\underline{\mathcal{X}}$ (in particular, $\mathcal{X}$ contains each injective).

Let $\mathcal{U}_b$ be the full subcategory of $\mathcal{CA}$ consisting of the complexes X which satisfy

- X is acyclic and left bounded,
- $X_n \in \mathcal{X}$, $\forall\, n \in \mathbf{Z}$ and X_n is injective $\forall\, n < 0$.

Endowed with the pointwise split conflations $\mathcal{U}_b$ is an exact subcategory of $\mathcal{CA}$ with enough injectives : the complexes homotopic to 0 lying in $\mathcal{U}_b$ (cf. 2.5).

Since the $X \in \mathcal{U}_b$ are acyclic, they admit conflations

$$Z_n \xrightarrow{j_n} X_n \xrightarrow{q_n} Z_{n-1}\ ,\ n \in \mathbf{Z}$$

such that $d_n = j_{n-1}\, q_n$, $\forall\, n$. The functor $Z_{-1} : \mathcal{U}_b \to \mathcal{A}$ is exact and preserves injectives. It induces an S-functor $Z_{-1} : \underline{\mathcal{U}_b} \to \underline{\mathcal{A}}$. On the other hand, the canonical S-functor

$$Q_b : \underline{\mathcal{U}_b} \to \mathcal{H}^b_{0]}\, \underline{\mathcal{X}}\ ,\ (X_n, d_n) \mapsto (X_n, \overline{d_n})$$

is an S-equivalence by the following lemma. By composing Z_{-1} with an S-quasiinverse of Q_b we obtain $F_b : \mathcal{H}^b_{0]}\, \underline{\mathcal{X}} \to \underline{\mathcal{A}}$. The assertion of 1.3 about the full faithfulness of F_b is easily established by the induction argument of [1].

Lemma. *Q_b is an S-equivalence.*

Proof. *1st step : The suspended category $\underline{\mathcal{U}_b}$ is generated by the objects aY, $Y \in \mathcal{X}$* (that is to say that any full suspended subcategory of $\underline{\mathcal{U}_b}$ which contains the aY, $Y \in \mathcal{X}$ contains each object of $\underline{\mathcal{U}_b}$, up to isomorphism) : For $X \in \underline{\mathcal{U}_b}$, let $X_{0]}$ be the 'factor complex' of X with $(X_{0]})_n = X_n$ for $n \geq 0$ and $(X_{0]})_n = 0$ for $n < 0$. Since the kernel of $X \to X_{0]}$ is left bounded and has injective components, we have $X \xrightarrow{\sim} a(X_{0]})$. The assertion follows, because $X_{0]}$ lies in the suspended category $\mathcal{H}^b_{0]}\, \mathcal{X}$, which is generated by the $Y \in \mathcal{X}$.

2nd step : The assertion : Since the $Q_b aY$, $Y \in \mathcal{X}$ generate $\mathcal{H}^b_{0]}\,\underline{\mathcal{X}}$ and since the suspension functors of both, $\underline{\mathcal{U}_b}$ and $\mathcal{H}^b_{0]}\,\underline{\mathcal{X}}$, are fully faithful, the argument of [1] shows that it is enough to check that the maps

$$\begin{array}{rcl} \underline{\mathcal{U}_b}(S^n aY, aY') & \to & \mathcal{H}^b_{0]}\,\underline{\mathcal{X}}(S^n Q_b aY, Q_b aY') \text{ and} \\ \underline{\mathcal{U}_b}(aY, S^n aY') & \to & \mathcal{H}^b_{0]}\,\underline{\mathcal{X}}(Q_b aY, S^n Q_b aY') \end{array}$$

are bijective for all $n \geq 0$ and all $Y, Y' \in \mathcal{X}$. For example, we have

$$\underline{\mathcal{U}_b}\,(S^n aY, aY') \xrightarrow{\sim} \underline{\mathcal{U}_b}\,(S^n aY, Y')\,,$$

and, since aY is nothing else than an injective resolution

$$\ldots \to 0 \to Y \to I_{-1} \to \ldots \to I_{-n} \to I_{-n-1} \to \ldots$$

of Y, the latter group identifies with the $(-n)$-th homology group of the complex

$$\ldots \leftarrow 0 \leftarrow \mathcal{A}\,(Y, Y') \leftarrow \mathcal{A}\,(I_{-1}, Y') \leftarrow \ldots \leftarrow \mathcal{A}\,(I_{-n}, Y') \leftarrow \ldots\,,$$

hence with

$$\underline{\mathcal{A}}\,(S^n Y, Y') = \begin{cases} \underline{\mathcal{A}}\,(Y, Y') & n = 0 \\ 0 & n > 0, \end{cases}$$

which was to be shown.

5. The filtered category

5.1 Let $\mathcal{A}$ be an exact category. The *filtered category* $\mathcal{FA}$ has the sequences

$$X = (X^0 \xrightarrow{i_X^0} X^1 \to \ldots \to X^p \xrightarrow{i_X^p} X^{p+1} \to \ldots)\,,\ p \in \mathbf{N}$$

of inflations of $\mathcal{A}$ as objects. The morphisms $f : X \to Y$ bijectively correspond to the sequences $f^p \in \mathcal{A}(X^p, Y^p)$ with $i_Y^p f^p = f^{p+1} i_X^p$, $\forall\, p$. We endow $\mathcal{FA}$ with the exact structure $\mathcal{FE}$ consisting of the pairs of composable morphisms (j, e) such that (j^p, e^p) is a conflation of $\mathcal{A}$, $\forall\, p$.

EXAMPLE. For each $X \in \mathcal{FA}$, there is a functorial conflation

$$X'' \xrightarrow{j} X' \xrightarrow{e} X$$

with the components

$$\coprod_{p \leq r-1} X^p \xrightarrow{j^r} \coprod_{q \leq r} X^q \xrightarrow{e^r} X^r\,,\ r \in \mathbf{N}\,,$$

where e^r is given by the canonical morphisms $X^q \to X^r$ and where j^r is given by

$$X^p \xrightarrow{[1\ -\theta]^t} X^p \oplus X^{p+1} \xrightarrow{\text{can}} \coprod_{q \leq r} X^q\,,\ \theta = i_X^p.$$

Observe that X'' and X' are sequences of split inflations (= inflations admitting a retraction).

LEMMA. *If $\mathcal{A}$ has enough injectives, then so does $\mathcal{FA}$. In this case, the injectives of $\mathcal{FA}$ are the sequences with injective components.*

PROOF. We first show that each sequence I with injective components is injective in $\mathcal{FA}$. The sequence I is isomorphic to a product of sequences of the form

$$J = (0 \to \ldots \to 0 \to J^n \xrightarrow{1} J^{n+1} \xrightarrow{1} J^{n+2} \to \ldots)\,,$$

where $n \in \mathbf{N}$ and J^n is injective in $\mathcal{A}$. It remains to be shown that such a J is injective. If $n > 0$, then, for $X \in \mathcal{FA}$, we have

$$\mathcal{FA}\,(X, J) \xleftarrow{\sim} \mathcal{FA}\,(X_{\geq n}, J)\,,$$

where $(X_{\geq n})^p = 0$ for $p < n$ and $(X_{\geq n})^p = \operatorname{Cok}(X^{n-1} \to X^p)$ for $p \geq n$. Since $X \mapsto X_{\geq n}$ is an exact functor, we may assume $n = 0$. Then we have

$$\mathcal{FA}\,(X, J) \xrightarrow{\sim} \varprojlim \mathcal{A}\,(X^p, J^0)\,,$$

where the transition maps $\mathcal{A}\,(X^p, J^0) \leftarrow \mathcal{A}\,(X^{p+1}, J^0)$ are surjective. By the Mittag-Leffler criterion [9, 0_{III}, 13.1], the functor $\mathcal{FA}\,(?, J)$ is exact.

Given $X \in \mathcal{FA}$, we now construct a conflation

$$X \xrightarrow{j} I \xrightarrow{e} Y\,,$$

where I has injective components. In particular, this conflation shows that an injective X has injective components. We first choose a conflation

$$X^0 \xrightarrow{j^0} I^0 \xrightarrow{e^0} Y^0$$

with an injective I^0. When $p \geq 0$ and (j^p, e^p) has been constructed, we form a diagram

$$\begin{array}{ccccc}
X^p & \xrightarrow{i_X^p} & X^{p+1} & \to & X_p^{p+1} \\
j^p \downarrow & & \downarrow \begin{bmatrix} k \\ l \end{bmatrix} & & \downarrow l \\
I^p & \xrightarrow{[1\ 0]^t} & I^p \oplus J & \xrightarrow{[0\ 1]} & J \\
e^p \downarrow & & \downarrow e^{p+1} & & \downarrow \\
Y^p & \xrightarrow{i_Y^p} & Y^{p+1} & \to & Y_p^{p+1},
\end{array}$$

where the first row is a conflation for i_X^p, the third column is a conflation with injective J and the morphism k is chosen such that $k i_X^p = j^p$. By the snake lemma, the diagram can be completed in such a way that the pair consisting of $j^{p+1} = [k\ l]^t$ and e^{p+1} is a conflation and i_Y^p is an inflation.

5.2 In addition, we now suppose that $\mathcal{A}$ *has enough injectives.* Let $\underline{\mathcal{A}}^{\mathrm{N}}$ be the category of sequences

$$A^0 \xrightarrow{\overline{a^0}} A^1 \to \ldots \to A^p \xrightarrow{\overline{a^p}} A^{p+1} \to \ldots\ ,\ p \in \mathbf{N}$$

of morphisms of $\underline{\mathcal{A}}$ and let $R : \underline{\mathcal{F}\mathcal{A}} \to \underline{\mathcal{A}}^{\mathrm{N}}$ be the functor which associates the sequence

$$X^0 \xrightarrow{\overline{i^0_X}} X^1 \to \ldots \to X^p \xrightarrow{\overline{i^p_X}} X^{p+1} \to \ldots$$

with $X \in \underline{\mathcal{F}\mathcal{A}}$.

LEMMA. *The functor R is an epivalence (i.e. it is full and dense and a morphism $\overline{f}$ is invertible if $R\overline{f}$ is invertible).*

PROOF. *1st step : R is full :* A morphism $RX \to RY$ is given by a sequence $\overline{g^p} \in \underline{\mathcal{A}}(X^p, Y^p)$ with $\overline{i^p_Y}\,\overline{g^p} = \overline{g^{p+1} i^p_X}$, $\forall p$. We inductively construct a sequence $f^p \in \mathcal{A}(X^p, Y^p)$ such that $\overline{f^p} = \overline{g^p}$ and $i^p_Y f^p = f^{p+1} i^p_X$, $\forall p$. We put $f^0 = g^0$. Suppose $f^0, \ldots, f^p$ have been constructed. Since $\overline{i^p_Y}\,\overline{f^p} = \overline{i^p_Y}\,\overline{g^p} = \overline{g^{p+1} i^p_X}$, there is an injective I and there are morphisms $h \in \mathcal{A}(X^p, I)$ and $k \in \mathcal{A}(I, Y^{p+1})$ such that $i^p_Y f^p - g^{p+1} i^p_X = kh$. Since i^p_X is an inflation, there is an $l \in \mathcal{A}(X^{p+1}, I)$ with $h = l\, i^p_X$. We put $f^{p+1} = g^{p+1} + kl$.

2nd step : R is dense : We can factor an arbitrary morphism $a : X \to Y$ as $a = si$, where

$$i = \begin{bmatrix} a \\ j \end{bmatrix} : X \to Y \oplus I,\ s = [1\ \ 0] : Y \oplus I \to Y$$

and $j : X \to I$ is an inflation with injective I. Observe that i is an inflation and that $\overline{s}$ is invertible. Thus a sequence of composable morphisms $\overline{a^p}$, $p \in \mathbf{N}$ is isomorphic to the sequence $\overline{i^p}$, $p \in \mathbf{N}$, where the inflations i^p result from the factorizations

$$a^0 = s^1\, i^0\ ,\ a^1\, s^1 = s^2\, i^1\ ,\ \ldots\ ,\ a^p\, s^p = s^{p+1}\, i^p \ldots\ ,\ p \in \mathbf{N}.$$

3rd step : A morphism $\overline{f} : X \to Y$ is invertible if $R\overline{f}$ is invertible : With the notations of example 5.1, f gives rise to a morphism of triangles

$$\begin{array}{ccccccc} X'' & \to & X' & \to & X & \to & SX'' \\ \downarrow \overline{f''} & & \downarrow \overline{f'} & & \downarrow \overline{f} & & \downarrow S\overline{f''} \\ Y'' & \to & Y' & \to & Y & \to & SY''. \end{array}$$

It remains to be shown that $\overline{f''}$ and $\overline{f'}$ are invertible. Clearly, the restriction of R to the full subcategory of $\underline{\mathcal{F}\mathcal{A}}$ consisting of the sequences of split inflations is fully faithful. Hence it is enough to show that $R\overline{f''}$ and $R\overline{f'}$ are invertible. By assumption, the components $\overline{f^p}$ of $R\overline{f}$ are invertible whence so are the components

$$\coprod_{p \leq r-1} \overline{f^p} : \coprod_{p \leq r-1} X^p \to \coprod_{p \leq r-1} Y^p\ ,\ r \in \mathbf{N}\,,$$

of $R\overline{f''}$ and similarly those of $R\overline{f'}$.

5.3 In addition, we now suppose that *each sequence* $X \in \mathcal{FA}$ *has a direct limit* $\varinjlim X^p$ *in* $\mathcal{A}$.

LEMMA. *The functor* $\varinjlim : \mathcal{FA} \to \mathcal{A}$ *is exact.*

PROOF. Let

$$A \xrightarrow{j} B \xrightarrow{e} C$$

be a conflation in $\mathcal{FA}$. Since $\mathcal{A}$ has enough injectives, it is enough to show that the induced sequence

$$0 \leftarrow \mathcal{A}\,(\varinjlim A^p, I) \leftarrow \mathcal{A}\,(\varinjlim B^p, I) \leftarrow \mathcal{A}\,(\varinjlim C^p, I) \leftarrow 0$$

is exact for each injective I of $\mathcal{A}$. Since the transition maps $\mathcal{A}\,(C^p, I) \leftarrow \mathcal{A}\,(C^{p+1}, I)$ are surjective, this follows from the Mittag-Leffler criterion [9, 0_{III}, 13.1] (or from 5.1: $\varinjlim$ has the diagonal functor as a right adjoint and the latter preserves injectives).

EXAMPLE. By forming limits, we obtain the conflation

$$\coprod_{p \in \mathrm{N}} X^p \xrightarrow{\iota} \coprod_{q \in \mathrm{N}} X^q \xrightarrow{\varepsilon} \varinjlim X^p$$

from the conflation (j, e) of example 5.1.

6. CONSTRUCTION OF F

6.1 We use the notations and hypotheses of 1.4. It is easy to see that $\underline{\mathcal{A}}$ has countable sums, too, and that the canonical functor $\mathcal{A} \to \underline{\mathcal{A}}$ and the suspension functor $S : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ commute with countable sums.

Observe further that $\mathcal{CA}$ has direct limits over sequences of (pointwise split) inflations and that these can be computed by taking the limit in each component.

Let $\mathcal{L}$ be the full subcategory of $\mathcal{CA}$ consisting of the $A = \varinjlim A^p$, where

$$A^0 \xrightarrow{i_0} A^1 \to \ldots A^p \xrightarrow{i^p} A^{p+1} \to \ldots\ , \ p \in \mathbf{N}$$

is a sequence of inflations with left bounded, acyclic A^p. By lemma 5.3 the $A \in \mathcal{L}$ are acyclic themselves (but they are not left bounded in general).

LEMMA.

a) *We have* $\mathcal{HA}\,(A, I) = 0$ *if* I *has injective components and* A *lies in* $\mathcal{L}$.

b) *For each complex* $K \in \mathcal{CA}$ *there is a triangle*

$$aK \to K \to iK \to SaK$$

in $\mathcal{HA}$, *where* iK *has injective components and* aK *lies in* $\mathcal{L}$.

Remark. This implies that the inclusion of the subcategory of complexes with injective components into $\mathcal{HA}$ admits the left S-adjoint $K \mapsto iK$. The essential image of $\mathcal{L}$ in $\mathcal{HA}$ is the kernel of this S-functor and therefore is a triangulated subcategory. If idempotents split in $\mathcal{A}$, then $\mathcal{L}$ is closed under isomorphisms in $\mathcal{HA}$ (cf. 4.1) and we conclude that $\mathcal{L}$ is closed under extensions in $\mathcal{CA}$.

Proof. a) If A is left bounded, the assertion holds according to 4.1 a). In the general case, example 5.3 provides us with a triangle

$$\coprod_{p\in\mathrm{N}} A^p \to \coprod_{q\in\mathrm{N}} A^q \to A \to S \coprod_{p\in\mathrm{N}} A^p$$

in $\mathcal{HA}$. By applying $\mathcal{HA}(?, I)$ to this triangle we obtain an exact sequence from which the assertion follows, because $\coprod SA^p \xrightarrow{\sim} S \coprod A^p$ and the A^p are left bounded.

b) We inductively construct a morphism

$$\begin{array}{ccccc} K_{[0} & \to & K_{[1} \to \ldots \to & K_{[p} \to & K_{[p+1} \to \ldots \\ f^0 \downarrow & & \downarrow f^1 & \downarrow f^p & \downarrow f^{p+1} \\ I^0 & \to & I^1 \to \ldots \to & I^p \to & I^{p+1} \to \ldots \end{array}$$

in $\mathcal{FCA}$ such that I^p has injective components and the mapping cone Cf^p is left bounded and acyclic $\forall\, p$. By definition $SaK := \varinjlim Cf^p$ then lies in $\mathcal{L}$ and, because countable sums of injectives of $\mathcal{A}$ are injective, $iK := \varinjlim I^p$ has injective components. We obtain $f^0 : K_{[0} \to I^0$ from lemma 4.1 b). Suppose that f^p has been constructed. We choose an acyclic complex

$$0 \to K_{p+1} \xrightarrow{\varepsilon} J_p \xrightarrow{d_p} J_{p-1} \xrightarrow{d_{p-1}} J_{p-2} \to \ldots$$

with injective J_n, $n \le p$. A classical argument of homological algebra [3, V, 1.1] shows that there is a sequence of morphisms $g_n \in \mathcal{A}(J_n, I^p_n)$, $n \le p$ such that $g_p\, \varepsilon = f^p_p\, d^K_{p+1}$ and $d_n g_n = g_{n-1} d_n$, $\forall\, n \le p$. In other words, g makes the square

$$\begin{array}{ccc} S^p K_{p+1} & \xrightarrow{d} & K_{[p} \\ e \downarrow & & \downarrow f^p \\ J & \xrightarrow{g} & I^p \end{array}$$

commutative in $\mathcal{CA}$, where J is the complex

$$\ldots \to 0 \to J_p \xrightarrow{d_p} J_{p-1} \to \ldots \to J_n \xrightarrow{d_n} J_{n-1} \to \ldots\ ,\ n \le p\,,$$

the morphism e is furnished by ε and the morphism d by d^K_{p+1}. We obtain $f^{p+1} : K_{[p+1} \to I^{p+1}$ by applying the mapping cone functor to the above square.

6.2 Let $\mathcal{U}$ be the full subcategory of $\mathcal{CA}$ consisting of the complexes X satisfying

- X lies in $\mathcal{L}$,

- $X_n \in \mathcal{X}$, $\forall\, n$ and X_n is injective $\forall\, n < 0$.

As in 4.2, we have the S-functors $Z_{-1} : \underline{\mathcal{U}} \to \underline{\mathcal{A}}$ and

$$Q : \underline{\mathcal{U}} \to \mathcal{H}_{0]}\, \underline{\mathcal{X}}\,,\ (X_n, d_n) \mapsto (X_n, \overline{d_n}).$$

According to the following lemma, Q is an S-equivalence. We put $F = Z_{-1}Q^{-}$, where Q^{-} is an S-quassiinverse of Q. Obviously, we have $F|\, \mathcal{H}^b_{0]}\, \underline{\mathcal{X}} \xrightarrow{\sim} F_b$.

LEMMA. *Q is an S-equivalence.*

PROOF. We first show that Q is fully faithful. Let $X, Y \in \underline{\mathcal{U}}$. If X lies in $\mathcal{U}_b$, we have

$$\underline{\mathcal{U}}\,(X, Y) \xleftarrow{\sim} \mathcal{H}\mathcal{A}\,(X, Y_{[N)}) \xleftarrow{\sim} \underline{\mathcal{U}}\,(X, a(Y_{[N)}))$$

for $N \gg 0$, hence the assertion follows from 4.2. In general, X is homotopy equivalent to $aX = \varinjlim X^p$, where

$$X^0 \xrightarrow{i^0} X^1 \to \ldots X^p \xrightarrow{i^p} X^{p+1} \ldots\,,\ p \in \mathbf{N}$$

is a sequence of inflations of complexes $X^p \in \mathcal{U}_b$ as the proof of 6.1 b) shows. By example 5.3, there is a triangle

$$\coprod_{p\in\mathrm{N}} X^p \to \coprod_{q\in\mathrm{N}} X^q \to X \to S \coprod_{p\in\mathrm{N}} X^p$$

in $\mathcal{H}\mathcal{A}$. Its 'image' in $\mathcal{H}\underline{\mathcal{A}}$ is isomorphic to a triangle

$$\coprod_{p\in\mathrm{N}} QX^p \to \coprod_{q\in\mathrm{N}} QX^q \to QX \to S \coprod_{p\in\mathrm{N}} QX^p.$$

The bijectivity of

$$Q(X, Y) : \mathcal{H}\mathcal{A}\,(X, Y) \to \mathcal{H}\underline{\mathcal{A}}\,(QX, QY)$$

now follows from the above special case by the 5-lemma.

In order to show that $X \in \mathcal{H}_{0]}\, \underline{\mathcal{X}}$ is in the image of Q, we choose a sequence

$$Y^0 \xrightarrow{i^0} Y^1 \to \ldots \to Y^p \xrightarrow{i^p} Y^{p+1} \to \ldots\,,\ p \in \mathbf{N}$$

of inflations of $\mathcal{U}_b$ whose image in $\mathcal{H}_{0]}\, \underline{\mathcal{X}}$ is isomorphic to the sequence of the $X_{[p}$, $p \in \mathbf{N}$ (4.2 and 5.2). Then the direct limit $\varinjlim Y^p = Y$ formed in $\mathcal{C}\mathcal{A}$ lies in $\mathcal{L}$. For each fixed $n \in \mathbf{N}$ and for $q > p \geq n$, the morphism $Y^p_n \to Y^q_n$ of $\mathcal{A}$ is a section with an injective cokernel. Since countable sums of injectives of $\mathcal{A}$ are injective, $Y^p_n \to Y_n$ is a section with an injective cokernel, too. Thus Y_n lies in $\mathcal{X}$ and Y in $\mathcal{U}$. Clearly, QY yields a limit envelope (2.4) of the QY^p. Hence $QY \cong X$.

6.3 Let

$$C^0 \xrightarrow{\overline{c^0}} C^1 \to \ldots \to C^p \xrightarrow{\overline{c^p}} C^{p+1} \to \ldots\,,\ p \in \mathbf{N}$$

be an admissible sequence (2.4) with limit envelope C in $\mathcal{H}_{0]}\, \underline{\mathcal{X}}$. For short we set $\mathcal{H} = \mathcal{H}^b_{0]}\, \underline{\mathcal{X}}$.

Lemma.

a) *There is a sequence of inflations*

$$A^0 \xrightarrow{i^0} A^1 \to \ldots \to A^p \xrightarrow{i^p} A^{p+1} \to \ldots\ ,\ p \in \mathbf{N}$$

in $\mathcal{A}$ such that the sequence of the $\overline{i^p}$ is isomorphic to the sequence of the $F\,\overline{c^p}$. For any such sequence we have $\varinjlim A^p \cong FC$ in $\underline{\mathcal{A}}$.

b) *There are short exact sequences*

$$0 \to \varprojlim{}^1 \mathcal{H}(SC^p, Y) \xrightarrow{\delta'} \mathcal{H}(C, Y) \xrightarrow{\text{can}} \varprojlim \mathcal{H}(C^p, Y) \to 0$$

and

$$0 \to \varprojlim{}^1 \underline{\mathcal{A}}(SFC^p, Y') \xrightarrow{\delta''} \underline{\mathcal{A}}(FC, Y') \xrightarrow{\text{can}} \varprojlim \underline{\mathcal{A}}(FC^p, Y') \to 0,$$

which are functorial in $Y \in \mathcal{H}$ and $Y' \in \underline{\mathcal{A}}$, respectively, and which, for $Y' = FY$, fit into a commutative diagram

$$\begin{array}{ccccccc}
0 \to & \varprojlim{}^1 \mathcal{H}(SC^p, Y) & \xrightarrow{\delta'} & \mathcal{H}(C, Y) & \xrightarrow{\text{can}} & \varprojlim \mathcal{H}(C^p, Y) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \varprojlim{}^1 \underline{\mathcal{A}}(SFC^p, FY) & \xrightarrow{\delta''} & \underline{\mathcal{A}}(FC, FY) & \xrightarrow{\text{can}} & \varprojlim \underline{\mathcal{A}}(FC^p, FY) & \to 0,
\end{array}$$

whose vertical morphisms are given by 'applying F'. Here the morphisms can *are induced by the canonical morphisms $C^p \to C$.*

Proof. a) We choose a sequence

$$X^0 \xrightarrow{j^0} X^1 \to \ldots \to X^p \xrightarrow{j^p} X^{p+1} \to \ldots\ ,\ p \in \mathbf{N}$$

of inflations in $\mathcal{U}$ whose image in $\mathcal{H}$ is isomorphic to the sequence of the $\overline{c^p}$ (6.2 and 5.2). As in the proof of the essential surjectivity of Q in 6.2, we see that $X = \varinjlim X^p$ lies in $\mathcal{U}$ and that QX is a limit envelope of the sequence of the QX^p. Thus there are compatible isomorphisms $\overline{f^p} : X^p \to Q^- C^p$ and $\overline{f} : X \to Q^- C$ in $\mathcal{HA}$. If we put $A^p = Z_{-1} X^p$ and $i^p = Z_{-1} j^p$, then the sequence of the $\overline{i^p}$ is isomorphic to the sequence of the $F\,\overline{c^p}$ and $\varinjlim A^p \xrightarrow{\sim} Z_{-1} X$ is isomorphic to FC in $\underline{\mathcal{A}}$. The rest of the assertion follows from 5.2. For the proof of b), we need the full exact subcategory $\mathcal{V}$ of $\mathcal{CA}$ consisting of the acyclic complexes whose components are injective in degrees < 0. We consider the triangle

$$\coprod X^p \xrightarrow{\bar{\iota}} \coprod X^q \xrightarrow{\bar{\varepsilon}} X \to S \coprod X^p$$

in $\underline{\mathcal{V}}$ which we construct from the sequence of the X^p according to example 5.3. Using the isomorphisms $\overline{f^p}$ and $\overline{f}$ we transform it into a triangle

$$\coprod Q^- C^p \xrightarrow{\overline{u}} \coprod Q^- C^q \xrightarrow{\overline{v}} Q^- C \to S \coprod Q^- C^p$$

in $\underline{\mathcal{V}}$. We now apply the functor $\underline{\mathcal{V}}(?, Q^- Y) = H$ to this triangle and consider the sequence

$$0 \to \operatorname{Cok} HS\overline{u} \to HQ^- C \to \operatorname{Ker} H\overline{v} \to 0$$

which we extract from the corresponding long exact sequence. By the definition of $\overline{u}$ and $\overline{v}$ it is isomorphic to

$$0 \to \varprojlim{}^1 \underline{\mathcal{U}}(SQ^- C^p, Q^- Y) \xrightarrow{\delta} \underline{\mathcal{U}}(Q^- C, Q^- Y) \xrightarrow{\text{can}} \varprojlim \underline{\mathcal{U}}(Q^- C^p, Q^- Y) \to 0$$

(cf. 2.2). We define δ' to be the composition

$$\varprojlim{}^1 \mathcal{H}(SC^p, Y) \xrightarrow{\alpha} \varprojlim{}^1 \underline{\mathcal{U}}(SQ^- C^p, Q^- Y) \xrightarrow{\delta} \underline{\mathcal{U}}(Q^- C, Q^- Y) \xrightarrow{\beta} \mathcal{H}(C, Y),$$

where α is provided by Q^- and β by Q. Clearly, Z_{-1} gives rise to an S-functor $\underline{\mathcal{V}} \to \underline{\mathcal{A}}$ which commutes with countable sums. The 'image' of the above triangle under Z_{-1} is isomorphic to a triangle

$$\coprod FC^p \to \coprod FC^q \to FC \to S \coprod FC^p.$$

As above, we derive a short exact sequence

$$0 \to \varprojlim{}^1 \underline{\mathcal{A}}(SFC^p, Y') \xrightarrow{\delta''} \underline{\mathcal{A}}(FC, Y') \xrightarrow{\text{can}} \varprojlim \underline{\mathcal{A}}(FC^p, Y') \to 0$$

from this triangle. It is clear that, for $Y' = FY$, Z_{-1} yields a morphism from (δ, can) to (δ'', can), and, by definition, Q^- yields a morphism from

$$0 \to \varprojlim{}^1 \mathcal{H}(SC^p, Y) \xrightarrow{\delta'} \mathcal{H}(C, Y) \xrightarrow{\text{can}} \varprojlim \mathcal{H}(C^p, Y) \to 0$$

to (δ, can).

6.4 The description of FX as a direct limit given in 1.4 follows at once from 6.3 a).

We want to prove 1.4 c). The pre-image

$$X^0 \xrightarrow{\overline{j^0}} X^1 \to \ldots \to X^p \xrightarrow{\overline{j^p}} X^{p+1} \to \ldots, \; p \in \mathbf{N}$$

of the sequence of the A^p under F_b is an admissible sequence (2.4). By 6.3 a) we have $FX \cong A$ for its limit envelope X.

7. Full faithfulness of F and $F^{\sim}$

7.1 We want to prove 1.4 a). Of course, we have $\underline{\mathcal{A}}\,(X, S^nY) = 0$ for $n > 0$ and $X, Y \in \underline{\mathcal{X}}$ and this implies the full faithfulness of F_b by the argument of [1]. Now let $X, Y \in \mathcal{H}_{0]}\,\underline{\mathcal{X}}$. If we apply 6.3 b) to the admissible sequence of the $X_{[p}$, $p \in \mathbf{N}$, we see that it is enough to show that F induces bijections

$$\mathcal{H}_{0]}\,\underline{\mathcal{X}}\,(X_{[p}, Y) \to \underline{\mathcal{A}}\,(FX_{[p}, FY) \text{ and } \mathcal{H}_{0]}\,\underline{\mathcal{X}}\,(SX_{[p}, Y) \to \underline{\mathcal{A}}\,(SFX_{[p}, FY)$$

for each p. Thus we may assume that X lies in $\mathcal{H}^b_{0]}\,\underline{\mathcal{X}}$. Choose $N \in \mathbf{N}$ such that $X_n = 0$ for all $n \geq N$. The conflation

$$Y_{[N} \to Y \to Y_{N+1]}$$

of $\mathcal{C}_{0]}\underline{\mathcal{X}}$ yields the triangle

$$FY_{[N} \to FY \to FY_{N+1]} \to SFY_{[N}.$$

Since S is fully faithful, $\underline{\mathcal{A}}\,(FX, ?)$ is a homological functor and the sequences

$$\underline{\mathcal{A}}\,(SFX, FY_{N+1]}) \to \underline{\mathcal{A}}\,(SFX, SFY_{[N}) \to \underline{\mathcal{A}}\,(SFX, SFY) \to \dots$$

and

$$\underline{\mathcal{A}}(SFX, FY_{N+1]}) \to \underline{\mathcal{A}}\,(FX, FY_{[N}) \to \underline{\mathcal{A}}\,(FX, FY) \to \underline{\mathcal{A}}\,(FX, FY_{N+1]})$$

are exact. It remains to be shown that

$$\underline{\mathcal{A}}(SFX, FY_{N+1]}) = 0 = \underline{\mathcal{A}}\,(FX, FY_{N+1]}).$$

By induction with respect to the greatest index n with $X_n \neq 0$, this easily follows from the assumption.

7.2 We prove the criterion for the full faithfulness of $F^\sim$ given in 1.5. Let $X \in \mathcal{X}$ and $Y \in \mathcal{H}_{0]}\,\underline{\mathcal{X}}$. According to 1.4, we can choose a sequence of inflations

$$A^0 \xrightarrow{i^0} A^1 \to \dots A^p \xrightarrow{i^p} A^{p+1} \to \dots\,,\; p \in \mathbf{N}$$

in $\mathcal{A}^\sim$ which is isomorphic to the sequence of the $F^\sim(SY)_{[p}$ in $\underline{\mathcal{A}^\sim}$. By remark B.2 a) we have

$$\underline{\mathcal{A}^\sim}\,(EX, \varinjlim A^p) \xleftarrow{\sim} \varinjlim \underline{\mathcal{A}^\sim}\,(EX, A^p)$$

hence

$$\underline{\mathcal{A}^\sim}\,(EX, SFY) \xleftarrow{\sim} \varinjlim \underline{\mathcal{A}^\sim}(EX, F^\sim(SY)_{[p}) = 0\,,$$

since $F^\sim |\, \mathcal{H}^b_{0]}\,\underline{\mathcal{X}}$ is fully faithful by the argument of [1]. Now the assertion follows from 1.4 a).

8. Existence of a right adjoint

8.1 Let $\mathcal{B}$ be a svelte additive category. A functor $F \in \operatorname{Mod}\mathcal{B}$ is *resolvable* iff it has a resolution by representable functors, i.e. iff there is an exact sequence

$$\ldots \to P_2 \to P_1 \to P_0 \to F \to 0$$

in $\operatorname{Mod}\mathcal{B}$ such that each P_i is representable [2, Ch. I, §2, Ex. 6]. It is not hard to establish that the full subcategory of resolvable functors is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms in $\operatorname{Mod}\mathcal{B}$.

Theorem. *Let $H : (\mathcal{H}_b\mathcal{B})^{op} \to \mathcal{A}b$ be a cohomological functor* [19] *such that the restriction of HS^n to $\mathcal{B}$ vanishes for all $n \ll 0$ and is a resolvable functor for all $n \in \mathbf{Z}$. There is an $X \in \mathcal{H}_+\mathcal{B}$ and an isomorphism between H and the restriction of $\mathcal{H}_+\mathcal{B}(?, X)$ to $\mathcal{H}_b\mathcal{B}$.*

Proof. For a cohomological functor G and $n \in \mathbf{N}$, we write $G|_n$ as an abbreviation of $GS^n|\,\mathcal{B}$. We may assume that $H|_n$ vanishes for all $n < 0$. We shall construct a sequence of positive complexes

$$K^0 \xrightarrow{k^0} K^1 \to \ldots \to K^p \xrightarrow{k^p} K^{p+1} \to \ldots\ ,\ p \in \mathbf{N}$$

and a sequence of morphisms

$$\varphi^p : \widehat{K^p} \to H \qquad (\widehat{X} = \mathcal{H}_b\mathcal{B}\,(?, X) \text{ for } X \in \mathcal{H}_b\mathcal{B})$$

such that $\varphi^{p+1}\widehat{k^p} = \varphi^p$, $\forall\, p$ and that $\varphi^p|_n$ is invertible for $p \geq n$. Thus the φ^p induce an isomorphism

$$\varinjlim \widehat{K^p} \xrightarrow{\sim} H.$$

The construction will show that k^p_n is invertible for $p \geq n$. Hence $K = \varinjlim K^p$ exists in $\mathcal{C}_+\mathcal{B}$ and we have isomorphisms

$$\mathcal{H}\mathcal{B}\,(?, K)|\,\mathcal{H}_b\mathcal{B} \xleftarrow{\sim} \varinjlim \widehat{K^p} \xrightarrow{\sim} H.$$

For the construction of K^0, we use the beginning of a resolution

$$\mathcal{B}\,(?, A) \xrightarrow{\mathcal{B}\,(?,d)} \mathcal{B}\,(?, B) \xrightarrow{\varepsilon} H|_0 \to 0$$

of $H|_0$ in $\operatorname{Mod}\mathcal{B}$. We define

$$K^0 = (\ldots \to 0 \to A \xrightarrow{d} B \to 0 \to \ldots)\,,$$

where B occurs in degree 0. There is a unique morphism $\eta : \widehat{B} \to H$ satisfying $(\eta B)(1_B) = (\varepsilon B)(1_B)$. The morphism η can also be characterized by the equation $\varepsilon = (\eta|_0)\,\omega$, where, for any $U \in \mathcal{B}$ and $n \in \mathbf{N}$, ω denotes the canonical isomorphism

$$\mathcal{B}\,(?, U) \xrightarrow{\sim} \widehat{S^n U}|_n\ ,\ f \mapsto S^n f.$$

Of course, we have $\eta\,\widehat{\overline{d}} = 0$. Because of the triangle

$$A \xrightarrow{\overline{d}} B \xrightarrow{\overline{u}} K^0 \to SA,$$

we can conclude that $\eta = \varphi^0\,\widehat{\overline{u}}$ for some $\varphi^0 : \widehat{K^0} \to H$. The commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathcal{B}(?,A) & \xrightarrow{\mathcal{B}(?,d)} & \mathcal{B}(?,B) & \xrightarrow{\varepsilon} & H|_0 & \to 0 \\
\downarrow \omega & & \downarrow \omega & & \uparrow \varphi^0|_0 & \\
\widehat{A}|_0 & \xrightarrow{\widehat{\overline{d}}|_0} & \widehat{B}|_0 & \xrightarrow{\widehat{\overline{u}}|_0} & \widehat{K^0}|_0 & \to 0
\end{array}$$

shows that $\varphi^0|_0$ is invertible.

For the construction of K^1, we choose an epimorphism

$$\chi : \mathcal{B}(?,C) \to H|_1$$

in $\operatorname{Mod}\mathcal{B}$. We define $\psi : \widehat{SC} \to H$ by $(\psi|_1)\,\omega = \chi$ and we consider

$$[\varphi^0\ \psi] : \widehat{K^0} \oplus \widehat{SC} \to H.$$

Clearly, $[\varphi^0\ \psi]|_1$ is an epimorphism. Since

$$\widehat{K^0}|_1 \xrightarrow{\sim} \operatorname{Ker}\mathcal{B}(?,d) \text{ and } \widehat{SC}|_1 \xleftarrow{\sim} \mathcal{B}(?,C)$$

are resolvable, there is an exact sequence

$$\mathcal{B}(?,D) \xrightarrow{\mu} (\widehat{K^0} \oplus \widehat{SC})|_1 \xrightarrow{[\varphi^0\ \psi]|_1} H|_1 \to 0$$

in $\operatorname{Mod}\mathcal{B}$. Let $\overline{f} : SD \to K^0 \oplus SC$ be such that $\mu = (\widehat{\overline{f}}|_1)\,\omega$. We put

$$K^1 = (\ldots \to 0 \to D \xrightarrow{f_1} K^0_1 \oplus C \xrightarrow{[d_1\ 0]} K^0_0 \to 0 \to \ldots)$$

and we choose $k^0 : K^0 \to K^1$ as the obvious 'embedding of the subcomplex' K^0. Since K^1 is the mapping cone over f, it fits into a triangle

$$SD \xrightarrow{\overline{f}} K^0 \oplus SC \xrightarrow{\overline{u}} K^1 \to SSD.$$

Because $[\varphi^0\ \psi]\,\widehat{\overline{f}} = 0$, it follows that $[\varphi^0\ \psi] = \varphi^1\,\widehat{\overline{u}}$ for some $\varphi^1 : \widehat{K^1} \to H$. This also implies $\varphi^1\,\widehat{\overline{k^0}} = \varphi^0$ since $u|\,K^0 = k^0$. Obviously, k^0 induces an isomorphism

$$\widehat{K^0}|_0 \xrightarrow{\sim} \widehat{K^1}|_0\,,$$

which implies that $\varphi^1|_0$ is invertible. That $\varphi^1|_1$ is invertible follows from the commutative diagram

$$\begin{array}{ccccccc}
\widehat{SD}|_1 & \xrightarrow{\widehat{\overline{f}}|_1} & (\widehat{K^0} \oplus \widehat{SC})|_1 & \xrightarrow{[\varphi^0\ \psi]|_1} & H|_1 & \to 0 \\
\| & & \| & & \uparrow \varphi^1|_1 & \\
\widehat{SD}|_1 & \xrightarrow{\widehat{\overline{f}}|_1} & (\widehat{K^0} \oplus \widehat{SC})|_1 & \xrightarrow{\widehat{\overline{u}}|_1} & \widehat{K^1}|_1 & \to 0\,,
\end{array}$$

whose first row is exact by the construction of $\overline{f}$ and whose second row is exact by the above triangle. Note that K^1_n vanishes for $n > 2$ and that $K^1|_2$ is a resolvable functor.

By essentially the same procedure, one constructs K^{p+1}, k^p and φ^{p+1} for $p \geq 1$ using the additional induction hypotheses that K^p_n vanishes for $n > p+1$ and that $K^p|_{p+1}$ is resolvable.

8.2 We want to prove 1.4 b). For $Y \in \underline{\mathcal{A}}$, we obtain a complex $X \in \mathcal{H}_{0]}\,\underline{\mathcal{X}}$ and a family $\varphi^p \in \underline{\mathcal{A}}\,(FX_{[p}, Y)$, $p \in \mathbf{N}$ from 8.1 such that, for $X' \in \mathcal{H}^b_{0]}\,\underline{\mathcal{X}}$ and $p \gg 0$, the map

$$\varphi^p_* \circ F(X', X_{[p}) : \mathcal{H}_{0]}\,\underline{\mathcal{X}}\,(X', X_{[p}) \to \underline{\mathcal{A}}\,(FX', Y)$$

is bijective. By 5.2 there is a $\varphi \in \underline{\mathcal{A}}\,(FX, Y)$ which 'extends' all the φ^p. In particular, we have a bijection

$$\varphi_* \circ F(X', X) : \mathcal{H}_{0]}\,\underline{\mathcal{X}}\,(X', X) \to \underline{\mathcal{A}}\,(FX', Y)$$

$\forall\, X' \in \mathcal{H}^b_{0]}\,\underline{\mathcal{X}}$. Let $X'' \in \mathcal{H}_{0]}\,\underline{\mathcal{X}}$. By applying Lemma 6.3 to the sequence $X''_{[p}$, $p \in \mathbf{N}$, we obtain the bijectivity of

$$\varphi_* \circ F(X'', X) : \mathcal{H}_{0]}\,\underline{\mathcal{X}}\,(X'', X) \to \underline{\mathcal{A}}\,(FX'', Y).$$

APPENDIX A : EXACT CATEGORIES

A.1 Motivated by [7], we exhibit a subset of Quillen's system of axioms [16] which is equivalent to the whole system. We use the terminology of [7] : Let $\mathcal{A}$ be an additive category. A pair (i, d) of composable morphisms

$$X \xrightarrow{i} Y \xrightarrow{d} Z$$

is *exact*, if i is a kernel of d and d a cokernel of i. Let $\mathcal{E}$ be a class of exact pairs closed under isomorphism and satisfying the following axioms Ex0, Ex1, Ex2 and Ex2^{op}. The *deflations* mentioned in these axioms are by definition the second components of the *conflations* $(i, d) \in \mathcal{E}$. The first components i are *inflations*.

Ex0 1_0 is a deflation.

Ex1 The composition of two deflations is a deflation.

Ex2 For each $f \in \mathcal{A}(Z', Z)$ and each deflation $d \in \mathcal{A}(Y, Z)$, there is a cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{d} & Z\,, \end{array}$$

where d' is a deflation.

Ex2^{op} For each $f \in \mathcal{A}(X, X')$ and each inflation $i \in \mathcal{A}(X, Y)$, there is a cocartesian square

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & & \downarrow f' \\ X' & \xrightarrow{i'} & Y', \end{array}$$

where i' is an inflation.

Proposition. *$(\mathcal{A}, \mathcal{E})$ is an exact category in the sense of* [16].

Proof. We first translate Quillen's axioms into our terminology :

a) For all $X, Z \in \mathcal{A}$, the pair

$$X \xrightarrow{[1\ 0]^t} X \oplus Z \xrightarrow{[0\ 1]} Z$$

is a conflation.

b) Axioms Ex1, Ex1^{op}, Ex2 and Ex2^{op} hold.

c) If the morphism d has a kernel and if de is a deflation for some morphism e, then d is a deflation.

c)^{op} If the morphism i has a cokernel and if ki is an inflation for some morphism k, then i is an inflation.

1st step : In the setting of Ex2 *(resp.* Ex2^{op} *) the pair*

$$Y' \xrightarrow{[-d'\ f']^t} Z' \oplus Y \xrightarrow{[f\ d]} Z \quad (\textit{resp. } X \xrightarrow{[-i\ f]^t} Y \oplus X' \xrightarrow{[f'\ i']} Y')$$

is a conflation : We consider Ex2. Let $(i, d) \in \mathcal{E}$. The morphism $i' : X \to Y'$ defined by $f' i' = i$, $d' i' = 0$ is obviously a kernel of d'. Thus we have a morphism of conflations

$$\begin{array}{ccccc} X & \xrightarrow{i'} & Y' & \xrightarrow{d'} & Z' \\ \| & & \downarrow f' & & \downarrow f \\ X & \xrightarrow{i} & Y & \xrightarrow{d} & Z. \end{array}$$

Now we form the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{d} & Z \\ i' \downarrow & & \downarrow j' & & \| \\ Y' & \xrightarrow{j} & E & \xrightarrow{e} & Z \\ d' \downarrow & & \downarrow e' & & \\ Z' & \xrightarrow{1} & Z', & & \end{array}$$

whose upper left square is cocartesian (Ex2^{op}). Obviously, e and e' are kernels of j and j', respectively, so that (j, e) and (j', e') are conflations. Since $i = f'\, i'$, we can define an isomorphism $g : E \to Z' \oplus Y$ by

$$g\, j' = \begin{bmatrix} 0 \\ 1_Y \end{bmatrix} \,,\; g\, j = \begin{bmatrix} -d' \\ f' \end{bmatrix}$$

($[h\ j']$ with $h\, d' = j'\, f' - j$ is an inverse of g). The pair (gj, eg^{-1}) equals

$$Y' \xrightarrow{[-d'\ f']^t} Z' \oplus Y \xrightarrow{[f\ d]} Z.$$

2nd step: axioms c) and c)^{op} : Let

$$Y' \xrightarrow{e} Y \xrightarrow{d} Z$$

be given as in c) and let $i : X \to Y$ be a kernel of d. By the first step,

$$[d\ \ de] : Y \oplus Y' \to Z$$

is a deflation. Hence, so is

$$[d\ \ 0] = [d\ \ de] \begin{bmatrix} 1_Y & -e \\ 0 & 1_{Y'} \end{bmatrix}.$$

Thus $i \oplus 1_{Y'}$ is an inflation, and, because of the cocartesian square

$$\begin{array}{ccc} X \oplus Y' & \xrightarrow{i\oplus 1} & Y \oplus Y' \\ {[1\ 0]} \downarrow & & \downarrow [1\ 0] \\ X & \xrightarrow{i} & Y, \end{array}$$

i is an inflation. Since $[d\ \ 0]$ is a cokernel of $i \oplus 1_{Y'}$, d is a cokernel of i, so (i, d) is a conflation.

3rd step : Axiom b) : Only Ex1^{op} remains to be proved. Let

$$X \xrightarrow{i} Y \xrightarrow{d} Z \text{ be a conflation and } Y \xrightarrow{j} Y'$$

an inflation. By Ex2^{op} and the first step, we have a bicartesion square

$$\begin{array}{ccc} Y & \xrightarrow{j} & Y' \\ d \downarrow & & \downarrow d' \\ Z & \xrightarrow{k} & Z'\,, \end{array}$$

where $[d'\ k]$ is a deflation. Because of the cartesian square

$$\begin{array}{ccc} Y' \oplus Y & \xrightarrow{1\oplus d} & Y' \oplus Z \\ {[0\ 1]} \downarrow & & \downarrow [0\ 1] \\ Y & \xrightarrow{d} & Z, \end{array}$$

$1_{Y'} \oplus d$ is a deflation, too. Consequently,

$$[d' \;\; k] \begin{bmatrix} 1_{Y'} & 0 \\ 0 & d \end{bmatrix} = [d' \;\; kd]$$

is a deflation by Ex1. We have $[d' \;\; kd] = d' \, [1_{Y'} \;\; j]$. Thus, by the second step, d' is a deflation, since d' has the kernel ji. Hence (d', ji) is a conflation.

4th step : Axiom a) : Because of the cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{1} & Z \\ \downarrow & & \downarrow \\ 0 & \rightarrow & 0, \end{array}$$

it follows from Ex0 and Ex2 that 1_Z is a deflation. Because of the cocartesian square

$$\begin{array}{ccc} X & \xrightarrow{[1\ 0]^t} & X \oplus Z \\ \downarrow & & \downarrow [0\ 1] \\ 0 & \longrightarrow & Z, \end{array}$$

the assertion now follows from Ex2^{op}.

A.2 PROPOSITION. (cf. [5, II]) *If $(\mathcal{A}, \mathcal{E})$ is a svelte, exact category, there is an equivalence $G : \mathcal{A}^{op} \to \mathcal{M}$ onto a full subcategory $\mathcal{M}$ of an abelian category such that $\mathcal{M}$ is closed under extensions and that $\mathcal{E}$ is formed by the composable pairs (i, d) inducing exact sequences*

$$0 \to GZ \xrightarrow{Gd} GY \xrightarrow{Gi} GX \to 0.$$

We sketch a proof (cf. [4, 7.3, Ex. G]) which is based on the localisation theory of abelian categories [5, III]. Let $\operatorname{Mod} \mathcal{A}$ be the category of contravariant functors from $\mathcal{A}$ to the category of abelian groups and let $\operatorname{Sex} \mathcal{A}$ be the full *subcategory of left exact functors*, i.e. the functors $F \in \operatorname{Mod} \mathcal{A}$ which transform conflations (i, d) into exact sequences

$$0 \to FZ \xrightarrow{Fd} FY \xrightarrow{Fi} FX.$$

We show first that $\operatorname{Sex} \mathcal{A}$ is an abelian category. Let $\mathcal{C} \subset \operatorname{Mod} \mathcal{A}$ be the full subcategory of *effaceable* functors, i.e. functors F such that, for each pair consisting of an object $Z \in \mathcal{A}$ and an element $z \in FZ$, there is a deflation $d : Y \to Z$ such that $(Fd)(z) = 0$. With [5, III, Prop. 8], we infer from

Ex0' Identities are deflations.

and Ex1 that $\mathcal{C}$ is a localising subcategory of $\operatorname{Mod} \mathcal{A}$, i.e. that $\mathcal{C}$ is closed under forming subobjects, quotients, extensions and direct limits in $\operatorname{Mod} \mathcal{A}$. From

Ex2' For each $f \in \mathcal{A}(Z', Z)$ and for each deflation $d \in \mathcal{A}(Y, Z)$, there is a commutative square

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{d} & Z\,, \end{array}$$

where d' is a deflation.

we derive that any effaceable subfunctor of a left exact functor vanishes and that any extension of a left exact functor by an effaceable functor splits. By [5, III, Lemme 1] this means that $\operatorname{Sex}\mathcal{A}$ consists of the objects in $\operatorname{Mod}\mathcal{A}$ which are closed with respect to $\mathcal{C}$. Thus $\operatorname{Sex}\mathcal{A}$ is equivalent to the abelian category $(\operatorname{Mod}\mathcal{A})/\mathcal{C}$. We claim that the equivalence $G : A \mapsto A^\wedge = \mathcal{A}(\,?, A)$ from $\mathcal{A}$ onto the full subcategory $\mathcal{M}$ of $\operatorname{Sex}\mathcal{A}$ consisting of the representable functors satisfies the requirements of the assertion. First, it is clear from the definition of $\operatorname{Sex}\mathcal{A}$ that for each conflation (i, d), the sequence

$$0 \to X^\wedge \xrightarrow{i^\wedge} Y \xrightarrow{d^\wedge} Z^\wedge \to 0$$

is exact in $\operatorname{Sex}\mathcal{A}$. Now let

$$0 \to X^\wedge \xrightarrow{j} F \xrightarrow{e} Z^\wedge \to 0$$

be an arbitrary exact sequence of $\operatorname{Sex}\mathcal{A}$. Since e is an epimorphism of $\operatorname{Sex}\mathcal{A}$, the cokernel $\operatorname{Cok} e = C$ formed in $\operatorname{Mod}\mathcal{A}$ is effaceable. In particular, there is a deflation $d : Y \to Z$ such that $Cd : CZ \to CY$ carries the image of 1_Z to 0. This means that there is a commutative diagram

$$\begin{array}{cccccccc} 0 \to & U^\wedge & \xrightarrow{i^\wedge} & Y^\wedge & \xrightarrow{d^\wedge} & Z^\wedge & \to 0 \\ & f \downarrow & & \downarrow g & & \| & \\ 0 \to & X^\wedge & \xrightarrow{j} & F & \xrightarrow{e} & Z^\wedge & \to 0, \end{array}$$

where $i = \operatorname{Ker} d$. Using Ex2^{op}, we form the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{i} & Y & \xrightarrow{d} & Z \\ f \downarrow & & \downarrow f' & & \| \\ X & \xrightarrow{i'} & V & \xrightarrow{d'} & Z\,, \end{array}$$

where the morphism d' defined by $d'i' = 0\,,\ d'f' = d$ is obviously a cokernel of i'. Since the embedding $\mathcal{A} \to \operatorname{Sex}\mathcal{A}$ carries the rows of the diagram to short exact sequences, the image of the left square must be cocartesian, whence there is an isomorphism

$$\begin{array}{cccccccc} 0 \to & X^\wedge & \xrightarrow{i'^\wedge} & V^\wedge & \xrightarrow{d'^\wedge} & Z^\wedge & \to 0 \\ & \| & & \downarrow & & \| & \\ 0 \to & X^\wedge & \xrightarrow{j} & F & \xrightarrow{e} & Y^\wedge & \to 0. \end{array}$$

Remark. It is not hard to show that Ex2 follows from Ex2' and Ex2^{op}. Thus the axioms given in 5.2 could be 'weakened' even more.

A.3 Let $\mathcal{A}$ be an additive category with a class of exact pairs $\mathcal{E}$. For each full subcategory $\mathcal{B}$ of $\mathcal{A}$, denote by $\mathcal{B} \cap \mathcal{E}$ the class of composable pairs (i,d)

$$X \xrightarrow{i} Y \xrightarrow{d} Z$$

in $\mathcal{E}$ such that X, Y and Z lie in $\mathcal{B}$.

Lemma. *$(\mathcal{A},\mathcal{E})$ is an exact category iff, for each svelte subcategory $\mathcal{U}$ of $\mathcal{A}$ there is a svelte exact category $(\mathcal{B},\mathcal{F})$ such that $\mathcal{B}$ is a full subcategory of $\mathcal{A}$ containing $\mathcal{U}$ and that $\mathcal{F}$ is a subclass of $\mathcal{E}$ containing $\mathcal{U} \cap \mathcal{E}$.*

Proof. If the condition is satisfied, then Ex0 and Ex1 clearly hold for $(\mathcal{A},\mathcal{E})$. In order to prove Ex2, we consider a $\mathcal{B}$ containing Y, Z, Z' and $\operatorname{Ker} d$. The cartesian square formed in $\mathcal{B}$ from d, f is cartesian in $\mathcal{A}$ as well, since, by the first step of the proof of A.1, the sequence

$$Y' \xrightarrow{[-d'\ f']^t} Z' \oplus Y \xrightarrow{[f\ d]} Z$$

lies in $\mathcal{F} \subset \mathcal{E}$ and therefore is an exact pair in $\mathcal{A}$.

In order to show that the condition is necessary, we construct an ascending chain $\mathcal{B}_n$, $n \in \mathbf{N}$ of full svelte subcategories of $\mathcal{A}$: The subcategory $\mathcal{B}_0$ contains all the objects of $\mathcal{U}$; the subcategory $\mathcal{B}_{n+1}$ contains each object B which occurs in a cartesian (resp. cocartesian) square

$$\begin{array}{ccc} B & \rightarrow & C \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{d'} & C' \end{array} \qquad \text{(resp. } \begin{array}{ccc} A' & \xrightarrow{i'} & B' \\ g\downarrow & & \downarrow \\ A & \rightarrow & B \end{array} \text{)}$$

where d' is a deflation and B', C, C' lie in $\mathcal{B}_n$ (resp. where i' is an inflation and A', A, B' lie in $\mathcal{B}_n$). Since the isomorphism class of such a B is determined by d' and f (resp. i' and g), $\mathcal{B}_{n+1}$ is svelte. Let $\mathcal{B}$ be the union of the $\mathcal{B}_n$, $n \in \mathbf{N}$. Clearly, $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$ and $(\mathcal{B}, \mathcal{B} \cap \mathcal{E})$ satisfies Ex0, Ex2, and Ex2^{op}. Now let d, e be two composable deflations (with respect to $\mathcal{B} \cap \mathcal{E}$) and let i be a kernel of d. We form the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{e'} & X \\ i'\downarrow & & \downarrow i \\ Y' & \xrightarrow{e} & Y. \end{array}$$

Clearly, i' is a kernel of de and therefore $(i', de) \in \mathcal{E}$. Since X' lies in $\mathcal{B}$ by definition, it follows that $(i', de) \in \mathcal{B} \cap \mathcal{E}$.

Appendix B : The countable envelope

B.1 Let $(\mathcal{A}, \mathcal{E})$ be an exact category. By definition, $(\mathcal{A}, \mathcal{E})$ is *countably complete* if

- each sequence $X \in \mathcal{FA}$ (5.1) has a direct limit $\varinjlim X^p$ in $\mathcal{A}$ and
- the functor $\varinjlim : \mathcal{FA} \to \mathcal{A}$ is exact.

The *countable envelope of* $\mathcal{A}$ is the category $\mathcal{A}^\sim$ with the same objects as $\mathcal{FA}$ in which morphisms $X \to Y$ bijectively correspond to the elements of

$$\varprojlim{}_p \varinjlim{}_q \mathcal{A}(X^p, Y^q)$$

(cf. [10, 8.2.5]) together with the class $\mathcal{E}^\sim$ consisting of the pairs of composable morphisms of $\mathcal{A}^\sim$ which are isomorphic to images of conflations of $\mathcal{FA}$ under the canonical functor $\mathcal{FA} \to \mathcal{A}^\sim$. The functor $E : \mathcal{A} \to \mathcal{A}^\sim$ associates the 'constant sequence' $i^p_{EA} = 1_A$, $\forall\, p$ with $A \in \mathcal{A}$.

PROPOSITION.

a) $(\mathcal{A}^\sim, \mathcal{E}^\sim)$ *is a countably complete exact category.*

b) For each countably complete exact category $\mathcal{B}$, *the functor*

$$E^* : Hom_{ex}(\mathcal{A}^\sim, \mathcal{B}) \to Hom_{ex}(\mathcal{A}, \mathcal{B}) \, , \; G \mapsto GE$$

induces an equivalence of the full subcategory of functors G *satisfying* $\varinjlim GX^p \xrightarrow{\sim} G \varinjlim X^p$, $\forall\, X \in \mathcal{FA}^\sim$ *onto* $Hom_{ex}(\mathcal{A}, \mathcal{B})$, *the category of exact functors from* $\mathcal{A}$ *to* $\mathcal{B}$.

c) The functor E *induces bijections of the extension groups. It preserves projectives and injectives.*

d) Countable sums of injectives of $\mathcal{A}^\sim$ *are injective. If* $\mathcal{A}$ *has enough injectives (resp. projectives), the class of countable sums of objects* EI, I *an injective of* $\mathcal{A}$ *(resp.* EP, P *a projective of* $\mathcal{A}$*), contains enough injectives (resp. projectives) for* $\mathcal{A}^\sim$.

REMARK. Note that the morphisms between two objects of $Hom_{ex}(\mathcal{A}, \mathcal{B})$ do not form a set in general.

We shall prove the proposition in B.2 - B.5. The following examples are based on remark B.6.

EXAMPLES. a) For an additive category $\mathcal{B}$ endowed with the split conflations, $\mathcal{B}^\sim$ is equivalent to the category $\mathcal{B}^+$ of 2.5. Moreover, we have

$$(\mathcal{C}^b_{0]}\mathcal{B})^\sim \xrightarrow{\sim} (\mathcal{C}_{0]}\mathcal{B})^\sim \xrightarrow{\sim} \mathcal{C}_{0]}\mathcal{B}^\sim \, .$$

b) Let k be a field and $\mathcal{S}$ a locally finite-dimensional k-category [6]. Let $\mathrm{mod}_{pf}\,\mathcal{S}$ be the category of 'pointwise finite modules', i.e. of k-linear functors $M : \mathcal{S}^{op} \to \mathrm{mod}\, k$ such that $\dim Mx < \infty$ for each $x \in \mathcal{S}$. The countable envelope $(\mathrm{mod}_{pf}\,\mathcal{S})^{\sim}$ identifies with the category of 'pointwise countable' modules M, i.e. k-linear functors $M : \mathcal{S}^{op} \to \mathrm{mod}\, k$ such that Mx has countable dimension for each $x \in \mathcal{S}$ (compare section 3).

B.2 We want to prove B.1 a). In a first step, we assume that $\mathcal{A}$ is svelte. The functor

$$L : \mathcal{F}\mathcal{A} \to \mathrm{Sex}\,\mathcal{A}\,,\ X \mapsto \varinjlim \mathcal{A}\,(?, X^p)$$

obviously induces a full embedding of $\mathcal{A}^{\sim}$ to $\mathrm{Sex}\,\mathcal{A}$ (A.2) and for each conflation (j, e) of $\mathcal{F}\mathcal{A}$, the induced sequence

$$0 \to LX \xrightarrow{Lj} LY \xrightarrow{Le} LZ \to 0$$

is exact. We shall show that an arbitrary short exact sequence

$$0 \to LX \xrightarrow{i} F \xrightarrow{d} LZ \to 0$$

of $\mathrm{Sex}\,\mathcal{A}$ is isomorphic to the image of a conflation of $\mathcal{F}\mathcal{A}$. Since $\mathrm{Sex}\,\mathcal{A}$ is abelian, it will then follow that $(\mathcal{A}^{\sim}, \mathcal{E}^{\sim})$ is an exact category.

By base change along the canonical morphism $\mathcal{A}\,(?, Z^r) \to LZ$, we obtain a short exact sequence

$$0 \to LX \to F^r \xrightarrow{d^r} \mathcal{A}\,(?, Z^r) \to 0$$

from (i, d) for each $r \in \mathbf{N}$. Since the cokernel of d^r formed in $\mathrm{Mod}\,\mathcal{A}$ is effaceable, there is a deflation $q^r : Q^r \to Z^r$ such that $\mathcal{A}\,(?, q^r) = d^r a^r$ for some $a^r : \mathcal{A}\,(?, Q^r) \to F^r$. Now let $P \in \mathcal{F}\mathcal{A}$ be the sequence of canonical morphisms

$$\coprod_{r \le p} Q^r \to \coprod_{r \le p+1} Q^r\,,\ p \in \mathbf{N}$$

and let $f \in \mathcal{F}\mathcal{A}\,(P, Z)$ be given by the components

$$f^p = [i_Z^{p-1}\, i_Z^{p-2} \ldots i_Z^0\, q^0, i_Z^{p-1} \ldots i_Z^1\, q^1, \ldots, q^p] : \coprod_{r \le p} Q^r \to Z^p\,,\ p \in \mathbf{N}.$$

By construction, f is a deflation of $\mathcal{F}\mathcal{A}$ whose image in $\mathrm{Sex}\,\mathcal{A}$ factors through d. Thus there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to LZ' & \xrightarrow{Lk} & LP & \xrightarrow{Lf} & LZ \to 0 \\ h \downarrow & & \downarrow & & \| \\ 0 \to LX & \xrightarrow{i} & F & \xrightarrow{d} & LZ \to 0 \end{array}$$

in $\mathrm{Sex}\,\mathcal{A}$, where k is a kernel of f in $\mathcal{F}\mathcal{A}$. The morphism h is given by a family

$$h^p \in \mathcal{A}\,(Z'^p, X^{\mu(p)})\,,\ \mu : \mathbf{N} \to \mathbf{N} \text{ some function,}$$

such that $h^{p+1}\, i^p_{Z'}$ and h^p have the same image in

$$\varinjlim_q \mathcal{A}\,(Z'^p, X^q)\,,\ \forall\, p.$$

Obviously, we may assume $\mu\,(p{+}1) > \mu\,(p), \forall\, p$. Then we can write h as $(Lc_\mu)^{-1}\, Lh'$, where $c_\mu : X \to X(\mu)$ has the components

$$i_X^{\mu\,(p),\,p}\ \ (i_X^{s,s} = 1_{X^s} \text{ and } i_X^{r,s} = i_X^{r-1}\, i_X^{r-2} \dots i_X^{s} \text{ for } r > s),$$

$X(\mu)$ is the sequence of inflations $i_X^{\mu\,(p+1),\,\mu\,(p)}$ and $h' \in \mathcal{FA}\,(Z', X(\mu))$ has the components h^p. By cobase change along h', we obtain from (k, f) the required conflation (j, e) whose image in $\operatorname{Sex}\mathcal{A}$ is isomorphic to (i, e)

$$\begin{array}{ccccccc} 0 \to & LX(\mu) & \overset{Lj}{\to} & LY & \overset{Le}{\to} & LZ & \to 0 \\ & Lc_\mu \uparrow & & \uparrow s & & \| & \\ 0 \to & LX & \overset{i}{\to} & F & \overset{d}{\to} & LZ & \to 0. \end{array}$$

This argument also shows that, for each inflation i of $\mathcal{A}^\sim$, there is an isomorphism s such that si is the image of a morphism j of $\mathcal{FA}$ whose components are inflations of $\mathcal{A}$. Thus, up to isomorphism, a sequence $X \in \mathcal{FA}^\sim$ is given by a sequence of morphisms $j^p \in \mathcal{FA}\,(X^p, X^{p+1})$ such that

$$j^{p,q} : X^{p,q} \to X^{p+1,\,q}$$

is an inflation for all p, q. The 'diagonal sequence' Y with

$$i_Y^p = i^p_{X^{p+1}} j^{p,p} : X^{p,p} \to X^{p+1,p+1}$$

supplies a direct limit of the X^p since, by definition of the morphisms of $\mathcal{A}^\sim$, we have

$$Y \xleftarrow{\sim} \varinjlim EX^{p,p} \xrightarrow{\sim} \varinjlim_p \varinjlim_q EX^{p,q} \xrightarrow{\sim} \varinjlim X^p.$$

We see that the embedding $\mathcal{A}^\sim \to \operatorname{Sex}\mathcal{A}$ commutes with forming direct limits of sequences of inflations. Since $\operatorname{Sex}\mathcal{A}$ has exact direct limits, it follows that $\varinjlim : \mathcal{FA} \to \mathcal{A}^\sim$ is exact.

Now let $(\mathcal{A}, \mathcal{E})$ be an arbitrary exact category. We want to show with the aid of lemma A.3 that $(\mathcal{A}^\sim, \mathcal{E}^\sim)$ is an exact category. Let $\mathcal{U} \subset \mathcal{A}^\sim$ be a svelte subcategory. Without restriction of generality, we may assume that $\mathcal{U}$ is even small. Let the full subcategory $\mathcal{U}' \subset \mathcal{A}^\sim$ contain $\mathcal{U}$ and, additionally, the terms of a conflation (j, e) of $\mathcal{FA}$ whose image in $\mathcal{A}^\sim$ is isomorphic to (i, d), for each conflation $(i, d) \in \mathcal{U} \cap \mathcal{E}^\sim$. Let $\mathcal{V} \subset \mathcal{A}$ be the full subcategory of the components X^p of sequences $X \in \mathcal{U}'$. By A.3 there is a svelte full exact subcategory $(\mathcal{B}, \mathcal{F})$ of $\mathcal{A}$ such that $\mathcal{V} \subset \mathcal{B}$ and $\mathcal{V} \cap \mathcal{E} \subset \mathcal{F}$. It is clear that $\mathcal{B}^\sim$, which we identify with a full subcategory of $\mathcal{A}^\sim$, contains $\mathcal{U}$ and that $\mathcal{F}^\sim$ contains the class $\mathcal{U} \cap \mathcal{E}^\sim$.

This construction also shows that, up to isomorphism, each conflation

$$X \xrightarrow{i} Y \xrightarrow{d} Z$$

of $\mathcal{F}\mathcal{A}^{\sim}$ lies in a subcategory $\mathcal{F}\mathcal{B}^{\sim}$ for some svelte full exact subcategory $\mathcal{B} \subset \mathcal{A}$. The above construction of $\varinjlim$ shows that the inclusion $\mathcal{B}^{\sim} \subset \mathcal{A}^{\sim}$ commutes with $\varinjlim$. We conclude that $\varinjlim : \mathcal{F}\mathcal{A}^{\sim} \to \mathcal{A}^{\sim}$ is well defined and exact.

REMARK. a) From the above construction of $\varinjlim : \mathcal{F}\mathcal{A}^{\sim} \to \mathcal{A}^{\sim}$, we conclude that for an exact functor $G : \mathcal{A}^{\sim} \to \mathcal{B}$ to a countably complete exact category $\mathcal{B}$, we have $\varinjlim GX^p \xrightarrow{\sim} G \varinjlim X^p$, $\forall\, X \in \mathcal{F}\mathcal{A}^{\sim}$ iff we have $G \varinjlim EX^p \xrightarrow{\sim} GX$, $\forall\, X \in \mathcal{A}^{\sim}$. In particular, we have

$$\mathcal{A}^{\sim}(EA, \varinjlim X^p) \xrightarrow{\sim} \varinjlim \mathcal{A}^{\sim}(EA, X^p)$$

for each $A \in \mathcal{A}$ and each $X \in \mathcal{F}\mathcal{A}^{\sim}$.

b) It is easy to check that the maps $f \mapsto (Lc_\mu)^{-1}\, Lf$ induce a bijection

$$\varinjlim \mathcal{F}\mathcal{A}(X, Y(\mu)) \to \mathcal{A}^{\sim}(X, Y).$$

Here μ runs through the partially ordered set of functions $\mu : \mathbf{N} \to \mathbf{N}$ with $\mu(p+1) > \mu(p)$, $\forall\, p$.

B.3 We want to prove B.1 b). We exhibit a fully faithful left adjoint of E^*. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. There is a unique functor $F^{\sim}$ which makes the square

$$\begin{array}{ccc} \mathcal{F}\mathcal{A} & \xrightarrow{\mathcal{F}F} & \mathcal{F}\mathcal{B} \\ L \downarrow & & \downarrow \varinjlim \\ \mathcal{A}^{\sim} & \xrightarrow{F^{\sim}} & \mathcal{B} \end{array}$$

commutative. $F^{\sim}$ is exact, we have

$$F^{\sim}EA = \varinjlim (FA = FA = \ldots) \xrightarrow{\sim} FA\,,\ \forall\, A \in \mathcal{A}$$

and, by remark B.2 a), we have $\varinjlim F^{\sim}X^p \xrightarrow{\sim} F^{\sim} \varinjlim X^p$, $\forall\, X \in \mathcal{F}\mathcal{A}^{\sim}$. On the other hand, if $\varinjlim GX^p \xrightarrow{\sim} G \varinjlim X^p$ holds for all $X \in \mathcal{F}\mathcal{A}^{\sim}$ for some exact functor $G : \mathcal{A}^{\sim} \to \mathcal{B}$, we have

$$(GE)^{\sim}X = \varinjlim \mathcal{F}(GE)X = \varinjlim GEX^p \xrightarrow{\sim} G \varinjlim EX^p \xrightarrow{\sim} GX.$$

B.4 We want to prove B.1 c). For a projective P of $\mathcal{A}$, the functor

$$\mathcal{A}^{\sim}(EP, Y) = \varinjlim \mathcal{A}(P, Y^q)$$

is exact in Y. If I is injective in $\mathcal{A}$, then EI is injective in $\mathcal{F}\mathcal{A}$ (5.1) and

$$\mathcal{A}^{\sim}(X, EI) \xleftarrow{\sim} \varinjlim \mathcal{F}\mathcal{A}(X, (EI)(\mu)) = \mathcal{F}\mathcal{A}(X, EI)$$

is an exact functor of X. That E induces bijections of the extension groups follows from the

LEMMA. *For $X, Y \in \mathcal{A}^{\sim}$ and $n \in \mathbf{N}$, we have*

$$\varinjlim \operatorname{Ext}^n_{\mathcal{FA}}(X, Y(\mu)) \xrightarrow{\sim} \operatorname{Ext}^n_{\mathcal{A}^{\sim}}(X, Y).$$

PROOF. Let $F_n X$ be the left hand side of the above isomorphism. By remark B.2 b), we have $F_0 X \xrightarrow{\sim} \mathcal{A}^{\sim}(X, Y)$. Since, for $n \geq 0$, the F_n form an exact ∂-functor which is effaceable for $n > 0$, the assertion follows from [8, 2.2.1].

B.5 We want to prove B.1 d). If $(I_n)_{n \in \mathrm{N}}$ is a family of injectives of $\mathcal{A}^{\sim}$, we have

$$\mathcal{A}^{\sim}(X, \coprod_{n \in \mathrm{N}} I_n) \xleftarrow{\sim} \varprojlim \mathcal{A}^{\sim}(EX^p, \coprod_{n \in \mathrm{N}} I_n) \xleftarrow{\sim} \varprojlim \coprod_{n \in \mathrm{N}} \mathcal{A}^{\sim}(EX^p, I_n)$$

because of $\varinjlim EX^p \xrightarrow{\sim} X$ and remark B.2 a). Since the transition maps

$$\coprod \mathcal{A}^{\sim}(EX^p, I_n) \leftarrow \coprod \mathcal{A}^{\sim}(EX^p, I_n)$$

are surjective, $\mathcal{A}^{\sim}(?, \coprod I_n)$ is an exact functor by the Mittag-Leffler criterion [9, 0_{III}, 13.1].

If $\mathcal{A}$ has enough projectives, it is clear from the conflations (example 5.3)

$$\coprod EX^p \xrightarrow{\iota} \coprod EX^q \xrightarrow{\varepsilon} X \ , \ X \in \mathcal{A}^{\sim}$$

that the class of countable sums of projectives EP, P projective in $\mathcal{A}$, contains enough projectives for $\mathcal{A}^{\sim}$.

If $\mathcal{A}$ has enough injectives, then, for each $X \in \mathcal{A}^{\sim}$, there is a conflation

$$X' \xrightarrow{j} J \xrightarrow{e} X$$

in $\mathcal{FA}$ such that J has injective components (5.1). In $\mathcal{A}^{\sim}$, J is isomorphic to a sum of objects EI, I an injective of $\mathcal{A}$. This implies the last assertion of B.1 d).

B.6 LEMMA. *In the situation of B.1 b), an exact functor $G : \mathcal{A}^{\sim} \to \mathcal{B}$ with $\varinjlim GX^p \xrightarrow{\sim} G \varinjlim X^p$, $\forall\, X \in \mathcal{FA}^{\sim}$ induces bijections*

$$\mathcal{A}^{\sim}(X, Y) \to \mathcal{B}(GX, GY) \ \textit{and} \ \operatorname{Ext}^1_{\mathcal{A}^{\sim}}(X, Y) \to \operatorname{Ext}^1_{\mathcal{B}}(GX, GY),$$

for all $X, Y \in \mathcal{A}^{\sim}$ iff the restriction $F = GE$ gives rise to bijections

$$\coprod \mathcal{A}(A, B_n) \to \mathcal{B}(FA, \coprod FB_n) \ \textit{and} \ \coprod \operatorname{Ext}^1_{\mathcal{A}}(A, B_n) \to \operatorname{Ext}^1_{\mathcal{B}}(FA, \coprod FB_n)$$

for all $A \in \mathcal{A}$ and all families $(B_n)_{n \in \mathrm{N}}$ in $\mathcal{A}$.

Remark. If $\mathcal{A}$ has enough projectives, the condition of the proposition is satisfied if F preserves projectives and the map

$$\coprod \mathcal{A}(P, Q_n) \to \mathcal{B}(FP, \coprod FQ_n)$$

is bijective for all projectives P and all families of projectives $(Q_n)_{n \in \mathrm{N}}$ in $\mathcal{A}$. If, moreover, the class of countable sums of objects FP, P a projective of $\mathcal{A}$, contains enough projectives for $\mathcal{B}$, then G is an equivalence. This implies the assertions of examples B.1 a) and b).

Proof. By lemma B.4, the condition is necessary. Suppose it is satisfied and let $Y \in \mathcal{A}^{\sim}$. We consider the conflation (example 5.3)

$$\coprod FY^p \to \coprod FY^q \to GY.$$

From the corresponding long exact sequence, we conclude that the maps

$$\varinjlim \mathrm{Ext}^n_{\mathcal{B}}(FA, FY^q) \to \mathrm{Ext}^n_{\mathcal{B}}(FA, GY)$$

are bijective for $n = 0, 1$. Consequently, G induces bijections

$$\mathrm{Ext}^n_{\mathcal{A}^{\sim}}(EA, Y) \to \mathrm{Ext}^n_{\mathcal{B}}(FA, GY)\ ,\ n = 0, 1.$$

We now consider the corresponding conflation

$$\coprod EX^p \xrightarrow{j} \coprod EX^q \xrightarrow{e} X$$

for $X \in \mathcal{A}^{\sim}$. Since $\mathcal{B}$ has exact countable sums, we have

$$\mathrm{Ext}^1_{\mathcal{B}}(\coprod FX^p, GY) \xrightarrow{\sim} \prod \mathrm{Ext}^1_{\mathcal{B}}(FX^p, GY).$$

By the 5-lemma, the assertion now follows from the long exact sequences associated with (j, e) and (Gj, Ge).

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