Commun. Korean Math. Soc. **21** (2006), No. 1, pp. 75–88

CHANGE OF SCALE FORMULAS FOR WIENER INTEGRAL OVER PATHS IN ABSTRACT WIENER SPACE

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ABSTRACT. Wiener measure and Wiener measurability behave badly under the change of scale transformation. We express the analytic Feynman integral over $C_0(B)$ as a limit of Wiener integrals over $C_0(B)$ and establish change of scale formulas for Wiener integrals over $C_0(B)$ for some functionals.

1. Introduction

Let (H, B, m) be an abstract Wiener space. Let $C_0(B) \equiv C_0([0,T], B)$ denote the set of abstract Wiener space valued continuous functions on $[0,T]$ which vanish at origin. From [10] it follows that $C_0(B)$ is a real separable Banach space with the norm

$$\|x\|_{C_0(B)} = \sup_{s\in[0,T]} \|x(s)\|_B$$

and the minimal σ-algebra making the mapping $x \to x(s)$ measurable consists of the Borel subsets of $C_0(B)$. Moreover the Brownian motion in B induces a probability measure m_B on $(C_0(B), \mathcal{B}(C_0(B)))$ which is mean-zero Gaussian.

The concept of the analytic Feynman integral has so far been defined on classical Wiener space $C_0[0,T]$ or abstract Wiener space B. But Yoo [12] defined the analytic Feynman integral over $C_0(B)$ and proved the existence theorems of this integral for functionals in the classes $\mathcal{S}''_{n,B}$ and $\mathcal{S}''_B$ which correspond to the classes $\mathcal{S}''_n$ and $\mathcal{S}''$ of functionals on classical Wiener space introduced by Cameron and Storvick [3, 4], respectively.

Received May 25, 2005.
2000 Mathematics Subject Classification: 28C20.
Key words and phrases: Wiener integral, Feynman integral, change of scale formula for Wiener integral.

Moreover, in [13, 14], Yoo, Lee and Kim continued to study the analytic Feynman integral over $C_0(B)$ and extended the results in [12].

On the other hand, it has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [2] and under translation [1]. However Cameron and Storvick [5] expressed the analytic Feynman integral for a rather large class of functionals as a limit of Wiener integrals. In doing so, they discovered nice change of scale formulas for Wiener integrals on classical Wiener space $(C_0[0,1], m_w)$ [6]. In [9, 15, 16, 18], Kim, Skoug, Yoo and Yoon extended these results to classical Yeh-Wiener space and to abstract Wiener space (H, B, m).

Recently Yoo, Song, Kim and Chang established [17] a change of scale formula for Wiener integrals of functions on abstract Wiener space which need not be bounded or continuous.

In this paper, we investigate on the relationship between the Wiener integral and the analytic Feynman integral for some functionals defined on $C_0(B)$. (See Theorems 2.4 and 3.2.) We also express the analytic Wiener integral as the limit of a sequence of Wiener integrals over $C_0(B)$ in Corollaries 2.5 and 3.3. Finally, in Corollaries 2.6 and 3.4, we establish change of scale formulas for Wiener integral over $C_0(B)$ of these functionals.

We turn now to introducing the definition of analytic Feynman integral over $C_0(B)$. We begin with introducing a concrete form of a Borel measure m_B on $C_0(B)$ [11].

Let $\vec{s} = (s_1, \ldots, s_n)$ be given with $0 = s_0 < s_1 < \cdots < s_n \leq T$ and let $T_{\vec{s}} : B^n \to B^n$ be defined by

$$T_{\vec{s}}(y_1, \ldots, y_n) = (\sqrt{s_1 - s_0}\,y_1, \ldots, \sqrt{s_k - s_{k-1}}\,y_k).$$

Define a Borel measure $\mu_{\vec{s}}$ on $\mathcal{B}(B^n)$ by $\mu_{\vec{s}}(E) = (\times_1^n m)(T_{\vec{s}}^{-1}(E))$ for every $E \in \mathcal{B}(B^n)$. Let $J_{\vec{s}} : C_0(B) \to B^n$ be the function defined by

$$J_{\vec{s}}(y) = (y(s_1), \ldots, y(s_n)).$$

For Borel subsets $E_1, \ldots, E_n$ of B, $J_{\vec{s}}^{-1}(\times_{k=1}^n E_k)$ is called I-set and then the collection $\mathcal{I}$ of all such I-sets is a semi-algebra. We define a set function m_B on $\mathcal{I}$ by

$$m_B(J_{\vec{s}}^{-1}(\times_{k=1}^n E_k)) = \mu_{\vec{s}}(\times_{k=1}^n E_k).$$

Then m_B is well-defined and countably additive on $\mathcal{I}$. Using the Caratheodory extension process, we have a Borel measure m_B on $C_0(B)$.

Now we introduce an integration formula over $C_0(B)$. This formula is easily obtained by the change of variable theorem [11].

THEOREM 1.1. *Let $\vec{s} = (s_1, \dots, s_n)$ be given with $0 = s_0 < s_1 < \cdots < s_n \le T$ and let $f : B^n \to \mathbb{C}$ be a Borel measurable function. Then*

$$\begin{aligned}(1.1)\qquad &\int_{C_0(B)} f(x(s_1), \dots, x(s_n))\, dm_B(x) \\ &\overset{*}{=} \int_{B^n} (f \circ T_{\vec{s}})(y_1, \dots, y_n)\, d(\times_1^n m)(y_1, \dots, y_n),\end{aligned}$$

where by $\overset{*}{=}$ we mean that if either side exists, then both sides exist and they are equal.

A subset E of $C_0(B)$ is said to be scale-invariant measurable provided αE is measurable for each $\alpha > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m_B(\alpha N) = 0$ for each $\alpha > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.).

Let $\mathbb{C}, \mathbb{C}_+$ and $\mathbb{C}_+^{\sim}$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part respectively.

Let F be a $\mathbb{C}$-valued measurable functional on $C_0(B)$ such that

$$J_F(\lambda) = \int_{C_0(B)} F(\lambda^{-1/2} x)\, dm_B(x)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J_F^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$, then $J_F^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0(B)$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0(B)}^{\text{anw}_\lambda} F(x)\, dm_B(x) = J_F^*(\lambda).$$

If the following limit exists for nonzero real q, then we call it the analytic Feynman integral of F over $C_0(B)$ with parameter q and we write

$$\int_{C_0(B)}^{\text{anf}_q} F(x)\, dm_B(x) = \lim_{\lambda \to -iq} \int_{C_0(B)}^{\text{anw}_\lambda} F(x)\, dm_B(x),$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

2. A change of scale formula for Wiener integral of functionals in $\mathcal{S}''_{n,B}$

We begin with this section by introducing the class of functionals that we work in this section.

Let $\Delta_n = \{(s_1,\dots,s_n) \in [0,T]^n : 0 = s_0 < s_1 < \cdots < s_n \le T\}$. Let $\mathcal{M}''_n = \mathcal{M}''_n(\Delta_n \times H^n)$ be the class of complex Borel measures on $\Delta_n \times H^n$ and let $\|\mu\| = \operatorname{var}\mu$, the total variation of $\mu \in \mathcal{M}''_n$.

Let $\{e_n\}$ be a complete orthonormal set in H such that the e_n's are in B^*, the dual of B. For each $h \in H$ and $y \in B$, a stochastic inner product $(\cdot,\cdot)^\sim$ on $H \times B$ is defined by

$$(h,y)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \langle h, e_j\rangle (y, e_j), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $(\cdot,\cdot)^\sim$ is a Borel measurable functional on $H \times B$, and if both h and y are in H, then Parseval's identity gives $(h,y)^\sim = \langle h,y\rangle$. Moreover, $y \to (h,y)^\sim$ is Gaussian with mean zero and variance $|h|^2$ [7, 8], that is,

$$\int_B \exp\{i(h,y)^\sim\}\, dm(y) = \exp\Big\{-\frac{1}{2}|h|^2\Big\}.$$

DEFINITION 2.1. Let $\mathcal{S}''_{n,B} = \mathcal{S}''_{n,B}(\Delta_n \times H^n)$ be the space of functionals of the form

$$F(x) = \int_{\Delta_n \times H^n} \exp\Big\{ i \sum_{k=1}^{n} (h_k, x(s_k))^\sim \Big\}\, d\mu(\vec{s},\vec{h}) \tag{2.1}$$

for s-a.e $x \in C_0(B)$ where $\mu \in \mathcal{M}''_n$. Here we take $\|F\|''_n = \inf\{\|\mu\|\}$, where the infimum is taken over all μ's so that F and μ are related by (2.1).

Yoo [12] proved the following existence theorem of the analytic Feynman integral over $C_0(B)$ for functionals in $\mathcal{S}''_{n,B}$.

THEOREM 2.2. *Let $F \in \mathcal{S}''_{n,B}$ be given by (2.1). Then F is analytic Feynman integrable over $C_0(B)$ and if q is a non-zero real number,*

$$\begin{aligned} &\int_{C_0(B)}^{\mathrm{anf}_q} F(x)\, dm_B(x) \\ &= \int_{\Delta_n \times H^n} \exp\Big\{ \frac{1}{2iq} \sum_{k=1}^{n} (s_k - s_{k-1}) \Big| \sum_{j=k}^{n} h_j \Big|^2 \Big\}\, d\mu(\vec{s},\vec{h}) \end{aligned} \tag{2.2}$$

for s-a.e. $y \in C_0(B)$. In particular if $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} &\int_{C_0(B)}^{\mathrm{anw}_\lambda} F(x)\, dm_B(x) \\ &= \int_{\Delta_n \times H^n} \exp\Big\{ -\frac{1}{2\lambda} \sum_{k=1}^{n} (s_k - s_{k-1}) \Big| \sum_{j=k}^{n} h_j \Big|^2 \Big\}\, d\mu(\vec{s},\vec{h}) \end{aligned} \tag{2.3}$$

for s-a.e. $y \in C_0(B)$.

The following lemma is easy to prove and we shall omit the proof. It also can be obtained from Lemma 3.7 of [17] by taking $t_k = 0$ for $k = 1, \ldots, l$.

LEMMA 2.3. *Let* λ *be a complex number with* $\operatorname{Re}\lambda > 0$, *let* $\{e_1, \ldots, e_l\}$ *be an orthonormal set in* H *and let* $h \in H$. *Then*

$$\tag{2.4} \begin{aligned} &\int_B \exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{l}[(e_j, y)^\sim]^2 + i(h,y)^\sim\Big\}\,dm(y) \\ =&\lambda^{-l/2}\exp\Big\{\frac{\lambda-1}{2\lambda}\sum_{j=1}^{l}\langle e_j, h\rangle^2 - \frac{1}{2}|h|^2\Big\}. \end{aligned}$$

Now we give a relationship between Wiener integral and analytic Feynman integral over $C_0(B)$ for functionals in $\mathcal{S}''_{n,B}$, that is, we express the analytic Feynman integral over $C_0(B)$ of F as the limit of a sequence of Wiener integrals over $C_0(B)$.

THEOREM 2.4. *Let* $F \in \mathcal{S}''_{n,B}$ *be given by* (2.1). *Let* $\{e_l\}$ *be a complete orthonormal set in* H. *Let* $\{\lambda_l\}$ *be a sequence of complex numbers with* $\operatorname{Re}\lambda_l > 0$ *such that* $\lambda_l \to -iq$, $q \neq 0$. *Then*

$$\tag{2.5} \int_{C_0(B)}^{\mathrm{anf}_q} F(x)\,dm_B(x) = \lim_{l\to\infty}\lambda_l^{ln/2}\int_{C_0(B)} F_{\lambda_l}(x)\,dm_B(x),$$

where

$$\tag{2.6} \begin{aligned} F_{\lambda_l}(x) = \int_{\Delta_n\times H^n} &\exp\Big\{\frac{1-\lambda_l}{2}\sum_{j=1}^{l}\sum_{k=1}^{n}\frac{[(e_j, x(s_k)-x(s_{k-1}))^\sim]^2}{s_k - s_{k-1}} \\ &+ i\sum_{k=1}^{n}(h_k, x(s_k))^\sim\Big\}\,d\mu(\vec{s}, \vec{h}). \end{aligned}$$

PROOF. Let $K_{\lambda_l}(x, \vec{s}, \vec{h})$ be the integrand in the expression (2.6). By Theorem 1.1,

$$\begin{aligned} &\int_{\Delta_n\times H^n}\int_{C_0(B)} |K_{\lambda_l}(x, \vec{s}, \vec{h})|\,dm_B(x)\,d\mu(\vec{s}, \vec{h}) \\ =&\int_{\Delta_n\times H^n}\int_{B^n}\exp\Big\{\frac{1-\operatorname{Re}\lambda_l}{2}\sum_{j=1}^{l}\sum_{k=1}^{n}[(e_j, y_k)^\sim]^2\Big\}\,d(\times_1^n m)(\vec{y})\,d\mu(\vec{s}, \vec{h}), \end{aligned}$$

where $\vec{y} = (y_1, \dots, y_n)$. But, by Wiener integration formula, the last expression is equal to

$$(2\pi)^{-ln/2} \int_{\Delta_n \times H^n} \int_{\mathbb{R}^{ln}} \exp\Big\{-\frac{\operatorname{Re}\lambda_l}{2} \sum_{j=1}^{l} \sum_{k=1}^{n} u_{j,k}^2 \Big\}\, d\vec{u}\, d\mu(\vec{s}, \vec{h})$$

which is finite since $\operatorname{Re}\lambda_l > 0$. Hence we apply Fubini theorem to obtain

$$\int_{C_0(B)} F_{\lambda_l}(x)\, dm_B(x) = \int_{\Delta_n \times H^n} \int_{C_0(B)} K_{\lambda_l}(x, \vec{s}, \vec{h})\, dm_B(x)\, d\mu(\vec{s}, \vec{h}).$$

Now we evaluate the Wiener integral over $C_0(B)$ on the right hand side of the last expression. Theorem 1.1 and a simple calculation show that

$$\begin{aligned}\int_{C_0(B)} K_{\lambda_l}(x, \vec{s}, \vec{h})\, dm_B(x) = \prod_{k=1}^{n} \int_B \exp\Big\{&\frac{1-\lambda_l}{2} \sum_{j=1}^{l} [(e_j, y_k)^{\sim}]^2 \\ &+ i \sum_{p=k}^{n} \sqrt{s_k - s_{k-1}}(h_p, y_k)^{\sim} \Big\}\, dm(y_k).\end{aligned}$$

Applying Lemma 2.3, we see that the single Wiener integral on the right hand side of the last expression is equal to

$$\lambda_l^{-l/2} \exp\Big\{\frac{\lambda_l - 1}{2\lambda_l} \sum_{j=1}^{l} (s_k - s_{k-1}) \langle e_j, \sum_{p=k}^{n} h_p \rangle^2 - \frac{1}{2}(s_k - s_{k-1}) \Big|\sum_{p=k}^{n} h_p\Big|^2 \Big\}$$

and so

$$\begin{aligned}&\int_{C_0(B)} K_{\lambda_l}(x, \vec{s}, \vec{h})\, dm_B(x) \\ &= \lambda_l^{-ln/2} \exp\Big\{\frac{\lambda_l - 1}{2\lambda_l} \sum_{k=1}^{n} \sum_{j=1}^{l} (s_k - s_{k-1}) \langle e_j, \sum_{p=k}^{n} h_p \rangle^2 \\ &\quad - \frac{1}{2} \sum_{k=1}^{n} (s_k - s_{k-1}) \Big|\sum_{p=k}^{n} h_p\Big|^2 \Big\}.\end{aligned} \tag{2.7}$$

Now, by the Parseval theorem,

$$\sum_{j=1}^{l} \langle e_j, \sum_{p=k}^{n} h_p \rangle^2 \to \Big|\sum_{p=k}^{n} h_p\Big|^2 \tag{2.8}$$

as $l \to \infty$, and so for sufficiently large l, the exponential function in (2.7) is bounded by 1. Moreover the constant function 1 is integrable with respect to μ. Hence by the dominated convergence theorem and (2.8), we have

$$\lim_{l\to\infty} \lambda_l^{ln/2} \int_{C_0(B)} F_{\lambda_l}(x)\, dm_B(x)$$
$$= \int_{\Delta_n \times H^n} \exp\Big\{ \frac{1}{2iq} \sum_{k=1}^{n} (s_k - s_{k-1}) \Big| \sum_{p=k}^{n} h_p \Big|^2 \Big\}\, d\mu(\vec{s}, \vec{h}).$$

By (2.2) in Theorem 2.2, the proof is completed. □

In our next corollary, we express the analytic Wiener integral over $C_0(B)$ of $F \in \mathcal{S}''_{n,B}$ as the limit of a sequence of Wiener integrals over $C_0(B)$.

COROLLARY 2.5. *Let F and $\{e_l\}$ be given as in Theorem 2.4. Let λ be a complex number with* $\operatorname{Re}\lambda > 0$. *Then we have*

$$\int_{C_0(B)}^{\mathrm{anw}_\lambda} F(x)\, dm_B(x) = \lim_{l\to\infty} \lambda^{ln/2} \int_{C_0(B)} F_\lambda(x)\, dm_B(x), \tag{2.9}$$

where $F_\lambda(x)$ is given by (2.6) with λ_l replaced by λ.

PROOF. To prove this corollary, we modify the proof of Theorem 2.4 by replacing "λ_l" by "λ" whenever it occurs and by replacing "$2iq$" by "-2λ" in the last step of the proof of Theorem 2.4. Finally by (2.3) in Theorem 2.2, we obtain the result. □

We are now ready to state a change of scale formula for Wiener integral over $C_0(B)$ for $F \in \mathcal{S}''_{n,B}$. The following corollary is easily obtained from Corollary 2.5.

COROLLARY 2.6. *Let F and $\{e_l\}$ be given as in Theorem 2.4 and let $\rho > 0$. Then we have*

$$\int_{C_0(B)} F(\rho x)\, dm_B(x) = \lim_{l\to\infty} \rho^{-ln} \int_{C_0(B)} F_{\rho^{-2}}(x)\, dm_B(x), \tag{2.10}$$

where $F_{\rho^{-2}}(x)$ is given by (2.6) with λ_l replaced by ρ^{-2}.

PROOF. By letting $\lambda = \rho^{-2}$ in (2.9), we have the result. □

Next we give an example to show that the change of scale formula (2.10) for Wiener integral over $C_0(B)$ hold for larger classes of functionals than $\mathcal{S}''_{n.B}$.

EXAMPLE 2.7. Let $\{e_l\}$ be a complete orthonormal set in H and let F be a functional on $C_0(B)$ defined by

$$F(x)=\int_{[0,T]\times H}\exp\{\alpha(h,x(T))^{\sim}\}\,d\mu(s,h)$$

for nonzero $h\in H$, where α is a real or complex number. Then the left hand side of (2.10) becomes

$$\begin{aligned}\int_{C_0(B)}F(\rho x)\,dm_B(x)&=\int_{[0,T]\times H}\int_B\exp\{\alpha\rho\sqrt{T}(h,y)^{\sim}\}\,dm(y)\,d\mu(s,h)\\&=\int_{[0,T]\times H}\exp\Big\{\frac{(\alpha\rho)^2}{2}T|h|^2\Big\}\,d\mu(s,h)\end{aligned}$$

and $F_{\rho^{-2}}(x)$ is given by

$$F_{\rho^{-2}}(x)=\int_{[0,T]\times H}\exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{j=1}^{l}\frac{[(e_j,x(T))^{\sim}]^2}{T}+i(h,x(T))^{\sim}\Big\}\,d\mu(s,h)$$

Next, we apply Theorem 1.1 and the techniques in the proofs of Theorem 3.1 and Lemma 3.6 of [17] to obtain

$$\begin{aligned}&\int_{C_0(B)}F_{\rho^{-2}}(x)\,dm_B(x)\\&=\int_{[0,T]\times H}\int_B\exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{j=1}^{l}[(e_j,y)^{\sim}]^2+\alpha\sqrt{T}(h,y)^{\sim}\Big\}\,dm(y)\,d\mu(s,h)\\&=\rho^l\int_{[0,T]\times H}\exp\Big\{\frac{(\alpha\rho)^2}{2}T\sum_{j=1}^{l}\langle e_j,h\rangle^2+\frac{\alpha^2T}{2}(|h|^2-\sum_{j=1}^{l}\langle e_j,h\rangle^2)\Big\}\,d\mu(s,h)\end{aligned}$$

Finally using the Parseval theorem, we obtain

$$\lim_{l\to\infty}\rho^{-l}\int_{C_0(B)}F_{\rho^{-2}}(x)\,dm_B(x)=\int_{[0,T]\times H}\exp\Big\{\frac{(\alpha\rho)^2}{2}T|h|^2\Big\}\,d\mu(s,h).$$

Thus we established that equation (2.10) is valid for all complex number α. Note that, if α is pure imaginary, $F\in\mathcal{S}''_{1,B}$. But if $\operatorname{Re}\alpha>0$, then F is an unbounded functional and so $F\notin\mathcal{S}''_{n,B}$ for any $n=1,2,\ldots$; recall that all functionals in $\mathcal{S}''_{n,B}$ are bounded. Hence we know that the change of scale formulas for Wiener integrals over $C_0(B)$ established in this section hold for larger classes of functionals than $\mathcal{S}''_{n.B}$.

3. A change of scale formula for Wiener integral of functionals $F(x) = G(x)\psi(x(T))$

Let $\mathcal{F}(B)$ be the class of functionals on B of the form

$$\psi(y) = \int_H \exp\{i(h,y)^\sim\}\, d\nu(h) \tag{3.1}$$

for $y \in B$ and $\nu \in \mathcal{M}(H)$, the class of complex Borel measures on H. We call $\mathcal{F}(B)$ as the Fresnel class.

In this section we discuss a change of scale formula for Wiener integral over $C_0(B)$ of functionals of the form $F(x) = G(x)\psi(x(T))$, where $G \in \mathcal{S}''_{n,B}$ is given by (2.1) and ψ is given by (3.1). We start with the existence theorem of the analytic Feynman integral for functionals we work in this section.

THEOREM 3.1. *Let $F(x) = G(x)\psi(x(T))$, where $G \in \mathcal{S}''_{n,B}$ is given by (2.1) and ψ is given by (3.1). Then F is analytic Feynman integrable over $C_0(B)$ and if q is a non-zero real number,*

(3.2)
$$\int_{C_0(B)}^{\mathrm{anf}_q} F(x)\, dm_B(x) = \int_{\Delta_n \times H^{n+1}} \exp\Big\{\frac{1}{2iq}\Big[\sum_{k=1}^{n}(s_k - s_{k-1})\Big|\sum_{j=k}^{n+1} h_j\Big|^2 + (T - s_n)|h_{n+1}|^2\Big]\Big\}\, d\mu_n(\vec{s}, (\vec{h}, h_{n+1}))$$

for s-a.e. $y \in C_0(B)$, where $d\mu_n(\vec{s}, (\vec{h}, h_{n+1})) = d\mu(\vec{s}, \vec{h})\, d\nu(h_{n+1})$. In particular, if $\lambda \in \mathbb{C}_+$

(3.3)
$$\int_{C_0(B)}^{\mathrm{anw}_\lambda} F(x)\, dm_B(x) = \int_{\Delta_n \times H^{n+1}} \exp\Big\{-\frac{1}{2\lambda}\Big[\sum_{k=1}^{n}(s_k - s_{k-1})\Big|\sum_{j=k}^{n+1} h_j\Big|^2 + (T - s_n)|h_{n+1}|^2\Big]\Big\}\, d\mu_n(\vec{s}, (\vec{h}, h_{n+1}))$$

for s-a.e. $y \in C_0(B)$.

PROOF. By (2.1) and (3.1), $F(x)$ is expressed as

$$F(x) = \int_{\Delta_n \times H^{n+1}} \exp\Big\{i\sum_{k=1}^{n+1}(h_k, x(s_k))^\sim\Big\}\, d\mu_n(\vec{s}, (\vec{h}, h_{n+1})), \tag{3.4}$$

where $s_{n+1} = T$ and $d\mu_n(\vec{s}, (\vec{h}, h_{n+1})) = d\mu(\vec{s}, \vec{h})\, d\nu(h_{n+1})$. From Fubini theorem and Theorem 1.1, it follows that for $\lambda > 0$,

$$\begin{aligned}
&\int_{C_0(B)} F(\lambda^{-1/2}x)\, dm_B(x) \\
(3.5)\quad &= \int_{\Delta_n \times H^{n+1}} \int_{B^{n+1}} \exp\Big\{ i\lambda^{-1/2} \sum_{k=1}^{n+1} \sum_{j=1}^{k} \sqrt{s_j - s_{j-1}} (h_k, y_j)^\sim \Big\} \\
&\quad d(\times_1^{n+1} m)(y_1, \ldots, y_{n+1})\, d\mu_n(\vec{s}, (\vec{h}, h_{n+1})).
\end{aligned}$$

Reordering the double summation in the last expression we see that

$$(3.6)\quad \sum_{k=1}^{n+1} \sum_{j=1}^{k} \sqrt{s_j - s_{j-1}} (h_k, y_j)^\sim = \sum_{k=1}^{n+1} \sum_{j=k}^{n+1} \sqrt{s_k - s_{k-1}} (h_j, y_k)^\sim .$$

Evaluating the Wiener integral in (3.5) together with (3.6), we have

$$\begin{aligned}
&\int_{C_0(B)} F(\lambda^{-1/2}x)\, dm_B(x) \\
&= \int_{\Delta_n \times H^{n+1}} \exp\Big\{ -\frac{1}{2\lambda} \sum_{k=1}^{n+1} (s_k - s_{k-1}) \Big| \sum_{j=k}^{n+1} h_j \Big|^2 \Big\}\, d\mu_n(\vec{s}, (\vec{h}, h_{n+1})).
\end{aligned}$$

But since $s_{n+1} = T$, we have

$$\begin{aligned}
&\int_{C_0(B)} F(\lambda^{-1/2}x)\, dm_B(x) \\
&= \int_{\Delta_n \times H^{n+1}} \exp\Big\{ -\frac{1}{2\lambda} \Big[\sum_{k=1}^{n} (s_k - s_{k-1}) \Big| \sum_{j=k}^{n+1} h_j \Big|^2 \\
&\quad + (T - s_n) |h_{n+1}|^2 \Big] \Big\}\, d\mu_n(\vec{s}, (\vec{h}, h_{n+1})).
\end{aligned}$$

The integrand in the last expression above is an analytic function of $\lambda \in \mathbb{C}_+$ and is bounded by the constant function 1. Moreover the constant function 1 is integrable with respect to μ_n. Using Cauchy theorem, Fubini theorem and Morera theorem, we have (3.3). Finally an application of the dominated convergence theorem enables us to pass the limit as $\lambda \to -iq$ and hence we obtain (3.2). □

The following theorem is a relationship between Wiener integral and analytic Feynman integral over $C_0(B)$ for functionals F of the form $F(x) = G(x)\psi(x(T))$.

THEOREM 3.2. *Let $F(x) = G(x)\psi(x(T))$, where $G \in \mathcal{S}''_{n,B}$ is given by (2.1) and ψ is given by (3.1). Let $\{e_l\}$ be a complete orthonormal set in H. Let $\{\lambda_l\}$ be a sequence of complex numbers with $\operatorname{Re}\lambda_l > 0$ such that $\lambda_l \to -iq$, $q \neq 0$. Then*

$$\int_{C_0(B)}^{\text{anf}_q} F(x)\, dm_B(x) = \lim_{l\to\infty} \lambda_l^{l(n+1)/2} \int_{C_0(B)} F_{\lambda_l}(x)\, dm_B(x), \tag{3.7}$$

where

$$\begin{aligned} F_{\lambda_l}(x) = \int_{\Delta_n \times H^n} \exp\Big\{ \frac{1-\lambda_l}{2} \sum_{j=1}^{l}\sum_{k=1}^{n+1} \frac{[(e_j, x(s_k) - x(s_{k-1}))^{\sim}]^2}{s_k - s_{k-1}} \\ + i \sum_{k=1}^{n} (h_k, x(s_k))^{\sim} \Big\}\, d\mu(\vec{s}, \vec{h}) \end{aligned} \tag{3.8}$$

with $s_{n+1} = T$.

PROOF. Let $K_{\lambda_l}(x, \vec{s}, \vec{h}, h_{n+1})$ be the integrand in the expression (3.8). By Theorem 1.1 and Wiener integration formula

$$\begin{aligned} &\int_{\Delta_n \times H^{n+1}} \int_{C_0(B)} |K_{\lambda_l}(x, \vec{s}, \vec{h}, h_{n+1})|\, dm_B(x)\, d\mu_n(\vec{s}, \vec{h}, h_{n+1}) \\ =&(2\pi)^{-l(n+1)/2} \int_{\Delta_n \times H^{n+1}} \int_{\mathbb{R}^{l(n+1)}} \exp\Big\{ -\frac{\operatorname{Re}\lambda_l}{2} \sum_{j=1}^{l}\sum_{k=1}^{n+1} u_{j,k}^2 \Big\} \\ &d\vec{u}\, d\mu_n(\vec{s}, \vec{h}, h_{n+1}) \end{aligned}$$

which is finite since $\operatorname{Re}\lambda_l > 0$. Hence we apply Fubini theorem to obtain

$$\begin{aligned} &\int_{C_0(B)} F_{\lambda_l}(x)\, dm_B(x) \\ =&\int_{\Delta_n \times H^{n+1}} \int_{C_0(B)} K_{\lambda_l}(x, \vec{s}, \vec{h}, h_{n+1})\, dm_B(x)\, d\mu_n(\vec{s}, \vec{h}, h_{n+1}). \end{aligned}$$

By the same method as in the proof of Theorem 2.4, we evaluate the Wiener integral over $C_0(B)$ on the right hand side of the last expression and obtain

$$\begin{aligned}(3.9)\qquad &\int_{C_0(B)} K_{\lambda_l}(x,\vec{s},\vec{h},h_{n+1})\,dm_B(x)\\ &=\lambda_l^{-l(n+1)/2}\exp\Big\{\frac{\lambda_l-1}{2\lambda_l}\sum_{k=1}^{n+1}\sum_{j=1}^{l}(s_k-s_{k-1})\langle e_j,\sum_{p=k}^{n+1}h_p\rangle^2\\ &\quad-\frac{1}{2}\sum_{k=1}^{n+1}(s_k-s_{k-1})\Big|\sum_{p=k}^{n+1}h_p\Big|^2\Big\}.\end{aligned}$$

Now, by the Parseval theorem,

$$(3.10)\qquad \sum_{j=1}^{l}\langle e_j,\sum_{p=k}^{n+1}h_p\rangle^2\to\Big|\sum_{p=k}^{n+1}h_p\Big|^2$$

as $l\to\infty$, and so for sufficiently large l, the exponential function in (3.9) is bounded by 1. Moreover the constant function 1 is integrable with respect to μ_n. Hence by the dominated convergence theorem and (3.10), we have

$$\begin{aligned}&\lim_{l\to\infty}\lambda_l^{l(n+1)/2}\int_{C_0(B)}F_{\lambda_l}(x)\,dm_B(x)\\ &=\int_{\Delta_n\times H^{n+1}}\exp\Big\{\frac{1}{2iq}\sum_{k=1}^{n+1}(s_k-s_{k-1})\Big|\sum_{p=k}^{n+1}h_p\Big|^2\Big\}\,d\mu_n(\vec{s},(\vec{h},h_{n+1})).\end{aligned}$$

By (3.2) of Theorem 3.1, the proof is completed. □

In our next corollary, we express the analytic Wiener integral over $C_0(B)$ of F as the limit of a sequence of Wiener integrals over $C_0(B)$.

Corollary 3.3. *Let F and $\{e_l\}$ be given as in Theorem 3.2. Let λ be a complex number with* $\operatorname{Re}\lambda>0$. *Then we have*

$$(3.11)\qquad \int_{C_0(B)}^{\text{anw}_\lambda}F(x)\,dm_B(x)=\lim_{l\to\infty}\lambda^{l(n+1)/2}\int_{C_0(B)}F_\lambda(x)\,dm_B(x),$$

where $F_\lambda(x)$ is given by (3.8) with λ_l replaced by λ.

Proof. To prove this corollary, we modify the proof of Theorem 3.2 by replacing "λ_l" by "λ" whenever it occurs and by replacing "$2iq$" by "-2λ" in the last step of the proof of Theorem 3.2. Finally by (3.3) in Theorem 3.1, we obtain the result. □

We are now ready to state a change of scale formula for Wiener integral over $C_0(B)$ of $F(x) = G(x)\psi(x(T))$. The following corollary is easily obtained from Corollary 3.3.

COROLLARY 3.4. *Let F and $\{e_l\}$ be given as in Theorem 3.2 and let $\rho > 0$. Then we have*

$$\int_{C_0(B)} F(\rho x)\, dm_B(x) = \lim_{l\to\infty} \rho^{-l(n+1)} \int_{C_0(B)} F_{\rho^{-2}}(x)\, dm_B(x), \tag{3.12}$$

where $F_{\rho^{-2}}(x)$ is given by (3.8) with λ_l replaced by ρ^{-2}.

PROOF. By letting $\lambda = \rho^{-2}$ in (3.11), we have the result. □

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