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CHANGES IN BACKGROUND RISK AND RISK TAKING BEHAVIOR

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Abstract

In this paper, we analyze the effect of a change in an unavoidable background risk on the optimal risk-taking strategy of a risk-averse individual. We first derive a necessary and sufficient condition to guarantee that a first-degree stochastic dominance shift in background risk makes risk-averters to behave in a more risk-averse way. This condition is stronger than decreasing absolute risk aversion. We next show that an increase in backround risk via second-degree stochastic dominance does not necessarily reduce the optimal demand for risky assets. Standard risk aversion is necessary - but not sufficient - to obtain intuitively acceptable comparative statics results. The corresponding weakest condition on the utility function to yield intuitive comparative statics is established.

Keywords: Background risk, standard risk aversion, prudence, temperance, portfolio selection, insurance demand.

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1. Introduction

Many authors have examined the effects of an increase in risk on the demand for instruments dealing with that risk (e.g. Meyer and Ormiston [1983,1985,1989], Black and Bulkley [1989], Hadar and Seo [1990], Ormiston [1992] and Gollier [1991]). Applications include the demand for risky assets, the purchase of insurance and decisions of the firm under uncertainty. It is well-known that risk aversiuon alone is not sufficient to generate unambiguous comparative statics with respect to changes in the level of riskiness, and each of the above papers seeks stronger restrictions on preferences and/or risk changes sufficient to generate unambiguous results.

Our paper is similar to that of Ormiston in that we consider both general first- and second-degree stochastic dominance changes in risk. However, unlike Ormiston, we consider changes in an independent background risk, rather than the "treated" risk. That such background risk actually exists seems apparent. For instance, one's human capital may be to a large degree inunsurable, making it exogenous to the problem. Changes in the riskiness of one's human capital are likely to affect market decisions for other risks. In one sense, our paper is similar in spirit to that of Hadar and Seo, who examine the comparative statics problem in a setting of multiple random variables, and who derive both necessary and sufficient conditions for unambiguous comparative static results. However, their model does not allow for an independent exogenous background risk.

In this paper, we derive conditions that are both necessary and sufficient for general first- and second-degree stochastic dominance changes in risk to yield unambiguous comparative statics results. These conditions are stronger than those required for a change from a riskless to a risky background wealth, which have been examined by Eeckhoudt and Kimball [1992]. The fact that we cannot extend results obtained for "an introduction of risk" to the case of a marginal increase in risk is a familiar feature of the theory of risk (Meyer and Ormiston [1983]). Indeed, seemingly paradoxical behavior is perhaps not paradoxical at all if we consider that "the more risk-averse of two individuals need not have the smaller certainty equivalent for a risk if another risk is present."(Pratt [1988])

The conditions we impose on preferences are fairly strong. They involve restrictions on the concavity of the rate of change of marginal utility. On the other hand, if we take as positive behavior that individuals behave in a more risk-averse manner when background wealth is riskier, these conditions are natural and point once again to the difficulties of modeling individual behavior under uncertainty.

We present the basic model in Section 2, whereas Section 3 is devoted to the case of changes in background risks which satisfy first degree stochastic dominance, a particular case of undesirable shifts in distribution. In Section 4, we consider the larger set of all undesirable changes in risk for risk-averse decision makers, i.e. second degree stochastic dominance. Conclusions,a s well as an extension to the case of nth-degree stochastic dominance, are presented in Section 5.

2. The Model

Consider a risk-averse decision maker with von Neumann-Morgenstern utility $u(w)$ and final wealth $\tilde{w}(\alpha) = w_0 + \alpha\tilde{x} + \tilde{y}$. Random variable $\tilde{y}$ is an exogenous and unavoidable "background" risk whose cumulative distribution is initially $F_1(y)$. There is another source of uncertainty due to the existence of an independent endogenous risk $\alpha\tilde{x}$. Parameter α is a variable under the control of the decision maker and $\tilde{x}$ is distributed according the cumulative distribution function (cdf) $G(x)$. For example, $w_0 + \tilde{y}$ might be the non capital income of an individual holding α shares of a stock whose return in excess of the risk-free rate is $\tilde{x}$. Alternatively, α can be viewed as the voluntary rate of retention of a risk generating a loss $-\tilde{x} + P$, where P is the full-insurance premium. We consider the impact on the optimal choice of α of a change in the distribution of the unavoidable risk $\tilde{y}$ from cdf F_1 to cdf F_2.

The decision maker facing the exogenous risk $\tilde{y}$ with distribution F_i, $i = 1, 2$, will select the level of the control variable α_i relative to the endogenous risk which maximizes his/her expected utility:

$$\alpha_i \in \arg\max_{\alpha} EU_i(\alpha) = E_i[u(\tilde{w}(\alpha))] = \int\int u(w_0 + \alpha x + y)\, dG(x) dF_i(y), \tag{1}$$

where E_i is the expectation operator assuming distribution F_i for $\tilde{y}$. Following Kihlstrom, Romer and Williams [1981] and Nachman [1982], the objective of the decision maker can be rewritten by defining indirect utility $v_i(w)$ as

$$v_i(w) = E_i[u(w + \tilde{y})] = \int u(w + y)\, dF_i(y). \tag{2}$$

Kihlstrom, Romer and Williams, and Kimball [1991b] have shown that these indirect utility functions inherit properties of the original utility function: they are monotone and concave; they exhibit decreasing absolute risk aversion (DARA) if u is DARA; they exhibit decreasing absolute prudence (DAP) if u is DAP. The problem (1) can now be rewritten as

$$\alpha_i \in \arg\max_{\alpha} EU_i(\alpha) = \int v_i(w_0 + \alpha x)\, dG(x). \tag{3}$$

This is the standard formulation of the single-risk model with v replacing u. This problem is concave, using the concavity of u and v_i. The usual first-order condition is[1]

$$E[v_i'(w_0 + \alpha_i\tilde{x})\tilde{x}] = 0. \tag{4}$$

It follows that a change in background risk can be analyzed as a change in attitude toward risk expressed by the shift from indirect utility v_1 to v_2. Arrow [1971] and Pratt [1964] have shown that a uniformly more risk-averse individual optimally selects a less risky position in the single-risk model. That means that if v_2 is a concave transformation of v_1, then $|\alpha_2|$ is less than $|\alpha_1|$, whatever the distribution of $\tilde{x}$.[2] Reciprocally, if v_2 is not a concave transformation of v_1, then it will always be possible to find a random variable $\tilde{x}$ such that $|\alpha_2|$ would be larger than $|\alpha_1|$. This result is summarized in the following Lemma:

[1] We assume u is continuously differentiable and examine only bounded solutions. Unbounded solutions are irrelevant for comparative statics analyses. See Gollier [1991] for necessary and sufficient conditions for the existence of a bounded solution for this problem.

[2] It is easily shown that both α_1 and α_2 have the same sign as $E[\tilde{x}]$.

Lemma: *Given any w_0 and any $\tilde{y}$, $|\alpha_2|$ is less than $|\alpha_1|$ for any distribution of $\tilde{x}$ independent of $\tilde{y}$ if and only if v_2 is a concave transformation of v_1.*

The initial problem thus simplifies to determining whether v_2 is uniformly more concave than v_1, i.e. if

$$R_2(w) \equiv -\frac{v_2''(w)}{v_2'(w)} = -\frac{E_2[u''(w_0 + \tilde{y})]}{E_2[u'(w_0 + \tilde{y})]} \geq -\frac{v_1''(w)}{v_1'(w)} = -\frac{E_1[u''(w_0 + \tilde{y})]}{E_1[u'(w_0 + \tilde{y})]} \equiv R_1(w), \tag{5}$$

for all w. R_i denotes the index of absolute risk aversion of the indirect utility function v_i. The solution to the comparative-statics problem thus requires examining the comparative effects of the change in the distribution of $\tilde{y}$ on $E[u'(w + \tilde{y})]$ and $E[u''(w + \tilde{y})]$ respectively.

3. First Degree Stochastic Dominance Changes in Background Risk

In this section, we consider a shift in distribution of background risk $\tilde{y}$ from F_1 to F_2, where F_2 is dominated by F_1 in the sense of first degree stochastic dominance (FSD), i.e. $F_2(y) \geq F_1(y)$ for all y. We will write $F_1\ FSD\ F_2$ to denote this dominance. The simplest case arises when the initial background wealth is non-random. We say that the background wealth $w_0 + \tilde{y}$ undergoes a global FSD change in distribution if $\tilde{y}$ initially equals y_0 with probability 1 and then takes a (random) value less than y_0 with probability 1. That is, $F_1(y) = F_2(y) = 1$ for $y \geq y_0$ and $F_1(y) = 0$ for $y < y_0$, while $F_2(y) > 0$ for some $y < y_0$. The following result extends the well-known property that a reduction in wealth makes individuals to behave in a more conservative way when the utility function is DARA.

Proposition 1: *If the background wealth undergoes a global FSD change in distribution, and if $u(w)$ exhibits decreasing absolute risk aversion, then the decision maker is uniformly more risk-averse under the new distribution of the background risk, i.e. v_2 is a concave transformation of v_1.*

Proof: (See Appendix)

By our Lemma, Proposition 1 implies that $|\alpha_2| < |\alpha_1|$. This result, which holds for a global FSD change in risk, is not true when considering a *marginal* FSD change in risk. In other words, when the initial background wealth is random, DARA is not sufficient to guarantee that a FSD change in background risk makes risk-averse decision makers to behave in a more conservative way. For example, consider an individual with the DARA utility function $u(w) = \ln(w)$, an initial distribution of $\tilde{y}$ of $(0, 1/2; 1, 1/2)$[3] and a final FSD-dominated distribution of $(0, 1/2; 0.5, 1/2)$. It is apparent that the indirect utility function

[3] This is a standard notation for a two-point lottery with payoffs 0 and 1 with probabilities 1/2.

$v_2(w) = 0.5\ln(w) + 0.5\ln(w+0.5)$ is not a concave transformation of $v_1(w) = 0.5\ln(w) + 0.5\ln(w+1)$. Indeed, $R_2(0.1) = 8.81$ whereas $R_1(0.1) = 9.24$.

Any FSD-shift in risk can be expressed by adding a noise $\tilde{\epsilon}$ which is nonpositive with probability 1. The fact that Proposition 1 cannot be extended to a risky initial background wealth is a consequence of the following observation by Pratt [1988]: the more risk-averse of two individuals need not to behave in a more risk-averse way if both individuals have an identical background wealth which is risky. The connexion with our analysis is obtained by considering an individual with utility $\hat{u}(w)$ defined as $\hat{u}(w) = E[u(w+\tilde{\epsilon})]$. By Proposition 1, he is more risk-averse than individual u, if u is DARA. But, as stressed by Pratt, this does not imply that utility $v_2(w) = E_1[u(w+\tilde{y}+\tilde{\epsilon})]$ is more concave than utility $v_1(w) = E_1[u(w+\tilde{y})]$.[4] In the following Proposition, we present the least restrictive conditions on utility functions which entails the same comparative static property for marginal FSD-shifts in background wealth than for global FSD-shifts.[5] We assume throughout the remainder of the paper that all distributions have support in $[a,b]$. We use the concept of absolute prudence as introduced by Kimball [1990]. We let $P(w)$ denote the measure of absolute prudence for $u(w)$, $P(w) = -u'''(w)/u''(w)$, and we let $P_i(w)$ denote the corresponding measure for $v_i(w)$.

Proposition 2: $\left[F_1 \text{ } FSD \text{ } F_2 \text{ implies } R_2(w) \geq R_1(w) \text{ for all } w\right]$ *if and only if*

$$\min_{y\in[a,b]} P(w+y) \geq \max_{y\in[a,b]} R(w+y). \tag{6}$$

Proof: (See Appendix)

Kimball [1990] argues that P should be positive and decreasing with wealth. Positive prudence is necessary for DARA. This is easily seenby noting the following property:

$$R'(w) = R(w)\,[R(w) - P(w)]. \tag{7}$$

Observe that absolute risk aversion is decreasing if and only if $P(w)$ is larger than $R(w)$. Therefore, DARA is necessary to verify condition (6), otherwise its LHS would be negative. But DARA is not sufficient. In Figure 1, we illustrate a DARA utility function which does not satisfy condition (6). For this utility function, there exist some FSD-dominated changes in background risk which generate a nonconcave transformation of the indirect utility function. Using Lemma 1, it follows that this DARA decision maker will increase his/her demand for the risky asset, for some distribution G of returns $\tilde{x}$.

[4] In short, the fact that $-\hat{u}''(w)/\hat{u}'(w)$ is larger than $-u''(w)/u'(w)$ for all w does not implies that $-E\hat{u}''(\tilde{w})/E\hat{u}'(\tilde{w})$ is larger than $-Eu''(\tilde{w})/Eu'(\tilde{w})$ for any distribution of $\tilde{w}$. This point was first made by Kihlstrom, Romer and Williams [1981].

[5] Note that Hadar and Seo [1990] consider a similar problem with $\tilde{w}(\alpha) = w_0 + \alpha\tilde{x} + (1-\alpha)\tilde{y}$, with $\tilde{x}$ and $\tilde{y}$ independent. They consider FSD shifts in $\tilde{x}$. Defining $\tilde{z} = \tilde{x} + \tilde{y}$ yields our model. However, shifts in $\tilde{z}$ then imply only shifts in $\tilde{x}$, since $\tilde{y}$ is unchanged, and $\tilde{x}$ and $\tilde{y}$ are not independent. Hence, the Hadar and Seo results do not apply in our model.

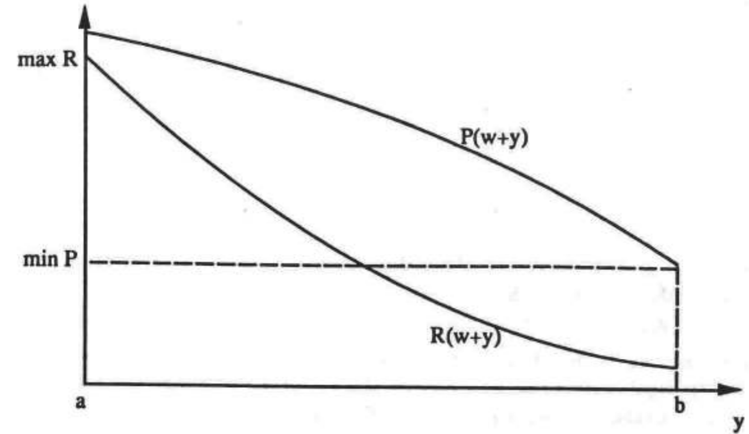

Figure 1: A case violating condition (6) despite DARA.

An informal proof of Proposition 2 is obtained by considering an initial distribution F_1 having the following characteristics: $\tilde{y}$ takes value y_0 with probability p, otherwise it takes value $\tilde{z}$ whose cdf is $H(z)$. It is then easy to verify that

$$\frac{\partial R_1}{\partial y_0}(w) = p\left[-\frac{u''(w+y_0)}{E_1[u'(w+\tilde{y})]}\right][R_1(w) - P(w+y_0)]. \tag{8}$$

The intuitively appealing comparative static result is obtained if the RHS is negative, i.e. if $P(w+y_0)$ is larger than $R_1(w)$. Selecting the y_0 which correponds to the smallest absolute prudence, property $\partial R_1/\partial y_0 < 0$ holds if $\min P \geq R_1(w)$. Since on the other hand, we can rewrite $R_1(w)$ as

$$R_1(w) = E_1[\gamma_1(w,\tilde{y})R(w+\tilde{y})], \tag{9}$$

with $\gamma_1(w,y) = u'(w+y)/E_1[u'(w+\tilde{y})]$ and $E_1[\gamma_1(w,\tilde{y})] = 1$, it appears that $R_1(w)$ is a weighted average of $R(w+\tilde{y})$. Therefore, one can find an initial distribution of $\tilde{y}$ which makes $R_1(w)$ as close as we want to the maximum of R. It follows that one can guarantee that a reduction in background wealth in at least one state of the world will increase risk aversion if and only if the smallest possible value of absolute prudence is larger than the largest value of absolute risk aversion. Otherwise it is possible to build an example where the FSD-change increases the optimal exposure to the freely chosen risk.

If we consider the set of all distributions, the necessary and sufficient condition (6) requires that the minimum of absolute prudence be larger than the maximum of absolute risk aversion, among all relevant levels of wealth. This is clearly a very strong condition in the sense that there are few utility functions which satisfy this condition. As seen before, the logarithmic utility function - and more generally all CRRA functions - does not satisfy it. The limit case is the set of exponential utility functions $u(w) = -exp(-Rw)$ for which v_2 is equal to v_1 up to a linear transformation, and $P(w) = R(w) = R = R_1(w) = R_2(w)$ for all

w. A non-limit case is obtained for example with the "one-switch" utility function $u(w) = k_1 w - e^{-k_2 w}$, as introduced by Bell [1988] , with $k_1 \geq 0$, $k_2 \geq 0$. In that case, condition (6) is verified since $P(w) = k_2$ and $\max_w R(w) = k_2$.

In the face of that negative result, one can try to find some restrictions on the FSD-change in distribution such that DARA is sufficient to guarantee that v_2 is a concave transformation of v_1. This is done in Proposition 3.

Proposition 3: *If utility u is DARA, then v_2 is a concave transformation of v_1 if one of the four following restrictions on the FS change in the background risk is satisfied:*

(i) an independent noise $\tilde{\epsilon}$ is added to the background wealth, where $\tilde{\epsilon}$ is negative with probability 1.

(ii) F_1 dominate F_2 in the sense of monotone likelihood ratio (MLR); i.e. there exists $y_0 \in [a,b]$ and a nonincreasing function $h : [y_0, b] \rightarrow R_+$ such that $dF_1(y) = 0$ for $y < y_0$ and $dF_2(y) = h(y)dF_1(y)$ for $y \geq y_0$. (Note that this property implies F_1 FSD F_2).

Proof: See the Appendix

Property (i) is basically an extension of Proposition 1. As a special case, we obtain the case where ϵ is a negative constant, i.e. $\tilde{y}$ is shifted by a constant amount. Property (ii) is a condition commonly encountered in the comparative-statics literature for a single risk when the distribution of this risk is endogenous.[6]. The MLR property implies FSD. As a special case, (ii) holds if the supports of F_1 and F_2 have an empty intersection. This case parallels Meyer's and Ormiston's [1985] "strong increase in risk", where probability mass is taken from its original support and moved outside it. Another special case of (ii) occurs if F_1 and F_2 have identical two-point supports, where F_2 puts more probability mass on the lower outcome than does F_1.

4. Second Degree Stochastic Dominance Changes in Background Risk

We assume in this section that the background risk undergoes an undesirable shift when taken in isolation, i.e. it undergoes a second degree stochastic dominance (F_1 SSD F_2) shift in distribution, i.e. $\int_a^t F_1(y)dy \leq \int_a^t F_2(y)dy \ \forall t \in [a,b]$. This means that the alternative distribution of the background risk is riskier than the original one in the sense of Rothschild and Stiglitz [1970]. Again, the case in which the initial background wealth is certain is much simpler. If the initial background wealth is risk-free and if the alternative distribution satisfies the SSD condition, we say that the background risk undergoes a global SSD-dominated change in distribution, or "an introduction of risk". Eeckhoudt and Kimball [1992] and Weil [1990] obtained the following general result for a global SSD change in risk.

[6] See for example Ormiston and Schlee [1991]. Machnes [1992] proves that a DARA producer reduces his level of production when facing a MLR-shift in background wealth.

Proposition 4: *If the background wealth undergoes a global SSD change in distribution, and if $u(w)$ exhibits decreasing absolute risk aversion and decreasing absolute prudence, then the decision maker is uniformly more risk-averse under the new distribution of the background risk, i.e. v_2 is a concave transformation of v_1.*

Proof: Follows from Eeckhoudt and Kimball [1992] and Proposition 2.

Decreasing absolute risk aversion and decreasing absolute prudence (DAP) are necessary and sufficient for standard risk aversion (SRA), as defined by Kimball [1991b]. DARA and DAP conditions are exhibited for example by nondegenerate exponential mixtures. This proposition indicates that standard risk aversion is sufficient to increase risk aversion of an individual facing a global increase in background risk. Ceteris paribus, this will have an adverse effect on the demand for risky assets and on the equilibrium price of risky assets (Weil [1990]).

We now consider the problem of determining whether this result remains true when considering a *marginal* increase in background risk rather than a global one. As in the case of FSD changes, the result obtained for global changes is not robust to the case of marginal changes. This is again a consequence of the analysis presented by Pratt [1988].In the face of this negative result, one can find additional restrictions on either the change in risk or the utility function in order to restore the comparative statics property which holds for intial safe wealth. Kimball [1991b] considers the first approach by defining the notion of "patently riskier". The next Proposition examines the other approach.

Proposition 5: $\left[F_1 \ SSD \ F_2 \text{ implies } R_2(w) \geq R_1(w) \text{ for all } w\right]$ *if and only if conditions (6) and*

$$\min_{y \in [a,b]} T(w+y) \ \geq \ \max_{y \in [a,b]} R(w+y), \tag{10}$$

are satisfied, where $T(w) = -u''''(w)/u'''(w)$ is the index of absolute temperance.

Proof: See the Appendix.

The concept of temperance is introduced by Kimball [1991a] to measure the degree of moderation in accepting risks. However, up to our knowledge, Proposition 5 provides the first example of interest in applying the index of absolute temperance. Parallel to equation (7), the derivative of the index of absolute prudence is easily shown to satisfy condition (11):

$$P'(w) = P(w)\ [P(w) - T(w)]. \tag{11}$$

Positive prudence and DAP imply that $T(w)$ is uniformly positive and larger than $P(w)$. Thus, standard risk aversion implies that $T(w) \geq P(w) \geq R(w) \geq 0$ for all w. DARA and DAP are clearly not sufficient for either condition (6) and or condition (10) to hold. It follows that standard risk aversion is not sufficient to guarantee that a marginal increase in background risk reduces the demand for risky assets. In Figure 2, we illustrate a utility function satisfying DARA and DAP, but which violates condition (10). Since condition

(10) is also necessary, this utility function with standard risk aversion is such that there exists at least one undesirable change in background risk which increases the optimal exposure to the endogenous risk, contrary to the intuition and to the result obtained by Eeckhoudt and Kimball [1992] for global increases.

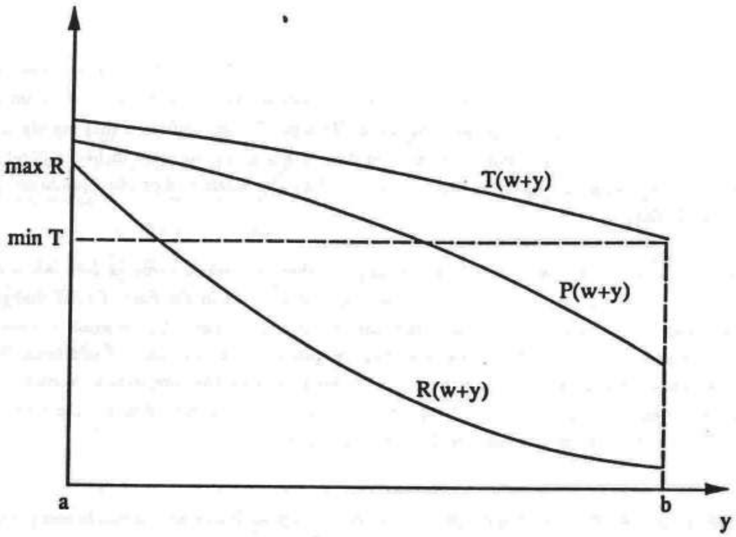

Figure 2: A case violating condition (10) despite DARA and DAP.

A heuristic proof of the necessity of condition (10) follows. Consider an initial background risk with an atom of probability p at $y = y_0$. Under the new distribution, this atom is split into two equal parts at $y_0 - \epsilon$ and $y_0 + \epsilon$. The indirect utility function v_2 is a function of ϵ:

$$R_1(w;\epsilon) = -\frac{E[u'']}{E[u']} = -\frac{(1-p)E[u''(w+\tilde{y}) \mid y \neq y_0] + \frac{p}{2}[u''(w+y_0-\epsilon) + u''(w+y_0+\epsilon)]}{(1-p)E[u'(w+\tilde{y}) \mid y \neq y_0] + \frac{p}{2}[u'(w+y_0-\epsilon) + u'(w+y_0+\epsilon)]},$$

implying that

$$\begin{aligned}\frac{\partial R_1}{\partial \epsilon}(w;\epsilon) &= \frac{p}{2}\frac{-E[u'][u'''(w+y_0+\epsilon) - u'''(w+y_0-\epsilon)] + E[u''][u''(w+y_0+\epsilon) - u''(w+y_0-\epsilon)]}{(E[u'(w+\tilde{y})])^2} \\ &\cong p\epsilon\,\frac{-E[u']u''''(w+y_0) + E[u'']u'''(w+y_0)}{(E[u'])^2} \\ &\cong p\epsilon\,\frac{u'''(w+y_0)}{E[u'(w+\tilde{y})]}\,[T(w+y_0) - R_1(w;\epsilon)] \qquad (12)\end{aligned}$$

The approximation is arbitrarily close for small enough ϵ. An increase in ϵ represents a mean-preserving spread of $\tilde{y}$. We know from Proposition 3 that positive prudence ($u'''(w+y_0) > 0$) is necessary, since FSD is

a particular case of SSD. The intuitively appealing comparative static result ($\partial R_1/\partial\epsilon > 0$) is thus obtained only if $T(w+y_0)$ is larger than $R_1(w;0)$. Otherwise, the mean-preserving spread applied to the unavoidable risk would lead the agent to be less risk averse, contrary to the intuition. Since $R_1(w;0)$ is a weighted average of $R(w+\tilde{y})$, condition (10) is necessary if we consider all possible initial unavoidable risks in $[a,b]$ and all possible undesirable changes in background risk.

Again, few utility functions satisfy condition (10) when considering unrestricted supports. The limit case is the set of exponential utility functions, since absolute risk aversion, prudence and temperance are constant and equal. We already know that a change in distribution of the background risk yields a linear transformation in the indirect utility function. For the one-switch utility function previously defined, condition (10) is automatically satisfied since $R(w) \leq k_2 = P(w') = T(w'')$ for all w, w', w''. Constant Relative Risk Aversion (CRRA) utility does not generally satisfy (6) or (10), but can within a restricted support as shown in the following example. Let u be CRRA with relative risk aversion $\gamma > 0$. It is straightforward to derive that $R(w) = \gamma w^{-1}$, $P(w) = (1+\gamma)w^{-1}$ and $T(w) = (2+\gamma)w^{-1}$. For the sake of illustration, suppose the infimum of the support is $a = 1$. Thus, $R_{max} = R(1) = \gamma$. Consequently, condition (6) is satisfied for the supremum of the support $b \leq (1+\gamma)\gamma^{-1}$ and (10) is satisfied for $b \leq (2+\gamma)\gamma^{-1}$. For the CRRA case, we note that (6) implies (10).[7] Thus, condition $b \leq (1+\gamma)\gamma^{-1}$ is both necessary and sufficient for SSD changes in background wealth to imply uniformly more risk aversion. In the special case where $\gamma = 1$, this requires that the higest possible value for $w+y$ should be no more than double the lowest possible value, i.e. $b \leq 2a$.

We recognize that condition (10) is a strong assumption which will be hard to verify. Our preliminary conclusion is thus that the reasonable argument that an increase in an unavoidable risk does lead risk-averters to reduce exposure to other independent risks is not a standard property in expected utility theory. The remainder of this section is devoted to restrictions on SSD changes in background risk which yield the intuitively appealing result under standard risk aversion.

Consider adding the risk $\tilde{\epsilon}$ to existing background risk $\tilde{w}$. When the noise is stochastically independent, standard risk aversion is sufficient for unambiguous comparative statics. It could also be the case that the added noise is stochastically dependent, but standard risk aversion remains sufficient. This is shown in the following Proposition. We use the concept of a deterministic transformation, as popularized by Sandmo [1971] and extended by Meyer and Ormiston [1989]. A deterministic transformation of a lottery replaces every payoff y of the lottery by a new payoff $t(y)$. Any change in risk can be represented by using a deterministic transformation of the initial random variable. The relationship is obtained by defining $t(y) \equiv F_2^{-1}(F_1(y))$. Sandmo introduces the concept of stretching which imposes $t(y)$ to be linear, with $t(E_1[\tilde{y}]) = 0$ and $t'(y) = \alpha \geq 1$. Thus, stretching represents a mean-preserving increase in risk.

Proposition 6: ***If utility satisfies standard risk aversion (DARA and DAP), then v_2 is a concave transformation of v_1, if one of the following restrictions on the SSD change in the background risk is satisfied:***

(i) an independent noise $\tilde{\epsilon}$ is added to the background wealth, where $E[\tilde{\epsilon}] \leq 0$;

[7] This is always the case for utility functions exhibiting decreasing absolute prudence.

(ii) F_2 is obtained from F_1 by stretching.

Proof: See Appendix.

Linear transformations (i.e. stretches) in background risk are a widespread phenomenon. For example, if $\tilde{y}$ is the net labor income, a change in the flat income tax rate represents a stretching of the after-tax income. If $\tilde{y}$ is the retained loss on a risk which is only partially insurable at coinsurance rate k, a change in the compulsory insurance rate is also an example of application of Proposition 6. In a similar way, one could also consider the case of an institutional investor who is prohibited to invest more than a given percentage of his wealth in a specific risky asset.

Conditions (6) and (10) are necessary and sufficient when we consider the set of all possible changes in risk which satisfy second degree stochastic dominance. It is only sufficient when we consider the subclass of mean-preserving increases in risk. Negative prudence (u' is concave) is a priori a good assumption to get the appealing result since the denominator of $R_1(w) = \frac{-Eu''(w+\tilde{y})}{Eu'(w+\tilde{y})}$ is reduced by a mean-preserving spread of $\tilde{y}$. On the contrary, when u' is convex - as assumed implicitly with DARA - the denominator of R_1 is increased by an increase in background risk. This has a (counter-intuitive) negative impact on risk aversion. Since u' is concave and $-u''$ is convex, any mean-preserving increase in risk of $\tilde{y}$ reduces $Eu'(w+\tilde{y})$ and increases $-Eu''(w+\tilde{y})$. Therefore, $R_1 = -Eu''/Eu'$ increases with any increase in background risk. Therefore, if the utility exhibits negative prudence and negative temperance ($u''' < 0$ and $u'''' < 0$) over the support of $w+\tilde{y}$, any increase in background risk leads the agent to reduce exposure to any independent risks. Thus, $u''' < 0$, $u'''' < 0$ presents another sufficient condition for the desired comparative static results for the subclass of mean-preserving increases in risk. These assumptions relative to negative prudence and negative temperance are unfamiliar[8]. Proposition 5 is not contradicted since negative prudence means increasing absolute risk aversion. Therefore the above condition leads agents to increase exposure to endogenous risk when facing a global FSD deterioration in background wealth. Notice that the opposite property is also true: if u''' and u'''' are uniformly positive, then any increase in background risk *reduces* implicit risk aversion. This means that positive prudence and negative temperance leads to the counter-intuitive result, for all mean-preserving spreads of the unavoidable risk.

5. Conclusion

We have shown that the relationship between more risk-averse types of behavior and increases in background risk as defined by general first- and second-degree stochastic dominance changes in the probability distribution are ambiguous unless one is willing to make "strong" assumptions about preferences. The restrictions on preferences are "strong" only in the sense of being stronger than commonly used restrictions as nonincreasing absolute risk aversion. On the other hand, if observed behavior indicates that people do take less risky positions in the presence of increased background risk, then these conditions should be viewed as natural.

[8]It does not even satisfy proper risk aversion which is a very natural assumption, as explained by Pratt and Zeckhauser [1987]. However, $u'''' < 0$ is necessary for decreasing absolute prudence.

Noting the proofs of Proposition 2 and 5 in the Appendix, it is straightforward (but extremely tedious) to generalize the methodology of our proofs, and to intuitively prove the general case:

Proposition 7: $\left[F_1\right.$ *nth-degree stochastically dominates F_2 implies $R_2(w) \geq R_1(w)$ for all w*$\left.\right]$ *if and only if*

$$\min_{y\in[a,b]} \frac{-u^{(k)}(w+y)}{u^{(k-1)}(w+y)} \geq \max_{y\in[a,b]} R(w+y), \quad \forall k = 3, ..., n+2$$

where $u^{(k)}(x)$ denotes the kth derivative of u, $\frac{d^k u}{dx^k}(x)$.

Situations involving background risks would seem to be the norm rather than the exception, in everyday life. Since many real-world phenomena also seem to create first- and/or second-degree stochastic dominance shifts, our results should prove useful in understanding how decision makers behave in settings of multiple risks.

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Appendix

Following Nachman [1979], define the probability measure of a (Borel measurable) subset B in $[a, b]$ as

$$\mu_i(B) = \int_{y \in B} \frac{u'(w+y)}{\int u'(w+x) dF_i(x)} dF_i(y).$$

In particular, write $\mu_i(y) \equiv \mu_i([a, y])$. Note that μ_i is the risk-adjusted distribution function for F_i given utility u (See Nachman [1979]).

Although it is straightforward to prove Proposition 1 directly, we choose to treat it as a special case after proving Proposition 2.

Proof of Proposition 2

Simple manipulations show that $R_2(w) \geq R_1(w)$ if and only if

$$\hat{R}(w) \equiv \frac{-\int_a^b u''(w+y) d(F_2 - F_1)}{\int_a^b u'(w+y) d(F_2 - F_1)} \geq \frac{-\int_a^b u''(w+y) dF_1}{\int_a^b u'(w+y) dF_1} \equiv R_1(w).$$

Integrating the numerator and denominator of $\hat{R}(w)$ by parts, we obtain

$$\hat{R}(w) \equiv \frac{-\int_a^b u'''(w+y)(F_2-F_1)dy}{\int_a^b u''(w+y)(F_2-F_1)dy} = \int_a^b P(w+y)d\eta(y)$$

where

$$\eta(y) \equiv \int_a^y \frac{u''(w+t)(F_2(t)-F_1(t))}{\int_a^b u''(w+s)(F_2(s)-F_1(s))ds}dt \equiv \int_a^y d\eta(t).$$

Note that $\eta(y)$ is also a probability measure aver $[a,b]$.

Let P_{min} and R_{max} denote the left- and right-hand sides of (6) respectively.

Proof of sufficiency: Suppose $R_{max} \leq P_{min}$ and F_1 FSD F_2. Thus,

$$\hat{R}(w) = \int_a^b P(w+y)d\eta(y) \geq P_{min} \geq R_{max} \geq \int_a^b R(w+y)d\mu_1(y) = R_1(w).$$

Proof of necessity: Suppose that $R_{max} > P_{min}$. Then $\exists x_0, y_0 \in [a,b]$ and $\epsilon > 0$ such that

$$R_{max} = R(w+x_0) > P(w+y) \quad \forall y \in N_\epsilon(y_0)$$

where $N_\epsilon(y_0)$ is the intersection of $(y_0-\epsilon, y_0+\epsilon)$ and $[a,b]$. It is an open (using relative topology) ball of radius ϵ in the support of $[a,b]$. Choose F_1 to be conditionally uniform on $[a,b]$ for $x \neq x_0$, but with an atom at x_0. Indeed, we can choose $dF_1(x_0)$ arbitrarily close to one. Thus,

$$\int_a^b R(w+y)d\mu_1(y) \; - \; R(w+x_0) \;\cong\; 0,$$

where we note that $\mu(.)$ also has an atom at x_0. Now choose F_2 such that

$$F_2 \begin{cases} > F_1 & \text{for } y \in N_\epsilon(y_0), \\ = F_1 & \text{otherwise.} \end{cases}$$

Thus $\eta(y) > 0$ if and only if $y \in N_\epsilon(y_0)$. Hence,

$$\hat{R}(w) = \int_a^b P(w+y)d\eta(y) < R(w+x_0) \cong \int_a^b R(w+y)d\mu_1(y) = R_1(w).$$

Since the above approximation can be made arbitrarily close, this implies that $\hat{R}(w) < R_1(w)$, or equivalently $R_2(w) > R_1(w)$. ∎

Proof of Proposition 1:

Recall that $dF_1(y_0) = 1$. Since F_1 FSD F_2, we have $d\eta(y) > 0$ only if $y < y_0$. Thus, $\hat{R}(w) = \int_a^{y_0} P(w+y)d\eta(y)$. Now DARA implies that $P(w+y) \geq R(w+y) \geq R(w+y_0) = R_1(w)$. It thus follows that $R_1(w) \leq R_2(w)$. ∎

Proof of Proposition 3:

(i) $R_1(w) = \frac{-v_1''(w)}{v_1'(w)} = \frac{-\int u''(w+y)dF_1(y)}{\int u'(w+y)dF_1(y)}$ and v_1 exhibits DARA since u does (see Kimball [1991b]). Thus, applying Proposition 1 to v_1 and noting that $v_2(w) = E[v_1(w\tilde{\epsilon})]$, the conclusion follows.

(ii) Observe that

$$R_2(w) - R_1(w) = \int R(w+y)d(\mu_2(y) - \mu_1(y))$$
$$= \int_a^{y_0} R(w+y)d\mu_2(y) \; - \; \int_{y_0}^b R(w+y)\left[\frac{u'(w+y)h(y)}{\int u(w+x)h(x)dF_1(x)} - \frac{u'(w+y)}{\int u'(w+x)dF_1(x)}\right]dF_1(x).$$

Since $h > 0$ and h is nonincreasing, $\exists y_1 \geq y_0$ such that the bracketted expression above is negative for all $y \geq y_1$. Consequently, since we assume DARA,

$$R_2(w) - R_1(w) \geq R(w+y_1)\int_a^b d(\mu_2(y) - \mu_1(y)) = 0. \; ∎$$

Proof of Proposition 5:

Let F_1 SSD F_2. Then F_1 can be obtained from F_2 via combinations of first-degree sttochastic dominance shifts plus mean-preserving second-degree stochastic dominance shifts (i.e. Rothschild and Stiglitz [1970] increases in risk). Proposition 2 shows that (6) must be part of any necessary and sufficient condition for desirable comparative static results. Hence, we only have to show that (10) must be added for mean-preserving increases in risk. Thus, let F_1 SSD F_2 such that $\int_a^b F_2(y) - F_1(y)dy = 0$. Define

$$\xi(y) = \int_a^y \frac{u'''(w+q)\left[\int_a^q (F_2(x)-F_1(x))dx\right]}{\int_a^b u'''(w+s)\left[\int_a^s (F_2(x)-F_1(x))dx\right]}dq$$

and note that $\xi(y)$ defines a probability measure.

Proof of sufficiency: Suppose $R_{max} \leq T_{min}$. Integrating the numerator and denominator of $\hat{R}(w)$ twice by parts, and rearranging yields

$$\hat{R}(w) = \int_a^b T(w+y)d\xi(y) \geq T_{min} \geq R_{max} \geq \int_a^b R(w+y)d\mu_1(y) = R_1(w).$$

Proof of necessity: Suppose $R_{max} > T_{min}$. Then $\exists x_0, y_0 \in [a,b]$ and $\epsilon > 0$ such that $R_{max} = R(w+x_0) > T(w+y)$ $\forall y \in N_\epsilon(y_0)$ where $N_\epsilon(y_0)$ is defined as in the proof of Proposition 2. Choose F_1 to be conditionally uniform on $[a,b]$ for $x \neq x_0$, but with an atom at x_0. Indeed, we can choose $dF_1(x_0)$ arbitrarily close to one. Thus,

$$\int_a^b R(w+y)d\mu_1(y) \; - \; R(w+x_0) \;\cong\; 0,$$

Choose F_2 such that $F_2(y) = F_1(y)\ \forall y \in [a,b] \mid N_\epsilon(y_0)$ and $F_2(y)$ is riskier than $F_1(y)$ in the sense of Rothschild and Stiglitz [1970]. Thus F_2 is obtained from F_1 by a series of mean-preserving spreads inside of $N_\epsilon(y_0)$. Then, we have, given the definition of $\xi(y)$,

$$\hat{R}(w) = \int_a^b T(w+y)d\xi(y) = \int_{N_\epsilon(y_0)} T(w+y)d\xi(y) < R(w+x_0) \cong \int_a^b R(w+y)d\mu_1(y) = R_1(w).$$

Since the above approximation can be made arbitrarily close, this implies that $\hat{R}(w) < R_1(w)$, or equivalently $R_2(w) > R_1(w)$. ∎

Proof of Proposition 6:

(i) This follows from Eeckhoudt and Kimball [1992, Proposition 2] and from our Proposition 1.

(ii) This follows by noting that the property of stretching makes the background risk "patently more risky", as defined by Kimball [1991b]. The result then follows directly from Kimball [1991b, Proposition 5] and from our Proposition 1. ∎

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