Geophys. J. R. astr. Soc. (1979) 58, 495–508

Changes in dimensions, stresses and gravitational energy of the Earth due to crystallization at the inner-core boundary under isochemical conditions

Heinz Häge and Gerhard Müller *Geophysical Institute, University of Karlsruhe, 7500 Karlsruhe, Federal Republic of Germany*

Received 1979 January 3; in original form 1978 July 4

Summary. If the inner-core boundary (ICB) is a phase boundary, not a chemical boundary, changes in core temperatures will lead to changes in the size of the inner core with corresponding changes in the size of the Earth. We have investigated theoretically some effects of crystallization at the ICB due to slow cooling of the core. The earth model consists of a homogeneous inner core, outer core and mantle, and the calculations are for the elastic and viscoelastic case, including self-gravitation. Complete solidification of the core is connected with a decrease of the Earth's radius by 5 km and with additional pressures in the core and mantle of the order of several kbar, under the assumption that the density change at the ICB is 1 per cent. Part of the released gravitational energy is converted into deformational energy, distributed throughout the Earth. The main part, more than 70 per cent, is converted into heat at the ICB: this is the work done by the hydrostatic pressure during contraction of the crystallizing material. It forms a part of the latent heat of crystallization that in previous estimates has been neglected. Assuming that: (1) the density change at the ICB is 1 per cent, (2) the radius (volume) of the inner core has grown to its present value in 4×10^9 yr, we obtain as an estimate for this part of latent heat 5.82×10^{11} (1.94×10^{11}) watt. This is 1.9 (0.6) per cent of the Earth's heat flux. If crystallization of the core would actually take place, the concentrated heat source at the ICB would delay cooling of the core and possibly maintain for long times temperature gradients in the outer core which are sufficient for thermal convection. The critical parameter for all effects studied is the (so far unknown) density change of core material upon crystallization under high pressures.

Introduction

In 1953 Jacobs suggested that the Earth has cooled from a completely molten state and solidification has begun at the centre of the Earth, forming the inner core. Verhoogen (1961) estimated that latent heat that would be released by solidification at the inner-core boundary (ICB) and discussed this heat as an energy source for driving the geodynamo. In this paper we consider the release of gravitational energy of the Earth which accompanies the volume contraction of the freezing core material and which contributes significantly to the latent heat. Further investigations concern changes in the Earth's radius and the production of stresses throughout the Earth.

This paper is based on the assumption that the ICB is a phase boundary in a medium of constant chemical composition. Then the melting temperature (or liquidus and solidus) is continuous, and temperature changes will shift the ICB. The question whether or not there is a discontinuous change in composition at the ICB could be decided, if the density change at this boundary were known; larger changes, of the order of a few per cent or more, would rule out an isochemical model. Currently, the density change is not known, in spite of attempts to determine this quantity from amplitudes of short period PKiKP waves (Bolt & Qamar 1970; Bolt 1972) and overtones of spherical oscillations (Derr 1969; Anderson & Hart 1976).

It is widely accepted that in the outer core a lighter constituent is present, possibly Si or S, besides the main constituent Fe or Fe–Ni. If this mixture extends down to the ICB crystallization may be connected with fractionation, i.e. with enrichment of the lighter constituent in the outer core (Braginski 1963; Gubbins 1977; Loper 1978). In this case a chemical change would occur at the ICB. All effects described in this paper would continue to be present, provided that there is an overall change of volume or density upon crystallization, and they would be augmented by the effects due to fractionation (Loper 1978; Müller & Häge 1979).

We begin with a description of the earth model used and the decomposition of the crystallization process into single steps, each one consisting in crystallization of a thin layer at the ICB. Then, each step is treated as a boundary–value problem in elastostatics. The effects of self-gravitation are included. The results are extended for a viscoelastic (Maxwell-type) earth. A final section deals with the release of gravitational energy and the conversion of this energy into deformational energy, distributed throughout the Earth, and into heat in the crystallizing layer. This heat is part of the latent heat of crystallization, but so far has not been included in estimates of the latter. Revised estimates of the power of this (still hypothetical) concentrated heat source at the ICB are given under the assumption that the inner core has grown to its present size during 4×10^9 yr.

Earth model and discretization of crystallization or melting

To study the effects mentioned above it is sufficient to use a simple earth model. We assumed the Earth being spherically symmetric and consisting of three homogeneous shells, the solid inner core, the liquid outer core, and the solid mantle. The densities are chosen such that they fit the mass and the moment of inertia of the real Earth. Under normal conditions the density of iron decreases by 3 per cent upon melting (McLachlan & Ehlers 1971). The high pressure at the ICB (about 3.4 Mbar) certainly reduces this value. We assume a density change at the ICB of 1 per cent in our numerical calculations. Table 1 shows the parameters of our model. The elastic parameters are mean values of the corresponding parts of the real Earth.

The starting point of our calculations is the present Earth, represented by the parameters in Table 1. Growth or melting of the inner core is a slow and continuous process. For the mathematical treatment it is decomposed into single steps. In each step a thin layer with thickness d (usually 1 km) crystallizes on to or melts from the inner core. This process is repeated until the whole outer core has crystallized or the whole inner core has melted. The elastic moduli λ, μ and k are kept constant, whereas the densities are slightly changed from

Table 1. Parameters of the starting earth model.

Parameter	Inner Core	Outer Core	Mantle
Radius (km)	r_i=1250.0	r_k=3480.0	r_e=6370.0
Density (g/cm^3)	ρ_i=11.11	ρ_k=11.0	ρ_m=4.45
Bulk modulus (kbar)	k_i=11 000	k_k=11 000	k_m=4 500
Shear modulus (kbar)	μ_i=2 200	μ_k=0.0	μ_m=2 000
Lamé's constant (kbar)	λ_i=9 530	λ_k=11 000	λ_m=3 160
P-Wave velocity (km/sec)	α_i=11.2	α_k=10.0	α_m=12.7

one step to the next in order to keep the masses of mantle and core constant. At the end of each step we compute the displacements, the stress increments and the new densities as well as the gravitational energy release. Displacements, stresses and energy must be accumulated.

Elastostatic theory for a gravitating earth

DIFFERENTIAL EQUATIONS

Crystallization or melting at the ICB causes radial displacements u which are governed by the differential equation for radial oscillations of the Earth (see, e.g. Bullen 1963). Since we consider very slow motions we neglect the acceleration term and obtain the following equilibrium equation:

$$\frac{d}{dr}\{(\lambda + 2\mu)u'\} + \frac{2(\lambda + 2\mu)}{r}u' + \left\{\frac{2\lambda'}{r} - \frac{2(\lambda + 2\mu)}{r^2} + \frac{4\rho g}{r}\right\}u = 0. \qquad (1)$$

Here r is the radial distance from the Earth's centre, g is gravity and a prime denotes differentiation with respect to r. In our case, where λ, μ and ρ are piecewise constant, equation (1) simplifies to

$$u'' + \frac{2}{r}u' - \left(1 - \frac{2gr}{\alpha^2}\right)\frac{2u}{r^2} = 0, \qquad (2)$$

where α is the P-wave velocity. Applying equation (2) to each of the three homogeneous layers, we find the following differential equations (γ = gravitational constant):

(a) Inner core (g strictly proportional to r):

$$u_i'' + \frac{2}{r}u_i' - \left(\frac{2}{r^2} - \frac{16\pi\gamma\rho_i}{3\alpha_i^2}\right)u_i = 0. \qquad (3)$$

(b) Outer core (g proportional to r with very good approximation):

$$u_k'' + \frac{2}{r}u_k' - \left(\frac{2}{r^2} - \frac{16\pi\gamma\rho_k}{3\alpha_k^2}\right)u_k = 0. \qquad (4)$$

(c) Mantle:

$$u_m'' + \frac{2}{r}u_m' - \left(\frac{2}{r^2} - \frac{4g}{\alpha_m^2 r}\right)u_m = 0. \qquad (5)$$

The general solution of equations (3) and (4) is a linear combination of the spherical Bessel functions of first order, j_1 and y_1,

$$u_{\substack{i\\k}}(r) = A_{\substack{i\\k}}\, j_1(\sqrt{c_{\substack{i\\k}}}\, r) + B_{\substack{i\\k}}\, y_1(\sqrt{c_{\substack{i\\k}}}\, r),$$

where (6)

$$c_{\substack{i\\k}} = 16\pi\gamma\rho_{\substack{i\\k}}/(3\alpha_{\substack{i\\k}}^2).$$

If we assume that g in the mantle is constant, which is true within about 15 per cent, the differential equation (5) can be solved by a series solution:

$$u_m(r) = \frac{4g}{\alpha_m^2}\, r \sum_{n=0}^{\infty} c_n \left\{ \frac{4g}{\alpha_m^2}(r - r_e) \right\}^n. \tag{7}$$

r_e is the Earth's radius, and c_0 and c_1 are integration constants which follow from the boundary conditions, together with the constants $A_{\substack{i\\k}}$ and $B_{\substack{i\\k}}$ in equation (6). The coefficients c_n for $n \geqslant 2$ can be determined from the recursion formula

$$c_n = -\frac{(n-1)(n+2)c_{n-1} + c_{n-2}}{n(n-1)X_e},$$

where

$$X_e = 4gr_e/\alpha_m^2.$$

The series (7) converges everywhere in the mantle.

BOUNDARY CONDITIONS

The boundary conditions for the radial displacement u require vanishing u at the Earth's centre, a discontinuous change at the ICB compatible with the volume change upon crystallization or melting, and continuity at the CMB. The boundary conditions for the additional radial stress $p_{rr} = (\lambda + 2\mu)u' + 2\lambda u/r$ require continuity at the ICB and the core–mantle boundary (CMB) and vanishing p_{rr} at the Earth's surface. Boundary conditions at the interfaces apply in the deformed position, in principle, but after linearization it is seen that the original radii can be taken as well. In the case of the ICB the original radius is $r_i + d$, since after crystallization ($d > 0$) or melting ($d < 0$) of a layer of thickness $|d|$ the particles at this radial distance are situated at the ICB. The relation between the displacements at the top and the bottom of the crystallizing or melting layer follows from the condition that the mass of this layer remains unchanged. For instance (see Fig. 1), in the case of crystallization the mass of the shell between the spheres 1 and 3, having the thickness d and the density ρ_k, equals the mass of the shell between the spheres 2 and 4, having the thickness d' and the density ρ_i. After linearization of this condition in the displacements at the bottom and the top of the crystallizing layer, u_i and u_k, the first of the equations (8) below is obtained. The condition for melting is treated in the same way. We find:

$$u_i(r_i) - \left(1 + \frac{d}{r_i}\right)^2 u_k(r_i + d) = \frac{r_i}{3}\left\{\left(1 + \frac{d}{r_i}\right)^3 - 1\right\}\frac{\rho_i - \rho_k}{\rho_i} \quad \text{(crystallization)}\ (d > 0)$$

$$u_k(r_i) - \left(1 + \frac{d}{r_i}\right)^2 u_i(r_i + d) = \frac{r_i}{3}\left\{1 - \left(1 + \frac{d}{r_i}\right)^3\right\}\frac{\rho_i - \rho_k}{\rho_i} \quad \text{(melting)}\ (d < 0). \tag{8}$$

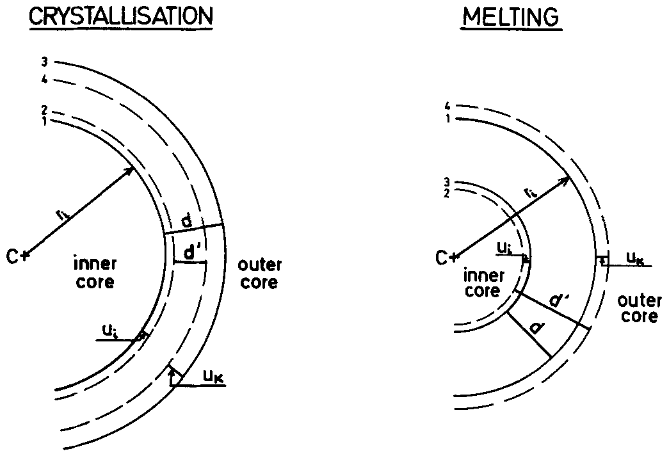

Figure 1. Conditions at the inner-core boundary (ICB). Before crystallization or melting the ICB is the sphere 1, and after crystallization (melting) the sphere 4 (2). The shell 1–3 crystallizes (melts) to give the shell 2–4.

RESULTS FOR DISPLACEMENTS AND STRESSES

Figs 2 to 4 show numerical results for the radial displacement and additional radial and horizontal stresses throughout the Earth, corresponding to crystallization of a 1 km thick layer of outer-core material on to the inner core; five different inner-core radii have been assumed. The maximum radial displacement is about -10 m and occurs at the top of the crystallizing layer. From there it decays towards the Earth's surface where the values are between -0.75 and -4.2 m. In the inner core displacement is approximately proportional to radial distance r. It is interesting to note that for r_i less than a critical radius $r_{ic} = 2929$ km the inner core is under compression ($u < 0$), whereas for $r_i > r_{ic}$ it is under extension ($u > 0$). The existence of a critical radius has already been noted by Love (1944, p. 143) in the case of a homogeneous sphere. If self-gravitation is neglected in the equilibrium equation (2), the whole core is under extension for all values of r_i. According to Figs 3 and

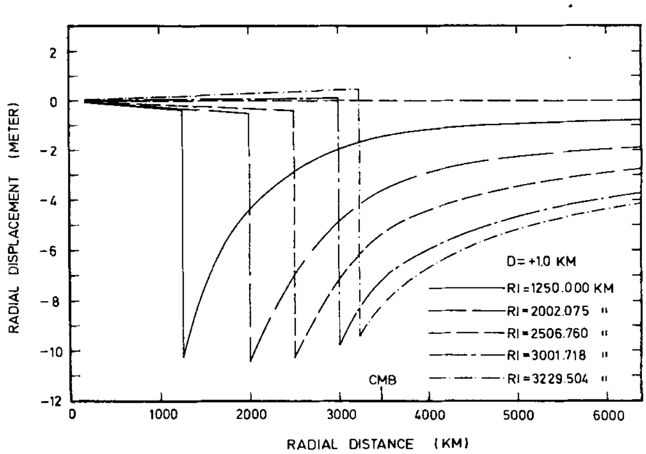

Figure 2. Radial displacement in the Earth upon crystallization of a layer of thickness 1 km on to the inner core, for five different inner-core radii. Density change upon crystallization is 1 per cent.

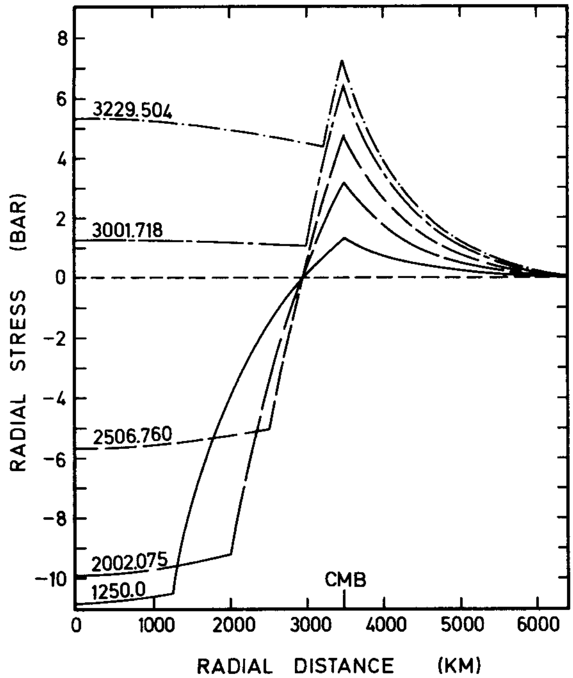

Figure 3. Same as Fig. 2 for radial stress.

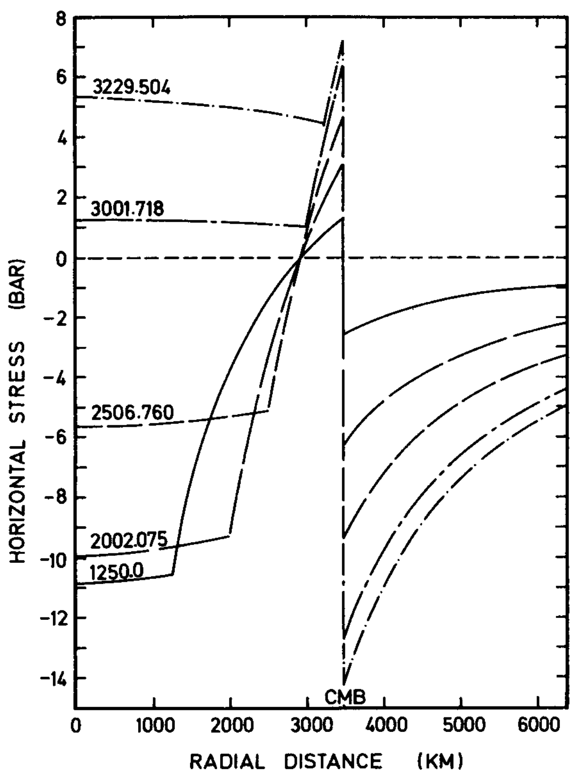

Figure 4. Same as Fig. 2 for horizontal stress.

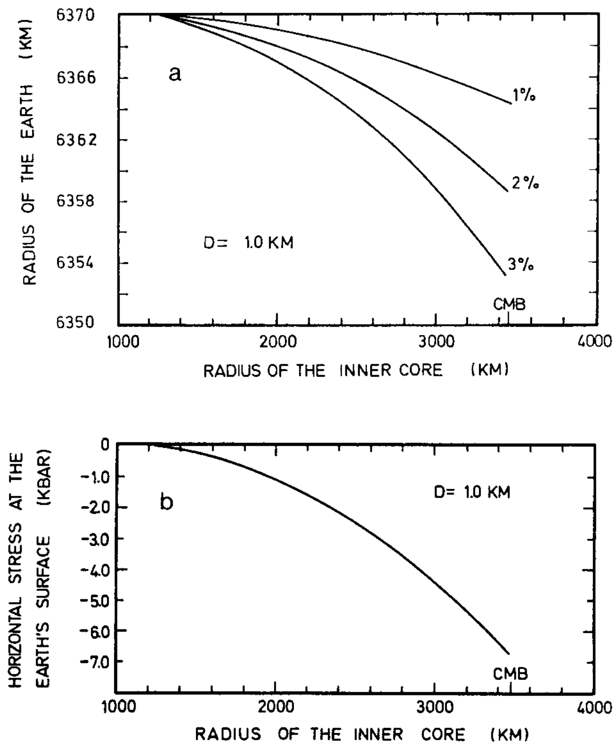

Figure 5. (a) Changes in Earth's radius by continuous crystallization of the core for three different percentage values of the density change upon crystallization. (b) Same as (a) for the horizontal stresses at the Earth's surface (density change is 1 per cent).

4, the additional stresses in the Earth are of the order of several bars; in the mantle they are extensional in radial and compressive in horizontal direction.

The effects of complete crystallization of the present outer core are found by accumulating the results presented in Figs 2 to 4. Fig. 5 shows the variation in the size of the Earth's radius and the horizontal stress at the Earth's surface. For a density change of 1 per cent at the ICB, the Earth's radius decreases by 5.7 km and horizontal compressive stresses of 6.7 kbar develop upon complete solidification of the outer core. The main results for complete melting of the present inner core are as follows (again for a density change of 1 per cent at the ICB): the Earth's radius increases by 0.32 km and horizontal extensional stresses of 0.38 kbar accumulate at the surface. More details are given by Häge (1977). Hence complete crystallization of the core from a molten initial state leads to a reduction of the Earth's radius by 6.0 km and to horizontal compressive stresses of 7.1 kbar at the surface. The size and non-isotropic nature of the additional stresses strongly depend on the model, which here is an elastic one. Stress relaxation which will occur under slow crystallization of the core is included in the viscoelastic model in the next section.

Viscoelastic earth model

A viscoelastic earth model allows a more realistic assessment of the stresses in the Earth due to slow deformation than an elastic earth model. For instance, isostatic adjustments of the Earth to surface loads are usually treated in the framework of such a model (Peltier

1974; Cathles 1975). In order to include time into the problem we assume that the thickness of the crystallizing or melting layer at the ICB is $D(t) = dH(t)$, where $H(t)$ is the unit step function. The problem is treated with the aid of the correspondence principle and the material functions of a Maxwell body (see, e.g. Cathles 1975):

$$L(s) = \frac{\lambda s + (2\mu/\eta)k}{s + 2\mu/\eta}$$

$$M(s) = \frac{\mu s}{s + 2\mu/\eta}. \tag{9}$$

Here λ, μ and k are the elastic moduli as used in the previous section and η is the viscosity. s is the complex Laplace transformation variable. In our case where gravity is included in the equilibrium equation we can only determine the final state of the viscoelastic earth, characterized by the radial displacement $u(r, t)$ for $t \to \infty$. The relaxation time is estimated from the solution for the case without gravity, which can be found exactly for all times. The correspondence principle yields the Laplace transform of $u(r, t)$,

$$\bar{u}(r, s) = F\{r, L(s), M(s)\}\frac{d}{s},$$

where $F(r, \lambda, \mu)d$ is the elastostatic solution derived in the previous section. The final value $u(r, \infty)$ of the displacement in the time domain is found from the final-value theorem of Laplace transformation:

$$u(r, \infty) = \lim_{s \to 0} s\,\bar{u}(r, s) = F\{r, L(0), M(0)\}d = F(r, k, 0)d.$$

Hence, the final state is obtained by solving the elastostatic problem for a liquid earth ($\mu = 0$) with the original distribution of bulk modulus k.

This general result means that we cannot expect complete stress relaxation, but that the stress distribution for large times is isotropic and can be described by an additional pressure which is superposed on the hydrostatic pressure due to gravity. This additional pressure is not a consequence of our relatively simple model of material behaviour, but results from the action of gravity. Therefore, if the process of continuous crystallization of the core continues, it will lead to slight departures from hydrostatic conditions. However, stresses will continue to be isotropic.

Numerical results for the final state of the gravitating Maxwell earth are given in Fig. 6 and Table 2. Fig. 6 shows the additional pressure, generated by an increase of inner-core radius of 1 km, and comparison is made with the immediate stresses which follow from Figs 3 and 4. In Table 2 dimensions and additional stresses in the Earth after complete solidification of the outer core are compiled. We conclude that stress relaxation mainly influences the stresses in the Earth and to a much lesser extent the changes in dimensions.

The viscoelastic boundary-value problem can be solved analytically for a non-gravitating Maxwell earth. The main result of these calculations is that all additional stresses in the Earth decay to zero with increasing time according to $\exp(-t/\tau)$, i.e. stress relaxation is complete. The relaxation time τ can be calculated by

$$\tau = 1.03 \times 10^{-20}\eta_m \text{ (yr)}$$

where η_m is the mantle viscosity in poise, for which extremal values are 10^{22} poise (Cathles 1975) and 10^{26} poise (MacDonald 1963). Thus τ ranges from about 10^2 to 10^6 yr. Since these relaxation times are short compared with the hypothesized times of core crystallization, the treatment by superposition of viscoelastic final states of single crystallization steps

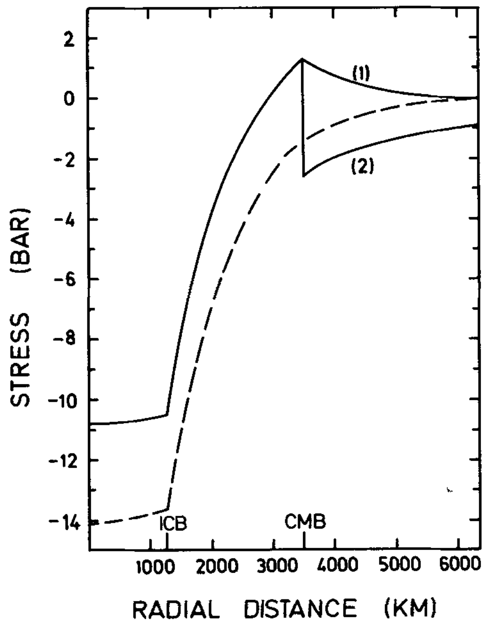

Figure 6. Stresses in a Maxwell earth due to crystallization of a layer with thickness 1 km at the inner-core boundary. Solid lines are immediate (elastic) values of: (1) radial and (2) horizontal stress. The dashed curve represents the long-time stresses after stress relaxation. The final stress distribution is isotropic and compressive.

is justified. The relaxation time for a gravitating earth probably has the same order of magnitude as the relaxation time for a non-gravitating earth.

Energy considerations

ENERGY CONSERVATION

In the remainder of this paper we investigate the conversion of gravitational energy into deformational energy and heat due to crystallization of a layer of thickness d at the ICB.

Table 2. Some parameters of the earth model before and after complete crystallization of the outer core (density change upon crystallization is 1 per cent). Final values correspond to superposition of visco-elastic final states, values in brackets to superposition of elastic initial states.

Parameter	Starting value	Final value	
Core radius r_k (km)	3480.0	3466.8	(3469.0)
Earth's radius r_e (km)	6370.0	6365.0	(6364.3)
Radial stress at r_k (kbar)	0	-9.1	(9.8)
Horizontal stress in the mantle at r_k (kbar)	0	-9.1	(-19.5)
Horizontal stress at r_e (kbar)	0	0	(-6.7)

Since the Earth as a whole contracts in this case, gravitational energy is released. For spherically symmetric radial displacements u this energy release is

$$G = -4\pi \int_0^{r_e} \rho g u r^2 \, dr. \tag{10}$$

u can be the elastic or viscoelastic displacement. In the second case G is time dependent; we are mainly interested in the viscoelastic final state. The integration in equation (10) is performed analytically in the core and numerically in the mantle.

The change D in deformational energy of the Earth is found by integrating the density D^* of deformational energy over the volume of the Earth:

$$D = \int_V D^* dV, \qquad D^* = \tfrac{1}{2} p_{ij} \epsilon_{ij} - p\theta.$$

Here, the ϵ_{ij} are strains, the p_{ij} additional stresses, p is hydrostatic pressure and θ dilatation. Since the p_{ij} are much less than p except in a thin (and hence negligible) layer at the surface of the Earth, we have with very good approximation $D^* = -p\theta$. D^* is positive upon contraction ($\theta < 0$). In our case $\theta = u' + 2u/r$, and we obtain

$$D = -4\pi \int_0^{r_e} p \left(u' + \frac{2}{r} u\right) r^2 \, dr. \tag{11}$$

The crystallizing layer is not included in equation (11), since there the volume change is not directly related to u, but to the density change from ρ_k in the liquid state to ρ_i in the solid state: $\theta = (\rho_k - \rho_i)/\rho_i$. During contraction of the crystallizing layer the work done by the hydrostatic pressure p is

$$Q = -\int_{\text{Layer}} p\theta \, dV = 4\pi r_i^2 d \frac{\rho_i - \rho_k}{\rho_i} p(r_i). \tag{12}$$

This work is converted into heat. This heat is part of the latent heat of crystallization; the second part which so far has been considered alone in the case of the ICB (see below) is due to the change in lattice upon crystallization. Numerical estimates given later show that equation (12) is an essential contribution to latent heat for density changes larger than about 0.5 per cent. The heat production per unit volume of the crystallizing layer is

$$Q^* = p(r_i)\,(\rho_i - \rho_k)/\rho_i.$$

Conservation of energy requires

$$G = D + Q, \tag{13}$$

which explicitly stated means that gravitational energy is converted into deformational energy, distributed throughout the Earth, and into heat which is released locally at the ICB.

Numerical results for the densities D^* and Q^* are given in Fig. 7 for the layer thickness $d = 1$ km and a density change upon crystallization of 1 per cent; for gravitational energy there is no density. The energies G, D and Q in this case are given in Table 3. They show that more than 70 per cent of the released gravitational energy is converted into heat in the crystallizing layer; this is true also for other values of the density change. The slight differences between G and the sum $D + Q$ are due to inaccuracies in the integration of equations (10) and (11) which is partly numerical, as mentioned above. In the case of a non-gravitating Maxwell earth it can be shown analytically that equation (13) is fulfilled for

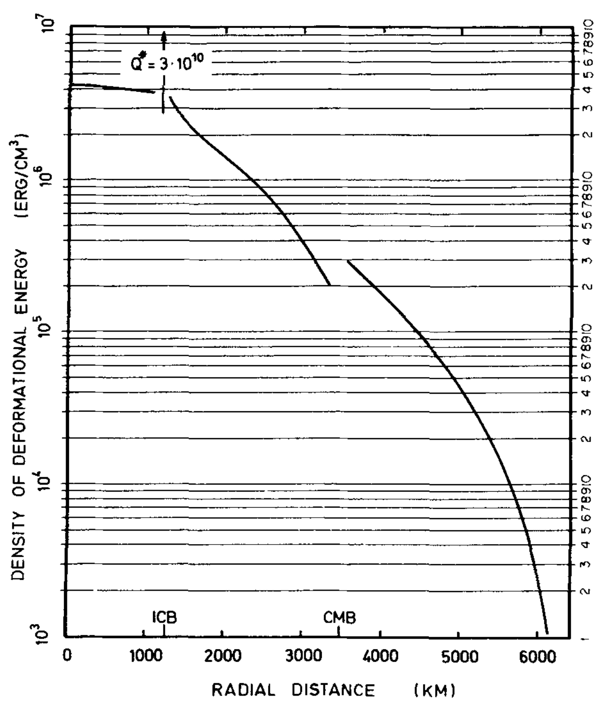

Figure 7. Density D^* of deformational energy in the Earth due to crystallization of a layer of thickness 1 km at the ICB (density change upon crystallization is 1 per cent). Calculation for viscoelastic final state of a gravitating earth. Q^* is the heat production per unit volume in the crystallizing layer.

large times, since then $D = 0$ because of complete stress relaxation, $u = 0$ for $0 \leqslant r \leqslant r_i$ and $u = \{(\rho_k - \rho_i)/\rho_i\}(r_i^2/r^2)d$ for $r > r_i$, and

$$G = 4\pi r_i^2 d \frac{\rho_i - \rho_k}{\rho_i} \int_{r_i}^{r_e} \rho g \, dr = 4\pi r_i^2 d \frac{\rho_i - \rho_k}{\rho_i} \{p(r_i) - p(r_e)\} = Q,$$

since $p(r_e) = 0$.

Table 3. Gravitational energy release G, deformational energy D and heat production Q due to crystallization of a layer with thickness 1 km at the ICB (density change upon crystallization is 1 per cent). Energy values in erg.

Energy	Elastic initial state	Viscoelastic final state
G	$7.60 \cdot 10^{32}$	$8.16 \cdot 10^{32}$
D_i (inner core)	$2.49 \cdot 10^{31}$	$3.23 \cdot 10^{31}$
D_k (outer core)	$5.09 \cdot 10^{31}$	$1.38 \cdot 10^{32}$
D_m (mantle)	$9.23 \cdot 10^{31}$	$5.63 \cdot 10^{31}$
$D = D_i + D_k + D_m$	$1.68 \cdot 10^{32}$	$2.26 \cdot 10^{32}$
Q	$5.87 \cdot 10^{32}$	$5.87 \cdot 10^{32}$
D + Q	$7.55 \cdot 10^{32}$	$8.13 \cdot 10^{32}$

LATENT HEAT

The latent heat of crystallization for an increase of inner-core radius r_i by d is (Verhoogen 1961)

$$Q_L = 4\pi r_i^2 d\rho_k \Delta H_m, \tag{14}$$

where ΔH_m is the specific heat of fusion, related to melting temperature T_m and melting entropy ΔS_m by $\Delta H_m = T_m \Delta S_m$. Under normal conditions $\Delta H_m = 65$ cal/g for iron. The main part of this heat is used to increase the internal energy; only a small amount of heat is needed for work against pressure during expansion from the solid to the liquid phase. Verhoogen extrapolated ΔH_m to the conditions at the ICB under the assumption that ΔS_m is pressure independent, such that ΔH_m is proportional to T_m. This effectively means that the work against pressure during expansion can be neglected also at very high pressures. We prefer the view that this work is additional to Verhoogen's estimates of ΔH_m which are 100 cal/g for T_m = 2780 K and 135 cal/g for T_m = 3750 K.

Table 4 gives the power of latent heat production at the ICB, both according to Verhoogen's ΔH_m estimates (q_0) and after inclusion of a contribution q_G from volume contraction or, in other words, from release of gravitational energy (q_L). Two different

Table 4. Power of latent heat production at the ICB in watts. Values of q_G are for a density change of 1 per cent at the ICB, $t = 4 \times 10^9$ yr.

Growth condition	Verhoogen's (1961) estimate q_0 T_m=2780 °K	T_m=3750 °K	Contribution q_G from gravitational energy	Revised estimate $q_L = q_0 + q_G$
$\dot{r}_i = \frac{r_i}{t}$	$8.96 \cdot 10^{11}$	$1.21 \cdot 10^{12}$	$5.82 \cdot 10^{11}$	$\sim 1.6 \cdot 10^{12}$
$\dot{V}_i = \frac{V_i}{t}$*	$2.98 \cdot 10^{11}$	$4.03 \cdot 10^{11}$	$1.94 \cdot 10^{11}$	$\sim 5.5 \cdot 10^{11}$

*This means $\dot{r}_i = \frac{r_i}{3t}$

growth conditions of the inner core are assumed: linear increase with time of radius r_i on the one hand and of volume V_i on the other to the present value during 4×10^9 yr. The contribution q_G from gravitational energy follows from equation (12):

$$q_G = 4\pi r_i^2 \frac{\rho_i - \rho_k}{\rho_i} p(r_i)\dot{r}_i,$$

where $\dot{r}_i$ is the rate of change of r_i. Similarly, q_0 follows from equation (14):

$$q_0 = 4\pi r_i^2 \rho_k \Delta H_m \dot{r}_i,$$

where ΔH_m is one of Verhoogen's estimates. The contribution q_G leads to significantly revised estimates of the power q_L of total heat production, as soon as the density change at the ICB is greater than about 0.5 per cent, since then q_G is greater than the uncertainty of q_0.

Discussion and conclusions

(1) The (hypothetical) process of continuous crystallization of the Earth's core releases latent heat at the ICB, and a considerable part of this heat may come from the Earth's gravitational energy. The controlling factor for the size of this part is the density change at

the ICB. Thus, more energy may be available to sustain thermal convection in the outer core and hence the geodynamo than assumed so far.

(2) Continuous crystallization of the core would point to cooling. Cooling by convective heat transport is an efficient way to reduce the temperature gradients from superadiabatic to adiabatic and thus to stop convection and regeneration of the Earth's magnetic field. However, cooling in our model is slowed down by the concentrated heat production at the ICB, which maintains elevated temperatures there and elevated temperature gradients in the outer core. Thus, a convective regime in the outer core may exist for long times.

(3) Among the other effects of continuous crystallization of the core, that have been studied theoretically in this paper, the more interesting ones are a reduction of the Earth's radius by 5 km and additional hydrostatic pressures of the order of several kbar in the core and the mantle, under the assumption that the core crystallizes completely and for a density increase of 1 per cent upon crystallization. Self-gravitation of the Earth has a stronger influence on these results than the viscosity.

(4) All effects, discussed in this paper, and the validity of the underlying assumption that there is no discontinuous change in chemical composition at the ICB, depend on the value of the density change at the ICB. It is highly desirable that new attempts be made to determine this quantity without hypotheses and reliability.

(5) Continuous growth of the inner core due to cooling could possibly also take place in a core with discontinuous change in chemical composition at the ICB, namely when crystallization is connected with fractionation, such that the ICB separates a heavier solid fraction from a liquid fraction which is enriched in lighter components such as Si or S. The more or less constant *P*-wave velocities above the ICB in many recent core models may point to such an enrichment when the ICB is approached (Alder & Trigueros 1977). Our results would also apply in this case, provided that the volumes, occupied by the same particles before and after (partial) crystallization, are different. The overall density change upon crystallization would be different from the density change at the ICB, whereas in our model both are the same. In such a core, gravitational energy may additionally be converted into the energy of mechanical convection, as discussed by Gubbins (1977) and Loper (1978), and the release of gravitational energy would power the geodynamo both via thermal and mechanical convection.

Acknowledgments

This work was supported in part by a grant from the Deutsche Forschungsgemeinschaft. The computations were performed at the computing centre of the University of Karlsruhe. We are grateful to Karl Fuchs and Horst Stöckl for critically reading the manuscript, and to Sieglinde Lohrengel for typing it.

Contribution No. 171, Geophysical Institute, University of Karlsruhe.

References

Alder, B. J. & Trigueros, M., 1977. Suggestion of a eutectic region between the liquid and the solid core of the earth, *J. geophys. Res.*, **82**, 2535–2539.

Anderson, D. L. & Hart, R. S., 1976. An earth model based on free oscillations and body waves, *J. geophys. Res.*, **81**, 1461–1475.

Bolt, B. A., 1972. The density distribution near the base of the mantle and near the earth's centre, *Phys. Earth planet. Int.*, **5**, 301–311.

Bolt, B. A. & Qamar, A., 1970. Upper bound to the density jump at the boundary of the earth's inner core, *Nature*, **228**, 148–150.

Braginski, S. I., 1963. Structure of the *F*-layer and reasons for convection in the Earth's core, *Dokl. Akad. Nauk, SSSR*, **143**, 8–10.

Bullen, K. E., 1963. *Introduction to the Theory of Seismology*, Cambridge University Press, London.

Cathles, L. M., 1975. *The Viscosity of the Earth's Mantle*, Princeton University Press.

Derr, J. S., 1959. Internal structure of the earth inferred from free oscillations, *J. geophys. Res.*, **74**, 5202–5220.

Gubbins, D., 1977. Energetics of the Earth's core, *Geophys. J. R. astr. Soc.*, **48**, 453–464.

Häge, H., 1977. Kristallisations–und Schmelzprozesse an der Grenze des inneren Kerns und ihre Auswirkungen auf Dimensionen, Spannungen, Tageslänge und Gravitationsenergie der Erde, *Diploma thesis*, University of Karlsruhe.

Jacobs, J. A., 1953. The earth's inner core, *Nature*, **172**, 297.

Loper, D. E., 1978. The gravitationally powered dynamo, *Geophys. J. R. astr. Soc.*, **54**, 389–404.

Love, A. E. H., 1944. *A Treatise on the Mathematical Theory of Elasticity*, Dover Publications, New York.

MacDonald, G. J. F., 1963. The deep structure of oceans and continents, *Rev. Geophys.*, **1**, 587–665.

McLachlan, Jr & Ehlers, E. G., 1971. Effect of pressure on the melting temperature of metals, *J. geophys. Res.*, **76**, 2780–2789.

Müller, G. & Häge, H., 1979. Comments on 'The gravitationally powered dynamo' by D. E. Loper, *Geophys. J. R. astr. Soc.*, **58**, 509–514.

Peltier, W. R., 1974. The impulse response of a Maxwell earth, *Rev. Geophys. Space Phys.*, **12**, 649–669.

Verhoogen, J., 1961. Heat balance of the earth's core, *Geophys. J. R. astr. Soc.*, **4**, 276–281.