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CHANNEL FLOW OF A VISCOUS FLUID IN THE BOUNDARY LAYER

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1. Introduction. Let u be the temperature, ψ be a stream function, and x and t be, respectively, the coordinates perpendicular and parallel to the channel walls. Ockendon [5] described the flow of a viscous fluid in the boundary layer by the following initial-boundary value problem:

$$\begin{aligned} &\psi_{xx} = e^u, \\ &u_{xx} + \psi_x u_t - \psi_t u_x = -e^u, \\ &u(x,0) = 0, \quad \psi(x,0) = \tfrac{1}{2}x^2\left(1 - \tfrac{1}{3}\beta^{-1/2}x\right) \quad \text{for } x > 0, \\ &u(0,t) = 0, \quad \psi(0,t) = 0, \quad \psi_x(0,t) = 0 = \psi_t(0,t) \quad \text{for } t > 0, \end{aligned}$$

where β, called the Nahme-Griffith number, is assumed to be large. For some positive constant a ($\leq 2\beta^{1/2}$), let $\Omega = (0,a) \times (0,T)$, and $\partial\Omega$ be the parabolic boundary $([0,a] \times \{0\}) \cup (\{0,a\} \times (0,T))$ of Ω, where $T \leq \infty$. Ockendon found that as $x \to \infty$, $u \sim t/x$ and $\psi \sim x^2/2$. Since β is very large, $u \to 0$ as $x \to 2\beta^{1/2}$. Thus, the model of the channel flow of a fluid with temperature-dependent viscosity in the boundary layer is simplified to the following degenerate parabolic problem (cf. Lacey [4], Stuart and Floater [6], and Floater [3]),

$$u_{xx} - xu_t = -e^u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where u may blow up at some finite T.

Recently, Floater [3] studied the problem (1.1). He approximated the forcing term e^u by u^p. Instead of zero initial temperature, he considered the following problem:

$$\begin{gathered} u_{xx} - xu_t = -u^p \quad \text{in } (0,1) \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0 \quad \text{for } x \in [0,1], \qquad u(0,t) = 0 = u(1,t) \quad \text{for } t > 0, \end{gathered} \tag{1.2}$$

where $p > 1$ and u_0 is assumed to be in $C^1([0,1])$ with $u_0(0) = 0 = u_0(1)$. He showed that if $1 < p \leq 2$ and u_0 is concave, then u blows up at the boundary $x = 0$. Numerical

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evidence by Stuart and Floater [6] indicates that blow-up occurs away from the boundary when $p > 2$.

Let q be any real number, and

$$L \equiv \frac{\partial^2}{\partial x^2} - x^q \frac{\partial}{\partial t}.$$

When $q = 0$, L is the ordinary heat operator. When $q \neq 0$, the coefficient of $\partial/\partial t$ in L becomes zero or infinity at $x = 0$, and we say that L is a degenerate parabolic operator. Floater [3] generalized the problem (1.2) by replacing the equation with

$$Lu = -u^p \quad \text{in } (0,1) \times (0,\infty)$$

for any $q > 0$, and he found that blow-up at the boundary $x = 0$ can occur when $1 < p \leq q + 1$.

To study the problem (1.1), we use a different approach. Let

$$h_r(u) = \left(1 - \frac{u}{r}\right)^{-r}.$$

We note that for each fixed r (> 0), $e^u < h_r(u)$ for $u \in (0, r)$. Also, $\lim_{r\to\infty} h_r(u) = e^u$. Let us consider the following problem:

$$Lu = -h_r(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

We see that the problem (1.1) is the limiting case of the problem (1.3) with $q = 1$ as r tends to infinity. We also note that for each fixed r, $h_r(0) = 1$, $h_r' > 0$, $h_r'' > 0$, $\lim_{u\to r^-} h_r(u) = \infty$, and $\int_0^r h_r(u)\,du = \infty$ for $r > 1$. This motivates us to study the more general degenerate parabolic quenching problem,

$$Lu = -f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.4}$$

for $q \neq 0$, where $f \in C^2([0, c))$ for some positive constant c such that $f(0) > 0$, $f' > 0$, $f'' \geq 0$, $\lim_{u\to c^-} f(u) = \infty$, and $\int_0^c f(u)\,du = \infty$. By quenching phenomena, we mean the blow-up of u_t at some finite T and existence of a unique critical length a^* (which is the length such that for $a < a^*$, u exists for all t (> 0), and for $a > a^*$, u reaches c somewhere at some finite T (and u_t blows up there)).

Chan and Kong [1] showed that the problem (1.4) has a unique classical solution u such that u and u_t are positive in Ω, and there exists a unique critical length a^*, which is the same as that for the case $q = 0$; if $\int_0^c f(u)\,du = M$ for some positive constant M and $a > a^*$, then u reaches c somewhere in

$$\begin{aligned} &[c^2/(2Ma), a/2] && \text{for } q > 0, \\ &[a/2, a - c^2/(2Ma)] && \text{for } q < 0, \end{aligned}$$

and $\lim_{(x,t)\to(x_0,T)} u_t(x,t) = \infty$ if $\lim_{(x,t)\to(x_0,T)} u(x,t) = c$ for some finite T.

In Sec. 2, we show that without assuming $\int_0^c f(u)\,du = M$, if $a > a^*$, then u reaches c somewhere in

$$\begin{aligned} &[0, a/2] && \text{for } q > 0, \\ &[a/2, a] && \text{for } q < 0; \end{aligned}$$

if $\int_0^c f(u)\,du = \infty$, then u_t becomes unbounded somewhere when u reaches c at some finite T.

2. Blow-up of u_t. The following result is given by Chan and Kong [1].

THEOREM 1. The problem (1.4) has a unique classical solution u such that u and u_t are positive in Ω.

We remark that if quenching does not occur, then Theorem 1 gives the existence result for all $t > 0$.

Let $\omega_- = (0, a/2) \times (0, T)$ and $\omega_+ = (a/2, a) \times (0, T)$. Since $u_t > 0$ in Ω, we use the strong maximum principle to prove the following lemma.

LEMMA 2. For the problem (1.4), (i) if $q > 0$, then $u_x(x,t) < 0$ in ω_+; (ii) if $q < 0$, then $u_x(x,t) > 0$ in ω_-.

Proof. (i) Let α be a positive number less than or equal to $a/2$, $\Omega_\alpha = (a - 2\alpha, a - \alpha) \times (0, T)$, and $y(x,t) = u(x,t) - u(2a - 2\alpha - x, t)$. For $x \in (a - 2\alpha, a - \alpha)$, we have $2a - 2\alpha - x > x$. It follows from $q > 0$ and $u_t > 0$ in Ω that

$$(L + f'(\xi))y < 0 \quad \text{in } \Omega_\alpha,$$

where $\xi(x,t)$ is between $u(x,t)$ and $u(2a - 2\alpha - x, t)$. Since $u = 0$ on the parabolic boundary $\partial\Omega$, we have $y(a - 2\alpha, t) \geq 0$, and $y(x, 0) = 0$. Also, $y(a - \alpha, t) = 0$. Hence, $y > 0$ in Ω_α by the strong maximum principle. In particular, for any ϵ such that $0 < \epsilon < \alpha$,

$$u(a - \alpha - \epsilon, t) > u(a - \alpha + \epsilon, t).$$

Thus, $u_x(a - \alpha, t) \leq 0$. From this,

$$u_x(x,t) \leq 0 \quad \text{for } x \in \left[\frac{a}{2}, a\right).$$

We note that

$$(u_x)_{xx} - x^q(u_x)_t + f'(u)u_x = qx^{q-1}u_t.$$

Since $q > 0$ and $u_t > 0$, we have

$$(L + f'(u))u_x > 0 \quad \text{in } \omega_+.$$

If u_x attains its maximum 0 at some point (say, (x^*, σ)) in ω_+, then by the strong maximum principle, $u_x(x,t) \equiv 0$ in $(a/2, a) \times (0, \sigma]$, and hence $(L + f'(u))u_x = 0$ there. This contradiction shows that $u_x(x,t) < 0$ in ω_+.

(ii) The proof is similar to that for (i).

From the proofs of Theorems 5 and 6 of Chan and Kong [1], the problem (1.4) has a unique critical length a^*. When $a > a^*$, it follows from Lemma 2 that u reaches c somewhere in $[0, a/2]$ for $q > 0$, and in $[a/2, a]$ for $q < 0$.

We modify the idea in the proof of Theorem 1 of Chan and Kwong [2] to prove the following result.

THEOREM 3. For the problem (1.4), suppose $\int_0^c f(u)\, du = \infty$. As $t \to T^-$, if $u(x,t) \to c^-$ at some $x \in [0, a]$, then $u_t(x,t) \to \infty$ at (at least) one of such x.

Proof. Suppose u_t is bounded on $\overline{\Omega}$. Let us choose a positive number K such that $u_t < K$ on $\overline{\Omega}$ and $K > 8c/a^2$.

We first consider the case $q > 0$. Since $\lim_{u \to c^-} f(u) = \infty$, there exists a constant β_1 ($> 3c/4$) such that

$$f(u) > (1 + a^q)K \quad \text{for } u \in [\beta_1, c). \tag{2.1}$$

It follows from Lemma 2(i) and $\int_0^c f(u)\,du = \infty$ that we can choose a constant τ_1 close to T such that $u(x, \tau_1)$ attains a local maximum at some point $x^* \in (0, a/2]$ with the properties $u(x^*, \tau_1) > \beta_1$ and

$$\int_{\beta_1}^{u(x^*,\tau_1)} f(s)\,ds > \frac{1 + a^q}{2}\left(\frac{4\beta_1}{a} + \frac{Ka^{1+q}}{8}\right)^2. \tag{2.2}$$

When $u > \beta_1$, $u_{xx} < a^q K - f(u)$. By (2.1), $u_{xx} < -K < 0$. Hence, there exists x_1 such that $x^* < x_1 < a$, $u(x_1, \tau_1) = \beta_1$, and $u_x(x, \tau_1) < 0$ in (x^*, x_1). For $x \in (x^*, x_1)$, we have $u(x, \tau_1) > \beta_1$. It follows from $q > 0$, $u_t < K$, and (2.1) that

$$\begin{aligned} u_{xx}(x, \tau_1) &< a^q K - f(u(x, \tau_1)) \\ &< -\frac{1}{1 + a^q} f(u(x, \tau_1)). \end{aligned}$$

Since $u_x(x, \tau_1) < 0$ in (x^*, x_1), we have

$$u_x(x, \tau_1) u_{xx}(x, \tau_1) > -\frac{1}{1 + a^q} f(u(x, \tau_1)) u_x(x, \tau_1).$$

Upon integrating from x^* to x_1, we obtain

$$\frac{u_x^2(x_1, \tau_1)}{2} > \frac{1}{1 + a^q} \int_{\beta_1}^{u(x^*,\tau_1)} f(s)\,ds.$$

By (2.2),

$$u_x^2(x_1, \tau_1) > \left(\frac{4\beta_1}{a} + \frac{Ka^{1+q}}{8}\right)^2.$$

Since $u_x(x_1, \tau_1) \le 0$, we have

$$u_x(x_1, \tau_1) < -\left(\frac{4\beta_1}{a} + \frac{Ka^{1+q}}{8}\right). \tag{2.3}$$

We claim that $a - x_1 > a/4$. For $x \in (x^*, x_1)$, we have $u(x, \tau_1) > \beta_1$. Thus, $u_{xx}(x, \tau_1) < -K$, and we have

$$\int_{x^*}^{x_1} \int_{x^*}^{x} u_{\xi\xi}(\xi, \tau_1)\,d\xi\,dx < -K \int_{x^*}^{x_1} \int_{x^*}^{x} d\xi\,dx,$$

which gives

$$\beta_1 - u(x^*, \tau_1) < -\frac{K}{2}(x_1 - x^*)^2.$$

Since $u < c$, $\beta_1 > 3c/4$, and $K > 8c/a^2$, we have $x_1 - x^* < a/4$. It follows from $x^* \in (0, a/2]$ that $a - x_1 = a - x^* - (x_1 - x^*) > a/4$.

Since $u_t < K$ and $f(u) > 0$, we have $u_{xx}(x, \tau_1) < a^q K$. By (2.3),

$$\int_{x_1}^{x_1+\frac{a}{4}} \int_{x_1}^{x} u_{\xi\xi}(\xi, \tau_1)\, d\xi\, dx < a^q K \int_{x_1}^{x_1+\frac{a}{4}} \int_{x_1}^{x} d\xi\, dx$$

gives

$$u\left(x_1 + \frac{a}{4}, \tau_1\right) < \beta_1 - \left(\frac{4\beta_1}{a} + \frac{K a^{1+q}}{8}\right)\left(\frac{a}{4}\right) + \frac{K a^q}{2}\left(\frac{a}{4}\right)^2 = 0.$$

This contradicts $u > 0$ in Ω.

Now, we consider the case $q < 0$. By Lemma 2(ii), $u(x,t) < c$ for $x \in (0, a/2)$. Thus, there exists a positive constant b ($< c$) such that $u(a/8, t) \le b$ for $t > 0$. Since $\lim_{u \to c^-} f(u) = \infty$, there exists a constant β_2 such that $\beta_2 > \max\{3c/4, b\}$ and

$$f(u) > \left[1 + \left(\frac{a}{8}\right)^q\right] K \quad \text{for } u \in [\beta_2, c). \tag{2.4}$$

It follows from Lemma 2(ii) and $\int_0^c f(u)\, du = \infty$ that we can choose a constant τ_2 close to T such that $u(x, \tau_2)$ attains a local maximum at some point $x_* \in [a/2, a)$ with the properties $u(x_*, \tau_2) > \beta_2$ and

$$\int_{\beta_2}^{u(x_*, \tau_2)} f(s)\, ds > \frac{1}{2}\left[1 + \left(\frac{a}{8}\right)^q\right]\left[\frac{8\beta_2}{a} + \frac{K}{2}\left(\frac{a}{8}\right)^{1+q}\right]^2. \tag{2.5}$$

When $u > \beta_2$, we have $x > a/8$. It follows from $q < 0$ and (2.4) that

$$u_{xx} < \left(\frac{a}{8}\right)^q K - f(u) < -K < 0.$$

Hence, there exists x_2 such that $a/8 < x_2 < x_*$, $u(x_2, \tau_2) = \beta_2$, and $u_x(x, \tau_2) > 0$ in (x_2, x_*). Then, it follows from $u_t < K$ and (2.4) that for $x \in (x_2, x_*)$,

$$\begin{aligned} u_{xx}(x, \tau_2) &< \left(\frac{a}{8}\right)^q K - f(u(x, \tau_2)) \\ &< -\frac{1}{1 + (a/8)^q} f(u(x, \tau_2)). \end{aligned}$$

Since $u_x(x, \tau_2) > 0$ in (x_2, x_*), we have

$$u_x(x, \tau_2) u_{xx}(x, \tau_2) < -\frac{1}{1 + (a/8)^q} f(u(x, \tau_2)) u_x(x, \tau_2).$$

Integrating from x_2 to x_*, we have

$$-\frac{u_x^2(x_2, \tau_2)}{2} < -\frac{1}{1 + (a/8)^q} \int_{\beta_2}^{u(x_*, \tau_2)} f(s)\, ds.$$

By (2.5),

$$u_x^2(x_2, \tau_2) > \left[\frac{8\beta_2}{a} + \frac{K}{2}\left(\frac{a}{8}\right)^{1+q}\right]^2.$$

Since $u_x(x_2, \tau_2) \geq 0$, we have

$$u_x(x_2, \tau_2) > \frac{8\beta_2}{a} + \frac{K}{2}\left(\frac{a}{8}\right)^{1+q}. \tag{2.6}$$

We claim that $x_2 > a/4$. For $x \in (x_2, x_*)$, we have $u(x, \tau_2) > \beta_2$. By (2.4), $u_{xx}(x, \tau_2) < -K$. Then,

$$\int_{x_2}^{x_*} \int_{x}^{x_*} u_{\xi\xi}(\xi, \tau_2)\, d\xi\, dx < -K \int_{x_2}^{x_*} \int_{x}^{x_*} d\xi\, dx,$$

which gives

$$\beta_2 - u(x_*, \tau_2) < -\frac{K}{2}(x_* - x_2)^2.$$

Since $u < c$, $\beta_2 > 3c/4$, and $K > 8c/a^2$, we have $x_* - x_2 < a/4$. It follows from $x_* \in [a/2, a)$ that $x_2 = x_* - (x_* - x_2) > a/4$.

Since $u_t < K$ and $f(u) > 0$, we have $u_{xx}(x, \tau_2) < (a/8)^q K$ for $x > a/8$. Thus,

$$\int_{x_2 - \frac{a}{8}}^{x_2} \int_{x}^{x_2} u_{\xi\xi}(\xi, \tau_2)\, d\xi\, dx < \left(\frac{a}{8}\right)^q K \int_{x_2 - \frac{a}{8}}^{x_2} \int_{x}^{x_2} d\xi\, dx$$

gives

$$u\left(x_2 - \frac{a}{8}, \tau_2\right) < \beta_2 - \frac{a}{8}\left[u_x(x_2, \tau_2) - \frac{K}{2}\left(\frac{a}{8}\right)^{1+q}\right].$$

By (2.6), $u(x_2 - a/8, \tau_2) < 0$. This contradicts $u > 0$ in Ω. Hence, the theorem is proved.

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