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Chaos and Entropy for Circle Maps

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Abstract. Our aim is to check that the notions of positive entropy, chaos in the sense of Devaney and ω-chaos are equivalent for the circle maps.

Some notions of 'chaos' are introduced in discrete dynamical systems. We are interested in relationships between those notions. For any continuous map of the compact interval into itself, Li([Li2]) showed that it is chaotic in the sense of Devaney if and only if it has positive entropy. On the other hand, for continuous maps of the circle into itself, Kuchta([Ku]) gave a characterization of chaos in the sense of Li-Yorke. In the present paper we show that the result due to Li is also true for maps of the circle.

Devaney([De]) noticed that any member in class of maps called 'chaotic' has properties of topological transitivity, the periodic points which are dense and sensitive dependence on initial conditions.

Let (X, d) be a compact metric space and $C(X)$ denote the set of continuous maps of X into itself. For $f \in C(X)$ we say that a set $A \subset X$ is *f-invariant* if $f(A) \subset A$, and that f is *topologically transitive* if for every pair of non-empty open sets U and V in X, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$. $f \in C(X)$ is said to be *chaotic in the sense of Devaney* if there is an f-invariant closed infinite set $D \subset X$ such that the following conditions hold:

(D1) $f|_D$ is topologically transitive and

(D2) $\mathrm{Per}(f|_D)$ is dense in D,

where $f|_D$ denotes the restriction of f to D and $\mathrm{Per}(f|_D)$ is the set of all periodic points of $f|_D$. D is called a *chaotic* set (see [De] and [Li2]). It is well known that if D is chaotic then $f|_D$ has sensitive dependence on initial conditions, i.e. there exists a $\delta > 0$ such that for any $x \in D$ there exists a sequence y_k of points in D which converges to x, and a sequence n_k of positive integers such that $d(f^{n_k}(y_k), f^{n_k}(x)) > \delta$ (cf. [BBCDS], [Si] and [GW]). Remark that if D is a subinterval of $\mathbf{R}$, then (D1) implies (D2) (cf. [Li1], [Si] and [VB]).

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On the other hand, for any continuous map f of the interval into itself, Li ([Li2]) introduced the following notion of ω-chaos, and showed that f is ω-chaotic if and only if f has positive entropy. A subset S of X is called an *ω-scrambled set* for f if, for any $x, y \in S$ with $x \neq y$, the following conditions hold:

(ω1) $\omega(x, f) \setminus \omega(y, f)$ is uncountable,

(ω2) $\omega(x, f) \cap \omega(y, f) \neq \emptyset$ and

(ω3) $\omega(x, f) \not\subset \mathrm{Per}(f)$,

where $\omega(x, f)$ is an ω-limit set of a point $x \in X$. f is called *ω-chaotic* if there exists an uncountable ω-scrambled set for f (see [Li2]).

Our aim is to show the following:

MAIN THEOREM. *Let f be a continuous map of the circle into itself. The following conditions are equivalent.*

(I) *f has positive topological entropy.*

(II) *There is an uncountable ω-scrambled set S such that $\bigcap_{x\in S} \omega(x, f) \neq \emptyset$.*

(III) *f is ω-chaotic.*

(IV) *There is an ω-scrambled set consisting of exactly two points.*

(V) *f is chaotic in the sense of Devaney.*

(VI) *There is a chaotic set D which contains an uncountable ω-scrambled set S.*

Let S^1 and I denote the unit circle and the interval respectively. For maps of the circle, Silverman showed that a topologically transitive map has a periodic point if and only if the entire space S^1 is chaotic (see [Si, Theorem 7.1]). Kuchta ([Ku]) proved remarkable results for continuous maps of the circle with zero topological entropy, which will be used later in the proof of Main Theorem (see Lemmas 5 and 10 stated below).

We denote by $h(f)$ the topological entropy of f. Let $f \in C(S^1)$ and $\deg(f)$ denote the degree of f. If $\deg(f) \neq 0, \pm 1$, then $h(f) \geq \log|\deg(f)| > 0$ (see [ALM, p. 263]). F denotes a lift of f. If $\deg(f) = 1$, then

$$\rho(F) = \left\{ \limsup_{n\to\infty} \frac{F^n(x) - x}{n} : x \in \mathbf{R} \right\}$$

is either one point or a compact interval, and moreover, f has periodic points whose periods are denominators of rational numbers contained in $\rho(F)$ (see [It]). It can be easily checked that $h(f) > 0$ if and only if $\rho(F)$ is not one point. (The proof is clear by Lemma 4 stated below.)

Applying Main Theorem, we have the following corollary.

COROLLARY. *Let $f \in C(S^1)$. Then,*

(1) *if* $\deg(f) \neq 0, \pm 1$, *then f is chaotic in the sense of Devaney,*

(2) *in the case when* $\deg(f) = 1$, *f is chaotic in the sense of Devaney if and only if $\rho(F)$ is not one point.*

Li and Yorke ([LY]) introduced another notion of chaos. We say that $f \in C(X)$ is *chaotic in the sense of Li-Yorke* if there is an uncountable set $C \subset X$ such that for any $x, y \in C$ with $x \neq y$, the following conditions hold:

(L1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$ and

(L2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

If $h(f) > 0$ ($f \in C(I)$ or $C(S^1)$), then f is chaotic in the sense of Li-Yorke. However, the converse is not true in general (cf. [Sm], [BC, §VI.3] and [Ku]).

We say that a point $x (\in I$ or $S^1)$ is *homoclinic* if there exists a point y ($x \neq y$) such that

$$f^n(y) = y\,, \quad x \in \bigcap_{\epsilon>0} \bigcup_{m\geq 0} f^{nm}(\{a : d(a, y) < \epsilon\}) \quad \text{and} \quad f^{nk}(x) = y$$

for some $n, k > 0$. In the case of $f \in C(I)$, we know that $h(f) > 0$ if and only if f has a homoclinic point (cf. [BC, Proposition III.21]). When $f \in C(S^1)$, $h(f) > 0$ implies that f has a homoclinic point. The converse, however, is not true (cf. [BCMN] and [BC, Theorem IX.28]).

REMARK. Let (X, d) be a compact metric space. Then $f \in C(X)$ is chaotic in the sense of Devaney if and only if f^n is chaotic in the sense of Devaney for $n > 0$.

Indeed, suppose that f^n is chaotic in the sense of Devaney. Let D' denote a chaotic set for f^n. Then it is easily checked that $D = \bigcup_{i=0}^{n-1} f^i(D')$ is a chaotic set for f. Conversely, suppose that f is chaotic in the sense of Devaney. Let D denote a chaotic set for f. Since $f|_D$ is topologically transitive, there exists a point $x \in D$ such that $\overline{\mathrm{Orb}(x, f)} = D$. Here $\bar{E}$ denotes the closure of E. Put $D' = \overline{\mathrm{Orb}(x, f^n)}$, and then it is easily checked that D' is a closed infinite set. Since $\overline{\mathrm{Orb}(x, f)} = \omega(x, f) = \bigcup_{i=0}^{n-1} \omega(f^i(x), f^n)$, we have $\overline{\mathrm{Orb}(f^i(x), f^n)} = \omega(f^i(x), f^n) = f^i(\omega(x, f^n)) = f^i(\overline{\mathrm{Orb}(x, f^n)})$ for $i \geq 0$. Then $f^n(D') = D'$ and $f^n|_{D'}$ is topologically transitive. Moreover, $\overline{\mathrm{Per}(f) \cap \omega(f^i(x), f^n)} = \omega(f^i(x), f^n)$ for $i \geq 0$, which implies $\overline{\mathrm{Per}(f^n) \cap D'} = D'$. Then $\mathrm{Per}(f^n|_{D'})$ is dense in D'. Thus D' is a chaotic set for f^n.

To show Main Theorem we decompose the class $C(S^1)$ into the following three disjoint sets:

$$\mathcal{F}_0 = \{f \in C(S^1) : h(f) = 0,\ \mathrm{Per}(f) = \emptyset\}\,,$$

$$\mathcal{F}_1 = \{f \in C(S^1) : h(f) = 0,\ \mathrm{Per}(f) \neq \emptyset\} \quad \text{and}$$

$$\mathcal{F}_2 = \{f \in C(S^1) : h(f) > 0\}\,.$$

It is known that $f \in \mathcal{F}_0$ if and only if f is topologically semiconjugated to an irrational rotation of the circle (see [Ku, Theorem B]), and that $f \in \mathcal{F}_2$ if and only if there exists a closed set Y and $n > 0$ such that $f^n(Y) = Y$ and $f^n|_Y$ is at most two-to-one topologically semiconjugated to the one-sided shift map (cf. [BC, Theorem IX.28] and [Li2]).

In proof of Main Theorem, implications (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) and (VI) $\Rightarrow$ (V) are obvious, and (I) $\Rightarrow$ (II) and (I) $\Rightarrow$ (VI) are obtained from the following lemma:

LEMMA 1. *If $f \in \mathcal{F}_2$ then there are a chaotic set D and an uncountable ω-scrambled set $S \subset D$ such that $\bigcap_{x \in S} \omega(x, f) \neq \emptyset$.*

The proof of Lemma 1 is similar to that in [Li2, Chapter 4] and so we omit the proof. Therefore, to conclude Main theorem we give only the proof of (IV) ⇒ (I) and (V) ⇒ (I), which are described by the following propositions.

PROPOSITION 2. *If $f \in \mathcal{F}_0 \cup \mathcal{F}_1$ then f is not chaotic in the sense of Devaney.*

PROPOSITION 3. *If $f \in \mathcal{F}_0 \cup \mathcal{F}_1$ then there is no ω-scrambled set consisting of exactly two points.*

We denote by $\pi : \mathbf{R} \to S^1$ the natural projection. For $f \in C(X)$ and $x \in X$ we denote by $\mathrm{Fix}(f)$ the set of all fixed points of f, and denote by $\mathrm{Orb}(x, f)$ the orbit of f.

LEMMA 4 [BC, Theorem IX.28]. *$f \in C(S^1)$ belongs to $\mathcal{F}_1$ if and only if $\mathrm{Per}(f) \neq \emptyset$ and for any periods n, k $(n < k)$ of periodic points of f, k/n is a power of two.*

LEMMA 5 [Ku, Lemma 2.3]. *Let $g \in \mathcal{F}_1$ with $\mathrm{Fix}(g) \neq \emptyset$. Then there exist $G : \mathbf{R} \to \mathbf{R}$ a lift of g and a compact interval $I' \subset \mathbf{R}$ with length greater than one such that $G(I') \subset I'$.*

LEMMA 6. *Let $f \in \mathcal{F}_1$. If $y_1, y_2 \in \omega(x, f)$ for $x \in S^1$, then $\omega(y_1, f) = \omega(y_2, f)$.*

PROOF. For $f \in \mathcal{F}_1$ there exists $n > 0$ such that $\mathrm{Fix}(f^n) \neq \emptyset$. Put $g = f^n$ and then $g \in \mathcal{F}_1$. By Lemma 5 there exist G a lift of g and a compact interval $I' \subset \mathbf{R}$ with length greater than one. Since $y_1, y_2 \in \omega(x, f) = \bigcup_{i=0}^{n-1} \omega(f^i(x), g)$, we have $z_1 = f^{\ell}(y_1), z_2 = f^m(y_2) \in \omega(x, g)$ for some ℓ, m. Then there exist $x' \in (\pi|_{I'})^{-1}(\{x\})$, $z_1' \in (\pi|_{I'})^{-1}(\{z_1\})$ and $z_2' \in (\pi|_{I'})^{-1}(\{z_2\})$ such that $z_1', z_2' \in \omega(x', G|_{I'})$. From Lemma 4 it is easily checked that each period of periodic points of $G|_{I'}$ is a power of two. By [BC, Proposition VI.7] we have

$$\omega(z_1, g) = \pi|_{I'}(\omega(z_1', G|_{I'})) = \pi|_{I'}(\omega(z_2', G|_{I'})) = \omega(z_2, g)\,.$$

Therefore,

$$\omega(y_1, f) = \bigcup_{i=0}^{n-1} \omega(f^i(z_1), g) = \bigcup_{i=0}^{n-1} \omega(f^i(z_2), g) = \omega(y_2, f)\,. \qquad \square$$

For $f \in \mathcal{F}_1$ we denote $\Lambda(f) = \bigcup_{x \in S^1} \omega(x, f)$ and $\Lambda^2(f) = \bigcup_{x \in \Lambda(f)} \omega(x, f)$.

LEMMA 7. *For $f \in \mathcal{F}_1$, $x \in \Lambda^2(f)$ implies $x \in \omega(x, f)$.*

PROOF. Let $x \in \Lambda^2(f)$. Then by the definition of $\Lambda^2(f)$, there are points $a, b \in S^1$ such that $x \in \omega(a, f)$ and $a \in \omega(b, f)$. Then we have $a, x \in \omega(b, f)$. Therefore, by Lemma 6, it holds that $(x \in)\omega(a, f) = \omega(x, f)$. $\square$

LEMMA 8. *If $f \in \mathcal{F}_1$ then $\Lambda(f) \setminus \Lambda^2(f)$ is a countable set.*

PROOF. Let n, g, G, I' be defined as in the proof of Lemma 6. Then the set $\Lambda(G|_{I'}) \setminus \Lambda^2(G|_{I'})$ is countable (cf. [Xi, Theorems 1 and 2]). Since

$$\Lambda(f) \setminus \Lambda^2(f) = \Lambda(g) \setminus \Lambda^2(g) \subset \pi(\Lambda(G|_{I'}) \setminus \Lambda^2(G|_{I'})),$$

the conclusion is obtained. □

THE PROOF OF PROPOSITION 2. Proposition 2 is true for $f \in \mathcal{F}_0$, and so we check the case of $f \in \mathcal{F}_1$. Suppose f has a chaotic set D for f. Since $f|_D$ is topologically transitive, there exists $a \in D$ such that $\overline{\mathrm{Orb}(a, f)} = D$. Since $\mathrm{Per}(f|_D)$ is dense in D, there exist $p_1, p_2 \in \mathrm{Per}(f|_D)$ such that the orbits for p_1 and p_2 are distinct. Then we have $\omega(p_1, f) \cap \omega(p_2, f) = \mathrm{Orb}(p_1, f) \cap \mathrm{Orb}(p_2, f) = \emptyset$. Since $p_1, p_2 \in \omega(a, f)$, Lemma 6 implies $\omega(p_1, f) = \omega(p_2, f)$. This is a contradiction. Therefore f is not chaotic in the sense of Devaney.

THE PROOF OF PROPOSITION 3 FOR $f \in \mathcal{F}_1$. The proof for maps of the circle is similar to that for maps of the interval (cf. [Li2]). Suppose $\{y, z\}$ $(y \neq z)$ is an ω-scrambled set. Then there exists $u \in \omega(y, f) \cap \omega(z, f)$. By Lemmas 6 and 7, $\omega(y, f) \cap \Lambda^2(f) = \omega(u, f) = \omega(z, f) \cap \Lambda^2(f)$. Thus $\omega(y, f) \setminus \omega(z, f) \subset \Lambda(f) \setminus \Lambda^2(f)$. By Lemma 8, $\Lambda(f) \setminus \Lambda^2(f)$ is countable, thus a contradiction. Therefore there is no ω-scrambled set consisting of exactly two points.

THE PROOF OF PROPOSITION 3 FOR $f \in \mathcal{F}_0$. Let $f \in \mathcal{F}_0$. We have $\deg(f) = 1$ since $f \in \mathcal{F}_0$ has no periodic points. We say that a set $\emptyset \neq A \subset X$ is a *minimal set* if $\omega(a, f) = A$ for every $a \in A$. To obtain the conclusion, we need the following Lemmas 9 and 10.

LEMMA 9 [Mi, Theorem A]. *Let $f \in \mathcal{F}_0$ and $F : \mathbf{R} \to \mathbf{R}$ be a lift of f. Then there exists an uncountable minimal set $A \subset S^1$ such that $F|_{\pi^{-1}(A)}$ is a nondecreasing function.*

LEMMA 10 [Ku, Corollary 3.4 and Lemma 3.7]. *Let f, F, A be as in Lemma 9. Then, for any $x, y \in \pi^{-1}(A)$, $(x, y) \cap \pi^{-1}(A) = \emptyset$ implies*

$$F([x, y]) = [F(x), F(y)] \quad \textit{and} \quad (F(x), F(y)) \cap \pi^{-1}(A) = \emptyset.$$

Let A be the minimal set constructed in Lemma 9. Then $\omega(x, f) = A$ for $x \in S^1$. Indeed, for $x \in S^1$ there exist $x' \in \pi^{-1}(\{x\})$ and $a', b' \in \pi^{-1}(A)$ such that $x' \in [a', b']$ and $(a', b') \cap \pi^{-1}(A) = \emptyset$. By using Lemma 10 inductively we have that

$$F^n([a', b']) = [F^n(a'), F^n(b')] \quad \text{and} \quad (F^n(a'), F^n(b')) \cap \pi^{-1}(A) = \emptyset \tag{1}$$

for any $n > 0$. Put $J = \pi([a', b'])$ and $a = \pi(a')$. By (1) it is easily checked that for $n, m \in \mathbf{N}$, $\mathrm{Int}(f^n(J)) = \mathrm{Int}(f^m(J))$ or $\mathrm{Int}(f^n(J)) \cap \mathrm{Int}(f^m(J)) = \emptyset$, where $\mathrm{Int}(Z)$ denotes the interior of Z. Since $\mathrm{Per}(f) = \emptyset$, we have $\mathrm{Int}(f^n(J)) \cap \mathrm{Int}(f^m(J)) = \emptyset$ for $n \neq m$, and so $\lim_{n\to\infty} d(f^n(x), f^n(a)) = 0$ because $f^n(x), f^n(a) \in f^n(J)$. Since A is a minimal set, $\omega(x, f) = \omega(a, f) = A$.

By the above fact it follows that $\omega(x, f) = \omega(y, f)$ for $x, y \in S^1$. This shows that there is no ω-scrambled set consisting of exactly two points.

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