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Chaos in a Model of Credit Cycles with Good and Bad Projects

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Abstract

We consider a credit cycle model introduced by Matsuyama, which is defined by a one-dimensional piecewise smooth map with upward, downward and flat branches. We offer a detailed analysis of this model for the case where asymptotic dynamics does not involve the flat branch, under the additional assumption that the production function is Cobb-Douglas. In particular, using skew tent map (which is a one-dimensional map defined by two linear functions) as a border collision normal form we obtain conditions of abrupt transition from an attracting fixed point to an attracting cycle or a chaotic attractor (cyclic chaotic intervals). These conditions allow us to describe the overall bifurcation structure of the parameter space of the map in a neighborhood of the boundary related to the border collision bifurcation of the fixed point. Such a structure confirms, in particular, that chaotic attractors of the considered map are robust, that is, they are persistent under parameter perturbations.

Keywords: one-dimensional piecewise smooth map, border collision bifurcation, skew tent map, borrower net worth, composition of credit flows, financial instability

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1. Introduction

The idea that market mechanisms are inherently dynamically unstable can be traced back at least to Goodwin [12]. Recent events have also renewed interest in the hypothesis that financial frictions are responsible not only for amplifying the effects of exogenous shocks but also for causing macroeconomic instability (see, e.g., [17] and [24]). Although a vast majority of the macroeconomics literature on financial frictions that follow the seminal work of [6] and [18] continue to study amplification effects of financial frictions within a setting that ensures the existence of a *stable steady state* toward which the economy would gravitate in the absence of recurring exogenous shocks, there exist several micro-founded, dynamic general equilibrium models of financial frictions, in which the steady state is *unstable* and persistent fluctuations occur *without exogenous shocks*; see, for example, [1], [3], [21]. The model developed by Matsuyama ([20], [22]) which we study in the present paper is in the same vein. It generates endogenous fluctuations of borrower net worth and aggregate investment. This model considers an overlapping-generation economy in which entrepreneurs arrive sequentially. When they arrive, they first sell their labor and other inputs to the production of the consumption good to acquire some net worth, which they could later use to finance their own projects or lend to finance the projects run by others. There are two types of projects, *the Good* and *the Bad*. The Good projects produce capital, which contributes to the production, together with labor and other inputs supplied by others who could undertake projects in the future. By competing for these inputs, more Good projects drive up the prices of these inputs, thereby improving the net worth of next generations of entrepreneurs who supply these inputs. This also means that they are subject to diminishing returns. In contrast, the Bad projects are independently profitable as they directly produce the consumption good. In other words, they do not require the inputs supplied by others. This means that they fail to improve the net worth of next generations of entrepreneurs, and that they are not subject to diminishing returns. However, they are subject to borrowing constraints because their revenues are not fully pledgeable, which means that the entrepreneurs need to have some net worth of their own to finance them. In this setting, [22] showed that the trajectory of the economy is described by a one-dimensional map in the entrepreneurs net worth, which consists of upward, downward, and flat branches, as follows:

- When the current net worth is low, the entrepreneurs cannot finance the Bad projects, because the borrowing constraint is too tight. All credit thus flows into the Good projects, even after the rate of return of the Good projects become lower than that of the Bad projects, and hence, a higher current net worth leads to a higher net worth in the next period. This explains the *upward branch* of the model.

- When the current net worth is in the intermediate range, the Bad projects are financed but still subject to the borrowing constraint, so that the rate of return of the Bad projects remain strictly higher than that of the Good projects. Thus, a higher current net worth, by easing the borrowing constraint of the Bad projects, thereby making them more appealing to the lenders, reduces the credit flow to the Good, which causes a net worth decline in the next period. This explains the *downward branch* of the model.

- When the current net worth is high, the Bad projects are no longer subject to the borrowing constraint, so that both Good and Bad projects earn the same rate of return in equilibrium. With the Good being subject to diminishing returns, all additional credit flow into the Bad, not at all to the Good, hence the net worth in the next period, is independent of the current net worth. This explains the *flat branch.*

Furthermore, as discussed in [30], when the production function is Cobb-Douglas, the map depends on four parameters, the share of capital in the Cobb-Douglas production function (α), the profitability of the Bad projects (B); the pledgeability of the Bad projects (μ); and the fixed investment size of the Bad projects (m). The bifurcation structure of this map differs significantly depending on whether the constant branch is involved into asymptotic dynamics. In the present paper we propose a detailed analysis of dynamics of the map in case when the constant branch does not participate in the asymptotic dynamics. Our companion paper [30] offers a detailed study of the case where all three branches are involved. It is characterized by periodicity regions related to superstable cycles existing due the constant branch of the function. It is shown that these regions are ordered according to the well known *U-sequence* characteristic for unimodal maps (first described in [23], see also [13]) which is adjusted to the considered map.

The map which defines the model belongs to a class of one-dimensional (1D for short) piecewise smooth continuous noninvertible maps. It possesses quite complicated dynamics which, depending on parameters, is characterized by not only attracting fixed points and cycles of any period but chaotic attractors as well. Our main purpose is to unfold the mechanisms governing transitions between such attractors under variation of parameters and to describe the overall bifurcation structure of the parameter space of the map. From the point of view of the nonlinear dynamics theory the main feature of the considered map is its *nonsmoothness*. In fact, as we mentioned above, the map is given by three different smooth functions whose definition regions are separated by two *border points* at which the system function is not differentiable. As a result, under variation of a parameter one can observe not only bifurcations typical for 1D smooth maps (such as, for example, flip bifurcation of a fixed point related to its eigenvalue crossing -1, or homoclinic bifurcation related to a contact of stable and unstable sets of a repelling fixed point), but *border collision bifurcations* as well which are characteristic for nonsmooth systems (see [15], [14], [25], [5]). Recall that a border collision bifurcation (BCB for short) occurs when an invariant set, for example, a fixed point or cycle, collides with a border point. The result of such a bifurcation can be a direct transition from an attracting fixed point to a chaotic attractor that is impossible in smooth systems. Such an abrupt transition to chaos in a 1D piecewise smooth map can be observed also due to a degenerate bifurcation which is related to eigenvalue of a fixed point (or cycle) crossing 1 or -1 in presence of a particular degeneracy of the system function. For example, a *degenerate flip bifurcation* (DFB for short) of a fixed point occurs when its eigenvalue crosses -1 and the related branch of the function at the bifurcation value is linear or linear fractional (see [29]). Note that general bifurcation theory of nonsmooth dynamical systems has not yet such a complete form as the one established for smooth systems. As an important advancement towards such a theory we refer to the books [32], [10]. Examples of piecewise smooth models coming from economic applications can be found in [9], [15], [27], [11], to cite a few.

As one of the main contributions of the present paper we get conditions under which in the Matsuyama model of credit cycles abrupt transitions from an attracting fixed point to an attracting cycle or to a chaotic attractor are observed. Such conditions are obtained with help of a 1D piecewise linear map defined by two linear functions, called *skew tent map*. The dynamics of the skew tent map are completely described depending on the slopes of the linear branches (see [16], [19]) that makes it possible to use this map as a *border collision normal form* ([26], [5], [28]).

Remarkably, the skew tent map helps to classify not only border collision bifurcations but homoclinic bifurcations as well which are responsible for particular transformations of chaotic attractors. In fact, it is known that one more distinctive feature of piecewise smooth maps is associated with *robust chaos* (see [4]). A chaotic attractor which is robust is characterized by persistence under parameter perturbations, that is, in the parameter space of the map there exists an open region, called chaotic domain, related to a chaotic attractor. We recall that in a 1D continuous map a chaotic attractor consists of n cyclic intervals, $n \geq 1$. Varying some parameter inside a chaotic domain one can observe bifurcations at which the number of intervals constituting the chaotic attractor changes. In particular, a *merging bifurcation* is related to the transition from $2n$- to n-cyclic chaotic attractor. It is caused by the first homoclinic bifurcation of a repelling cycle with negative eigenvalue, located at the immediate basin boundary of the attractor. An *expansion bifurcation* occurs when a chaotic attractor abruptly increases in size filling the complete absorbing interval due to the first homoclinic bifurcation of a repelling cycle with positive eigenvalue (see [2] for details). Using skew tent map we describe merging and expansion bifurcations occurring in the considered map.

The paper is organized as follows. In Sec.2 we introduce the map, list its fixed points and obtain conditions of their stability. The parameter region we are interested in is confined by three boundaries. One of them is related to a contact of the absorbing interval with the border point of the map and two other boundaries are related to the bifurcations of a fixed point associated with the downward branch of the map. Namely, crossing one of such boundaries a border collision bifurcation of this fixed point occurs results of which are summarized in Proposition 1 in Sec.3. The second boundary is related to the flip bifurcation described in Proposition 2 in Sec.4. The overall bifurcation structure of the parameter space is discussed in Sec.5. Sec.6 concludes.

2. Description of the map, its fixed points and their bifurcations

We consider a 4-parameter family of 1D piecewise smooth maps defined as

$$T : w \mapsto \ T(w) = \begin{cases} w^{\alpha} & \text{if } 0 < w < w_c, \\ \left[\frac{1}{\mu\beta}\left(1 - \frac{w}{m}\right)\right]^{\frac{\alpha}{1-\alpha}} & \text{if } w_c < w < w_\mu, \\ \beta^{\frac{\alpha}{\alpha-1}} & \text{if } w \geq \max\left\{w_c, w_\mu\right\}, \end{cases} \tag{1}$$

where α, β, μ and m are real parameters such that

$$0 < \alpha, \mu < 1, \ \beta \equiv B\frac{1-\alpha}{\alpha} > 0, \ 1 < m < \frac{1}{1-\alpha}, \tag{2}$$

w_c and w_μ are the border points defined by

$$w_c^{1-\alpha} = \frac{1}{\mu\beta} \max\left\{1 - \frac{w_c}{m}, \mu\right\}, \qquad w_\mu = m(1-\mu). \tag{3}$$

The map T describes the dynamic trajectory of the entrepreneur *net worth* w in a credit cycle model, first introduced in [20], under the additional assumption that the aggregate production function is Cobb-Douglas (see [30]). The branches of the map T are defined as follows:

$T_L(w) \equiv w^{\alpha}$ (the upward branch);
$T_M(w) \equiv \left[\frac{1}{\mu\beta}\left(1-\frac{w}{m}\right)\right]^{\frac{\alpha}{1-\alpha}}$ (the downward branch);
$T_R(w) \equiv \beta^{\frac{\alpha}{\alpha-1}} \equiv w_B$ (the flat branch).

The map T in the simplest case is given by the branches $T_L(w)$ and $T_R(w)$ only with the border point $w_c = (w_B)^{1/\alpha}$. The boundary in the parameter space defined by

$$\beta = (m(1-\mu))^{\alpha-1} \tag{4}$$

is related to the appearance of the middle branch in the definition of T. Namely, for $\beta > (m(1-\mu))^{\alpha-1}$ the map T can be written in the following form:

$$T : w \mapsto T(w) = \begin{cases} T_L(w) = w^{\alpha} & \text{if } 0 \le w \le w_c, \\ T_M(w) = \left[\frac{1}{\mu\beta}\left(1-\frac{w}{m}\right)\right]^{\frac{\alpha}{1-\alpha}} & \text{if } w_c < w < w_{\mu}, \\ T_R(w) = w_B & \text{if } w > w_{\mu}. \end{cases} \tag{5}$$

Note that T maps $(0,1]$ into itself, so that we restrict T on $(0,1]$ from now on.

Let us recall fist the simplest bifurcation conditions presented in [30] related to existence and stability of the fixed points of the map T. We illustrate the corresponding regions and bifurcation curves in Fig.1 which shows the bifurcation diagram of map T in the (μ,β)-parameter plane.

The fixed points associated with the upward, downward and flat branches of the map T are denoted w_L^*, w_M^* and w_R^*, respectively. The fixed point $w_L^* = 1$ exists and is globally attracting for the parameter values belonging to the region

$$\mathbf{A}: \qquad \beta \le \max\left\{\frac{1}{\mu}\left(1-\frac{1}{m}\right), 1\right\}, \tag{6}$$

two boundaries of which correspond to BCBs of w_L^*, namely, for

$$BC_{LM}: \qquad \beta = \frac{1}{\mu}\left(1-\frac{1}{m}\right), \tag{7}$$

we have $w_L^* = 1 = w_M^*$, and for

$$BC_{LR}: \qquad \beta = 1, \tag{8}$$

the equality $w_L^* = 1 = w_R^*$ holds. The fixed point $w_R^* = w_B$ (which is obvioulsy superstable) exists for the parameter region

$$1 < \beta < (m(1-\mu))^{1-\frac{1}{\alpha}}.$$

At the boundary $\beta = 1$ (denoted as BC_{LR}) we have $w_R^* = w_L^* = 1$. If the parameter point crosses BC_{LR} we observe a border collision leading from the superstable fixed point w_R^* to the stable fixed point w_L^*.[1] The region of existence of w_R^* is divided by the boundary given in (4) in two subregions:

$$\mathbf{B}: \qquad 1 < \beta < (m(1-\mu))^{\alpha-1},$$

$$\mathbf{C}: \qquad (m(1-\mu))^{\alpha-1} < \beta < (m(1-\mu))^{1-\frac{1}{\alpha}},$$

(see Fig.1). While at the boundary

$$BC_{MR}: \qquad \beta = (m(1-\mu))^{1-\frac{1}{\alpha}} \tag{9}$$

we have $w_R^* = w_\mu = w_M^*$, so that BC_{MR} is related to one more border collision of w_R^*. The fixed point w_M^* exists if $w_c \leq w_M^* \leq w_\mu$ that holds for

$$\beta \geq \max\left\{\frac{1}{\mu}\left(1-\frac{1}{m}\right), (m(1-\mu))^{1-\frac{1}{\alpha}}\right\}. \tag{10}$$

Both boundaries of this parameter region are related to the border collision of w_M^*, namely, at the boundary BC_{LM} (see (7)) $w_M^* = 1 = w_L^*$, as already mentioned. The possible results of this BCB are described in Proposition 1 below. While at the boundary BC_{MR} (see (9)) we have $w_M^* = w_\mu = w_R^*$. Crossing BC_{MR} in a generic case we observe either persistence border collision, or flip BCB[2] (see [30]).

The fixed point w_M^* may become unstable via a flip bifurcation (see Proposition 2 below). The flip bifurcation curve of w_M^* is given by

$$FB_M: \qquad \beta = \frac{\alpha}{\mu}(m(1-\alpha))^{1-\frac{1}{\alpha}}. \tag{11}$$

[1] Border collision at which neither kind nor stability properties of the colliding invariant set change is called *persistence border collision.*

[2] Border collision of a fixed point due to which the fixed point changes stability while a 2-cycle emerges from the border point is called *flip BCB.* Simillarly to the smooth flip bifurcation a flip BCB can be sub- or supercritical. Note, however, that it is not related to an eigenvalue passing through -1. Moreover, it may result in a chaotic attractor that is impossible for the smooth flip bifurcation.

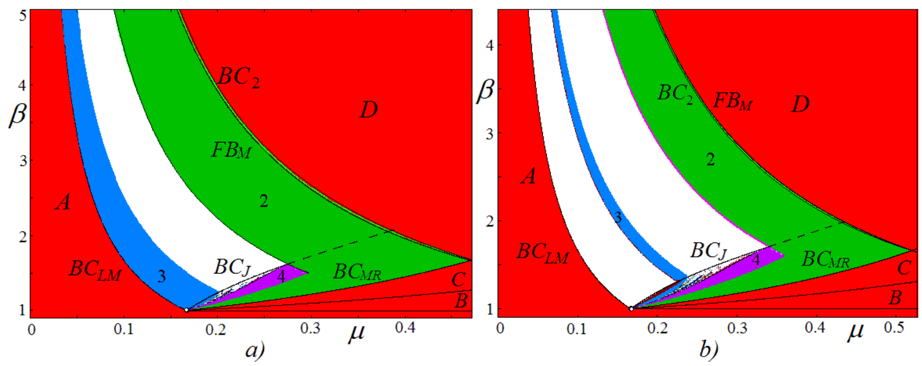

Figure 1: 2D bifurcation diagram in the (μ, β)-parameter plane at $m = 1.2$ and $\alpha = 0.47$ in *a)*, $\alpha = 0.52$ in *b)*.

So, for parameter values belonging to the region

$$\mathbf{D}: \qquad \beta > \max\left\{\frac{\alpha}{\mu}(m(1-\alpha))^{1-\frac{1}{\alpha}}, (m(1-\mu))^{1-\frac{1}{\alpha}}\right\}$$

(see Fig.1) there exists the locally attracting fixed point w_M^*.

We have the following two possibilities for an invariant absorbing interval J of map T:

1) In the absorbing interval J only the functions $T_L(w)$ and $T_M(w)$ are defined, that holds for parameter values belonging to the region

$$\mathbf{E\text{-}I}: \begin{cases} \beta < \frac{\alpha}{\mu}(m(1-\alpha))^{1-\frac{1}{\alpha}}, \\ \beta > \max\left\{\frac{1}{\mu}\left(1-\frac{1}{m}\right),\ 1-\frac{1}{\mu}+\frac{1}{\mu}(m(1-\mu))^{1-\frac{1}{\alpha}}\right\} \end{cases} \tag{12}$$

In such a case $J = [T^2(w_c), T(w_c)]$.

2) All the three functions, $T_L(w)$, $T_M(w)$ and $T_R(w)$, are involved in J, that holds in the region

$$\mathbf{E\text{-}II}: \begin{cases} \beta > (m(1-\mu))^{1-\frac{1}{\alpha}}, \\ \beta < \min\left\{1-\frac{1}{\mu}+\frac{1}{\mu}(m(1-\mu))^{1-\frac{1}{\alpha}},\ \frac{\alpha}{\mu}(m(1-\alpha))^{1-\frac{1}{\alpha}}\right\} \end{cases} \tag{13}$$

In such a case $J = [T(w_\mu), T(w_c)] = [w_B, T(w_c)]$.

The boundary between the two regions corresponds to the contact of J with the border point w_μ, occurring when the condition $T(w_c) = w_\mu$ is satisfied, leading to the curve BC_J having the following equation:

$$BC_J: \qquad \beta = 1 - \frac{1}{\mu} + \frac{1}{\mu}(m(1-\mu))^{1-\frac{1}{\alpha}}. \tag{14}$$

The bifurcation structure of the region E-II formed by the periodicity regions related to superstable cycles of the map T (existing due to its flat branch) is described in [30]. In the following we first describe the border collision and flip bifurcations of the fixed point w_M^* in detail and then we discuss the overall bifurcation structure of the region E-I.

3. Crossing the curve BC_{LM}: BCB of the fixed point

Let us consider first the BCB of the fixed point w_M^*, occurring when a parameter point crosses the boundary BC_{LM} given in (7) of the region E-I. To describe the possible results of this BCB we can use the skew tent map defined by

$$q : x \mapsto q(x) = \begin{cases} a_l x + \varepsilon & \text{if } x \leq 0, \\ a_r x + \varepsilon & \text{if } x > 0, \end{cases} \tag{15}$$

as a border collision normal form. This approach is based on the following statement (see [26], [5], [29]): *For a family of 1D piecewise smooth continuous maps $g : x \mapsto g(x, c)$ depending smoothly on a parameter c and having a border point $x = d$, suppose that*

$$g(d, c^*) = d \tag{16}$$

and let

$$a_l^* = \lim_{x \uparrow d} \frac{d}{dx} g(x, c^*), \quad a_r^* = \lim_{x \downarrow d} \frac{d}{dx} g(x, c^*). \tag{17}$$

Then in the generic case the border collision occurring in the map g as c varies through c^* is of the same kind as the one occurring in the skew tent map (15) as ε varies through 0 *at $(a_l, a_r) = (a_l^*, a_r^*)$.*

Clearly, this statement refers to the border collision of a fixed point $x = x^*$ of the map g (its existence before or/and after the collision follows from the conditions of the statement).[3] Generic case means that at $c = c^*$ the fixed point $x = x^*$ of the map g undergoes only one bifurcation, i.e. a codimension-one

[3]The skew tent map can be also used as a border collision normal form for a BCB of an n-cycle of the map g, in which case the statment has to be applied to the map g^n and its fixed point corresponding to the periodic point of g colliding with the border point.

BCB. An example of codimention-two bifurcation is when a border collision and a flip bifurcation occur simultaneously at the same point in the parameter space (in fact, this can happen in the map T, as we discuss later). For the detailed classification of the possible BCBs in the skew tent map and explanation how to use this map as a border collision normal form we refer to [2].

So, to construct a normal form for the border collision occurring in the map T when its fixed point collides with the border point w_c (in which case $w_M^* = w_L^* = w_c = 1$) we have to evaluate the left- and right-side derivatives of T at $w = 1$ for the parameter values belonging to the boundary BC_{LM} given in (7):

$$a_l^* = \lim_{w \uparrow 1} \frac{d}{dx} T(w) = \alpha, \qquad a_r^* = \lim_{w \downarrow 1} \frac{d}{dx} T(w) = -\frac{\alpha}{(1-\alpha)(m-1)}. \tag{18}$$

The relation between a point belonging to BC_{LM} and the parameters a_l, a_r of the skew tent map is given by

$$(a_l, a_r) = \left(\alpha, -\frac{\alpha}{(1-\alpha)(m-1)} \right),$$

so, if a parameter point moves along the boundary BC_{LM} the related point in the (a_l, a_r)-parameter plane moves along the curve denoted $\mathcal{B}_m$:

$$\mathcal{B}_m : \qquad a_r = -\frac{a_l}{(1-a_l)(m-1)}. \tag{19}$$

Recall that the curve BC_{LM} is valid for $\beta = B^{\frac{1-\alpha}{\alpha}} > 1$, that is, for $\alpha < \frac{B}{B+1}$. Moreover, $\alpha > 1 - \frac{1}{m}$ (see (2)). Thus, the curve $\mathcal{B}_m$ is to be considered in the range

$$1 - \frac{1}{m} < a_l < \frac{B}{B+1}, \quad \text{or} \quad \frac{-B}{m-1} < a_r < -1, \tag{20}$$

which is nonempty for $B > m - 1$.

Let us recall in short the curves forming the bifurcation structure in the (a_l, a_r)-parameter plane of the skew tent map for any $\varepsilon > 0$.[4] Let q_n denote a cycle of period n, $n \geq 2$, of the skew tent map. The stability region of q_n is

[4]For details see, for example, [19], [2].

bounded from above by the curve ϕ_n and from below by the curve ψ_n defined as

$$\phi_n \quad : \qquad a_r = -\frac{1-a_l^{n-1}}{(1-a_l)a_l^{n-2}}, \tag{21}$$

$$\psi_n \quad : \qquad a_r = \frac{-1}{a_l^{n-1}}, \tag{22}$$

(see Fig.2a). The curve ϕ_n is related to the fold BCB[5] leading to the appearance of the basic cycle[6] q_n and its complementary cycle[7] $\widetilde{q}_n$. The curve ψ_n is related to the DFB of q_n leading to $2n$-cyclic chaotic intervals $Q_{n,2n}$, $n \geq 3$, where the first index n means that this chaotic attractor is born due to a DFB of the n-cycle, while $2n$ indicates that the chaotic intervals constituting the attractor are $2n$-cyclic. The transitions $Q_{n,2n} \Rightarrow Q_{n,n}$ (merging bifurcation) and $Q_{n,n} \Rightarrow Q_1$ (expansion bifurcation) take place crossing the curves γ_n and $\widetilde{\gamma}_n$, respectively, whose equations are given by

$$\gamma_n \quad : \qquad a_l^{2(n-1)} a_r^3 - a_r + a_l = 0, \tag{23}$$

$$\widetilde{\gamma}_n \quad : \qquad a_l^{n-1} a_r^2 + a_r - a_l = 0. \tag{24}$$

For the description of merging and expansion bifurcations we refer to [2]. The curves γ_n and $\widetilde{\gamma}_n$ are related to the first homoclinic bifurcation of the cycles q_n and $\widetilde{q}_n$, respectively. There is also a set of curves σ_{2^i}, $i \geq 0$, given by

$$\sigma_{2^i} \; : \qquad \left(a_l^{\delta_i} a_r^{\delta_{i+1}}\right)^2 + (a_l/a_r)^{(-1)^{i+1}} - 1 = 0, \tag{25}$$

where $\delta_i = (2^i - (-1)^i)/3$. The curve σ_{2^i} for $i \geq 1$ corresponds to the first homoclinic bifurcation of harmonic 2^i-cycle, causing the merging bifurcation

[5]Border collision at which two fixed points (one attracting and one repelling, or both repelling) simultaneously collide with the border point and disappear after the collision is called *fold BCB*. It is worth to emphasize that a fold BCB is not associated with an eigenvalue passing through 1.

[6]For a 1D piecewise smooth map defined on two partitions, L and R, an n-cycle with symbolic sequence LR^{n-1} or RL^{n-1} for any $n \geq 2$ is called basic. The basic cycle q_n of the skew tent map (15) for $\varepsilon > 0$ has symbolic sequence RL^{n-1}. It can be shown that only such cycles can be stable.

[7]The symbolic sequences of two complementary cycles differ by one symbol. The symbolic sequence of the cycle $\widetilde{q}_n$ which is complementary to the basic cycle q_n is $RL^{n-2}R$.

$Q_{2,2^{i+1}} \Rightarrow Q_{2,2^i}$, and the curve σ_1 $(i = 0)$ is related to the first homoclinic bifurcation of the fixed point leading to the merging bifurcation $Q_{2,2} \Rightarrow Q_1$. The curves σ_{2^i} for $i \to \infty$ are accumulating to the point $(a_l, a_r) = (1, -1)$ (see Fig.2a).

Using the bifurcation curves of the skew tent map we can state the following

Proposition 1. *Consider the map T given in (5) for some fixed parameter values satisfying (2), and let $\beta = (1-1/m)/\mu$ (the boundary BC_{LM}). Consider the bifurcation structure of the (a_l, a_r)-parameter plane of the skew tent map (15) for $\varepsilon > 0$, defined by the curves (21)-(25), and let $(a_l, a_r) = (a_l^*, a_r^*)$ as in (18). Then the BCB occurring in the map T when its parameter point crosses transversely the boundary BC_{LM} leads from the attracting fixed point w_L^* to the following attractor:*

- *n-cycle g_n, $n \geq 2$, if (a_l^*, a_r^*) is below ϕ_n and above ψ_n;*
- *$2n$-cyclic chaotic intervals $G_{n,2n}$, $n \geq 3$, if (a_l^*, a_r^*) is below ϕ_n, ψ_n, and above γ_n;*
- *n-cyclic chaotic intervals $G_{n,n}$, $n \geq 3$, if (a_l^*, a_r^*) is below ϕ_n, γ_n and above $\widetilde{\gamma}_n$;*
- *2^i-cyclic chaotic intervals $G_{2,2^i}$, $i \geq 1$, if (a_l^*, a_r^*) is below ϕ_2, ψ_2, σ_{2^i} and above $\sigma_{2^{i-1}}$;*
- *Otherwise, the attractor is chaotic interval $G_1 = [T^2(w_c), T(w_c)]$.*

To illustrate this proposition we present in Fig.2a the bifurcation structure of the (a_l, a_r)-parameter plane of the skew tent map together with the curves $\mathcal{B}_m$ for different values of m, and in Fig.2b it is shown the 2D bifurcation diagram in the (μ, α)-parameter plane for $m = 1.05$, $B = 1.5$, where the curve BC_{LM} corresponds to the curve $\mathcal{B}_{1.05}$.

Let us associate the regions which the curve $\mathcal{B}_{1.05}$ intersects (see Fig.2a) with the attractors which are born when the curve BC_{LM} is crossed (see Fig.2b). First note that due to (20) the curve $\mathcal{B}_{1.05}$ is valid for $-30 < a_r < -1$. Starting from the point p_0' of $\mathcal{B}_{1.05}$ with $a_r = -1$, the curve $\mathcal{B}_{1.05}$ intersects (moving from above to below) the curve ψ_2 at the point p_1', the curves σ_2 and σ_1 at the

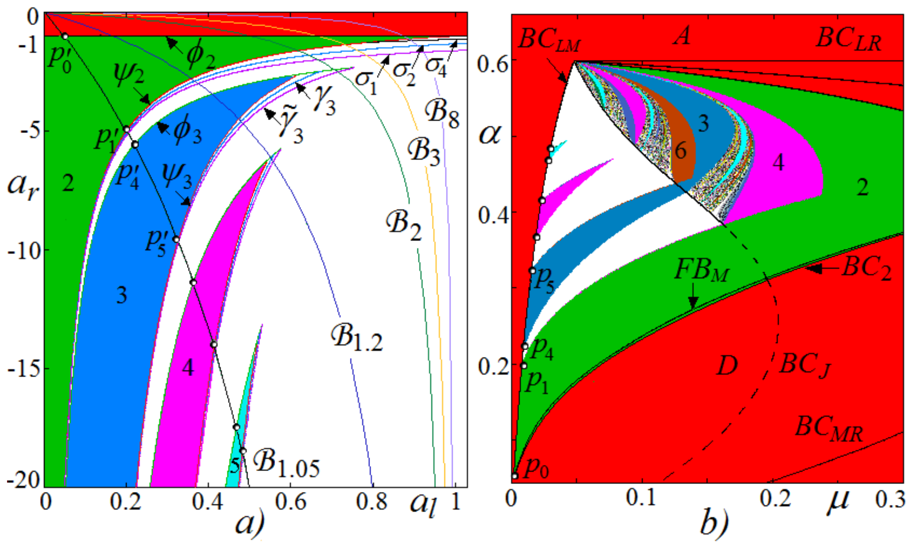

Figure 2: *a)* Bifurcation structure of the (a_l, a_r)-parameter plane of the skew tent map, where the border collision curves $\mathcal{B}_m$ are shown for $m = 1.05$, 1.2, 2, 3 and 8; *b)* Bifurcation structure of the (μ, α)-parameter plane of the map T at $m = 1.05$, $B = 1.5$.

points p'_2, p'_3, the curve ϕ_3 at the point p'_4, ψ_3 at p'_5, γ_3 at p'_6, $\widetilde{\gamma}_3$ at p'_7, and so on, up to the intersection with the curve $\widetilde{\gamma}_5$ at the point p'_{15}. It can be checked that $\mathcal{B}_{1.05}$ does not intersect any other bifurcation curve. Substituting (19) to the related equation (21)-(25), we obtain the a_l-coordinates of the intersection points, that is, $a_l = \alpha \equiv \alpha_j$, $j = 0, ..., 15$, which then can be substituted to (7) (recall that $\beta = B\frac{1-\alpha}{\alpha}$). In such a way we obtain the corresponding points p_i of the curve BC_{LM} (see Fig.2*b*). Namely, the α-coordinates of the points p_j are the following: $\alpha_0 = 0.047619$, $\alpha_1 \approx 0.199961$, $\alpha_2 \approx 0.201786$, $\alpha_3 \approx 0.203248$, $\alpha_4 \approx 0.218205$, $\alpha_5 \approx 0.322973$, $\alpha_6 \approx 0.324797$, $\alpha_7 \approx 0.326245$, and so on. The intersection point of BC_{LM} and BC_{LR} is $(\mu, \alpha) = (0.047619, 0.6)$ related to the end point of $\mathcal{B}_{1.05}$ with $a_r = -30$.

Let $BC_{LM}|_{p_j}^{p_{j+1}}$ denote an open arc of the curve BC_{LM} bounded by the points p_j and p_{j+1}. Now we can state, for example, that if the parameter point crosses the arc $BC_{LM}|_{p_0}^{p_1}$ then an attracting 2-cycle g_2 is born due to this BCB, because the related arc $\mathcal{B}_{1.05}|_{p'_0}^{p'_1}$ belongs to the stability region of the 2-cycle of the skew tent map. Similarly we can conclude that crossing $BC_{LM}|_{p_1}^{p_2}$, $BC_{LM}|_{p_2}^{p_3}$

and $BC_{LM}|_{p_3}^{p_4}$ leads to chaotic intervals $G_{2,4}$, $G_{2,2}$ and G_1, respectively, while crossing $BC_{LM}|_{p_4}^{p_5}$ leads to an attracting 3-cycle g_3, and so on.

Analyzing Fig.2a one can conclude also that for larger values of m less periodicity regions are intersected by $\mathcal{B}_m$. For example, the curve $\mathcal{B}_2$ intersects only the 2-periodicity region (which is in fact intersected by $\mathcal{B}_m$ for any m), thus, besides an attracting 2-cycle only chaotic attractors can appear due to the BCB. It is clear also that for fixed B the interval of valid values of α (see (20)) decreases for increasing m.

As one more example of application of the Proposition 1 we can check that for $m = 1.2$, $\alpha = 0.47$ the point $(a_l, a_r) = (a_l^*, a_r^*)$ belongs to the 3-periodicity region (see the curve $\mathcal{B}_{1.2}$ in Fig.2a at $a_l = 0.47$), thus, such a BCB of w_L^* leads to the attracting 3-cycle g_3, that is confirmed by Fig.1a, while for $m = 1.2$, $\alpha = 0.52$ the point $(a_l, a_r) = (a_l^*, a_r^*)$ is in the region related to a one-piece chaotic attractor, thus, in Fig.1b the crossing BC_{LM} leads from the fixed point w_L^* to a chaotic attractor G_1.

4. Crossing the curve FB_M: flip bifurcation of the fixed point

Let us consider now the flip bifurcation of the fixed point x_M^* which occurs if the parameter point crosses the boundary of the region $\mathbf{D}$, the curve FB_M given in (11). As we show below, this bifurcation can be supercritical, subcritical or degenerate as illustrated in Fig.3 by means of 1D bifurcation diagrams.

Namely, in Fig.3a one can see that decreasing μ a pair of 2-cycles (g_2 attracting and $\widetilde{g}_2$ repelling) are born due to a fold BCB before the subcritical flip bifurcation of the fixed point. So, in the interval between these two bifurcations the attracting fixed point w_M^* coexists with the 2-cycles g_2 and $\widetilde{g}_2$. Then, if we continue to decrease μ, at the subcritical flip bifurcation the fixed point w_M^* loses stability merging with $\widetilde{g}_2$ so that after the bifurcation the map T has the attracting 2-cycle g_2 and the repelling fixed point. The DFB of w_M^* illustrated in Fig.3b also leads to an attracting 2-cycle g_2, but the characteristic feature of this bifurcation is that at the bifurcation value any point of the interval $[w_c, T(w_c)]$, except for the fixed point w_M^*, is 2-periodic, including the

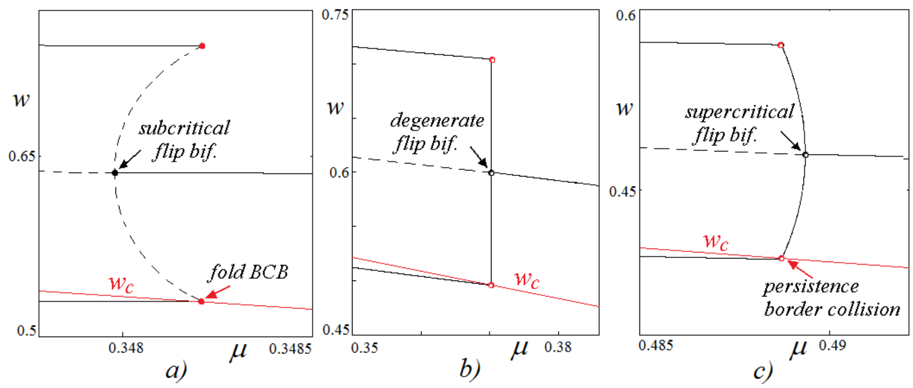

Figure 3: 1D bifurcation diagrams illustrating subcritical *a)*, degenerate *b)* and supercritical *c)* flip bifurcation of the fixed point w_M^*. Here $m = 1.2$ and $\alpha = 0.47$, $\beta = 2.25$ in *a)*, $\alpha = 0.5$, $\beta = 2.25$ in *b)*, $\alpha = 0.6$, $\beta = 2$ in *c)*.

end points of this interval. Thus, we have $T^2(w_c) = w_c$, that is, the BCB of the 2-cycle g_2 occurs simultaneously with the DFB of w_M^*. As for the supercritical flip bifurcation (see Fig.3c) note that soon after this bifurcation the attracting 2-cycle g_2 changes its symbolic sequence, from MM to LM, due to a persistence border collision. That is, one periodic point of the 2-cycle crosses the boundary w_c (from the region M to the region L) so that a border collision occurs, but the attractor is a 2-cycle before the bifurcation with symbolic sequence MM and persists as a 2-cycle after the bifurcation, with symbolic sequence LM.

Proposition 2. *The flip bifurcation of the fixed point w_M^* of the map T defined in (5) occurs for parameter values satisfying (2) and (10) at $\beta = \alpha(m(1-\alpha))^{1-\frac{1}{\alpha}}/\mu$ (the boundary FB_M). The flip bifurcation of w_M^* is supercritical for $\alpha > 0.5$, subcritical for $\alpha < 0.5$ and degenerate for $\alpha = 0.5$.*

To prove this statement we have to check the sign of $(T_M^2)'''(w)$ evaluated at the fixed point w_M^* for the bifurcation parameter value, namely, if we have $(T_M^2)'''(w_M^*) < 0$ then the flip bifurcation is supercritical, while for $(T_M^2)'''(w_M^*) > 0$ it is subcritical (see, e.g., [31]). In the case of a DFB (when it is $(T_M^2)'''(w_M^*) = 0$), it is enough to show that $T_M^2(w) \equiv w$ occurs in an interval around w_M^* (see [29]).

In order to simplify calculations let us introduce a change of variable, $x :=$

$(1 - w/m)$, and let also $\gamma = \alpha/(1-\alpha)$, $C = (\mu\beta)^\gamma/m$. Now the middle branch T_M of the map T has the form $t(x) = 1 - Cx^\gamma$, and its fixed point satisfies $x_M^* = 1 - C(x_M^*)^\gamma$. It is easy to see that at the flip bifurcation value we have $x_M^* = \alpha$. Using this equality after some algebraic computations and rearrangements we get

$$(t^2)'''(x_M^*) = (\gamma C)^2(1-\gamma)(x_M^*)^{2(\gamma-2)}(1+\gamma),$$

so that the sign of this expression depends on γ, namely, $(t^2)'''(x_M^*) < 0$ for $\gamma > 1$, and $(t^2)'''(x_M^*) > 0$ for $\gamma < 1$. Coming back to the map T and the original parameters we conclude that for $\alpha > 0.5$ we have $(T_M^2)'''(w_M^*) < 0$, thus, the flip bifurcation is supercritical, while for $\alpha < 0.5$ the inequality $(T_M^2)'''(w_M^*) > 0$ holds, so that the flip bifurcation is subcritical. For $\alpha = 0.5$ corresponding to $\gamma = 1$ we have $C = 1$, so that

$$t^2(x) = 1 - C(1 - Cx^\gamma)^\gamma|_{C=1,\gamma=1} \equiv x.$$

Thus, the flip bifurcation is degenerate. For the map T this means that any point of the absorbing interval, except for the fixed point w_M^*, is 2-periodic. The absorbing interval in such a case is $J = [w_c, T(w_c)]$ for the parameter region E-I, and $J = [w_B, T(w_B)]$ for the region E-II.

As we can see, all the bifurcation sequences described above include a border collision of a 2-cycle. Let us consider it in more details. The condition which is to be satisfied is

$$T_M \circ T_L(w_c) = w_c$$

and the related boundary in the parameter space is denoted BC_2:

$$BC_2 : \qquad \left[\frac{1}{\mu\beta}\left(1 - \frac{w_c^\alpha}{m}\right)\right]^{\frac{\alpha}{1-\alpha}} = w_c. \tag{26}$$

(See, for example, the curves BC_2 and FB_2 shown in case of subcritical flip bifurcation of w_M^* in Fig.1a and supercritical in Fig.1b). To see the result of this bifurcation we can use the skew tent map as a normal form for the border collision of the related fixed point of the map T^2. For this we need to evaluate the left- and right-side derivatives of T^2 at $w = w_c$ for the parameter values belonging to BC_2. Obviously, $a_l^* = (T_M \circ T_L)'(w_c) < 0$ and $a_r^* = (T_M^2)'(w_c) > 0$, and the skew tent map (15) with $\varepsilon < 0$ can to be used as a normal form. However, it is easy to show that bifurcation structure of the (a_l, a_r)-parameter plane for $\varepsilon < 0$ is symmetric with respect to $a_l = a_r$ to the one for $\varepsilon > 0$. Thus, we can use the results related to dynamics of the skew tent map presented in the previous section considering the symmetric point $(a_l, a_r) = (a_r^*, a_l^*)$. In particular, one can check that $a_l^* = (T_M \circ T_L)'(w_c) > -1$ for

$$w_c^\alpha \left(1 + \frac{\alpha^2}{1-\alpha}\right) < m \tag{27}$$

and $a_r^* = (T_M^2)'(w_c) > 1$ for $\alpha < 0.5$. The point $(a_l, a_r) = (a_r^*, a_l^*)$ with $a_l > 1$ and $0 < a_r < 1$ belongs to the region at which the skew tent map has an attracting and repelling fixed points (in Fig.2a a small part of this region can be seen), and a fold BCB occurs in the skew tent map if ε passes through 0. Thus, in the map T^2 also a fold BCB occurs. For the map T this means that the border collision occurring at BC_2 is also a fold BCB leading to a pair of 2-cycles, an attracting g_2 and a repelling $\widetilde{g}_2$, with symbolic sequences LM and MM, respectively. We can check also that crossing BC_2 for $\alpha = 0.5$ always leads to one attracting 2-cycle. To see this, note that the curve FB_M at $\alpha = 0.5$ is defined by

$$FB_M|_{\alpha=0.5}: \qquad \mu\beta = \frac{1}{m},$$

and the branches of the map T are $T_L(w) = \sqrt{w}$ and $T_M(w) = m - x$ with the border point $w_c = \left(-1 + \sqrt{1+4m}\right)^2/4$. We have $(T_M^2)'(w_c) = 1$, while $(T_M \circ T_L)'(w_c) > -1$, where the last inequality holds for $m > 3/4$, that is always true given that $m > 1$. Thus, the 2-cycle born due to this bifurcation (with symbolic sequence LM) is attracting. For $\alpha > 0.5$ we have $(T_M^2)'(w_c) < 1$ and $(T_L \circ T_M)'(w_c) > -1$ (for the parameter values satisfying (27)), so that due to collision with $w = w_c$ the 2-cycle remains attracting and only changes its symbolic sequence from MM to LM (persistence border collision). If the condition (27) does not hold, that is, if $(T_L \circ T_M)'(w_c) < -1$, then the crossing of the curve BC_2 leads to two repelling 2-cycles and to a chaotic attractor. An example of such a bifurcation is shown in Fig.4.

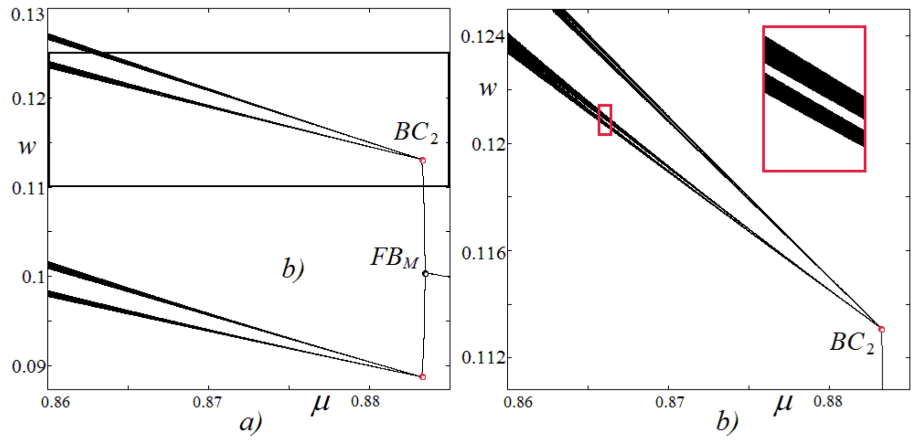

Figure 4: 1D bifurcation diagram in the map T for $\alpha = 0.9, m = 1.005$, $\beta = 1.315$, $\mu \in [0.86, 0.885]$ is shown in *a)*, and its enlargments are in *b)*. Here the BCB of the 2-cycle leads to 8-cyclic chaotic intervals.

Suppose that the map T has an attracting 2-cycle $g_2 = \{w_1, w_2\}$ with symbolic sequence LM. Let us obtain the condition of its flip bifurcation. First, from $T_M \circ T_L(w_1) = w_1$ we get that $w_1 = [(1 - w_1^\alpha/m)/\mu\beta]^{\frac{\alpha}{1-\alpha}}$. Then, from $(T_M \circ T_L)'(w)|_{w=w_1} = -1$ we get $w_1^\alpha = m(1-\alpha)/(\alpha^2 - \alpha + 1)$, so that the flip bifurcation of g_2 occurs for

$$FB_2 : \qquad \mu\beta = \frac{\alpha^2}{\alpha^2 - \alpha + 1} \left(\frac{(\alpha^2 - \alpha + 1)}{m(1-\alpha)} \right)^{\frac{1-\alpha}{\alpha^2}}. \tag{28}$$

Note that for $\alpha = 0.5$ the curve FB_2 is given by

$$FB_2|_{\alpha=0.5} : \qquad \mu\beta = \frac{3}{4m^2}.$$

5. Overall bifurcation structure of the region E-I

In this section we discuss the overall bifurcation structure of the region E-I defined in (12). The bifurcation structure of the region E-II defined in (13) is studied in detail in [30]. Recall that the region E-I is confined by the boundaries BC_{LM} (7), FB_M (11) and BC_J (14). Using Proposition 1 which describes the dynamics of the map T in a neighborhood of the curve BC_{LM} we can state which bifurcation curves issue from this boundary, namely, from the points

$p_j,\ j=0,...,l$ (where l depends on the parameters). Recall that these points correspond to the intersection points of the curve $\mathcal{B}_m$ (19) with the bifurcation curves (21)-(25) of the skew tent map.

Note that all the points p_j are codimention-two bifurcation points, for which, as we have already mentioned, the skew tent map does not help to state precisely which attractor is born after the BCB. Consider, for example, the codimension-two bifurcation point p_0, at which the BCB of the fixed point occurs simultaneously with its flip bifurcation, that is, the fixed point is (one-side) nonhyperbolic. Such a point is called *border-flip* codimention-two bifurcation point. It is shown in [8], focusing, in particular, on the geometric shapes of the bifurcation curves around a border-flip point, that in general three bifurcation curves are issuing from such a point, among which one is a curve related to the smooth bifurcation and the other two curves are BCB curves. In fact, in Fig.2b we see that besides the curve BC_{LM} two more curves issue from the border-flip point p_0, namely, the curve FB_M corresponding to the subcritical flip bifurcation of the fixed point w_M^* and the curve BC_2 related to the fold BCB of the 2-cycle. Clearly, if the curve BC_{LM} is crossed at the point p_0, then the parameter point can enter to the narrow region bounded by the curves BC_2 and FB_M, where an attracting 2-cycle coexists with the attracting fixed point. Such a coexistence obviously cannot be classified using only the skew tent map. In fact, any border-flip point of BC_{LM} corresponding to the intersection of the BCB curve $\mathcal{B}_m$ and DFB curve ψ_n, $n\geq 2$ (as, e.g., the points p_1 and p_5 indicated in Fig.2b), is an issue point of two curves, namely, a flip bifurcation curve FB_n and a border collision curve BC_{2n}.

Let us suppose that the curve $\mathcal{B}_m$ crosses an n-periodicity region of the skew tent map, for $n\geq 3$, that is, there is an arc $\mathcal{B}_m|_{p'_j}^{p'_{j+1}}$ belonging to this region (as shown in Fig.2a for several values of m). A neighborhood of the curve BC_{LM} in such a case is shown schematically in Fig.5. According to Proposition 1 in the one-side neighborhood of the arc $BC_{LM}|_{p_j}^{p_{j+1}}$ there must be a region related to an attracting n-cycle g_n of the map T (to simplify, the region related to the attracting cycle g_n is denoted in Fig.5 in the same way as the cycle, that is,

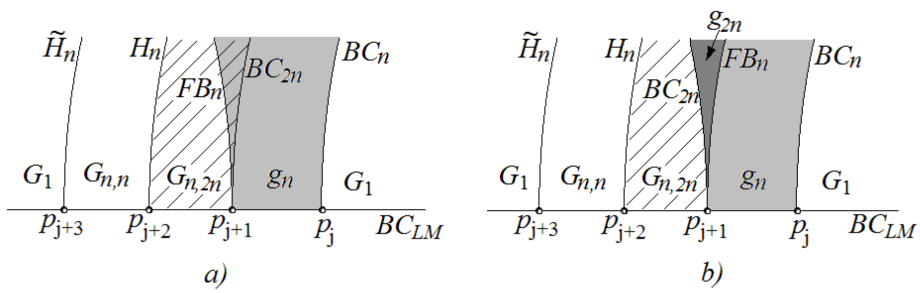

Figure 5: A neighborhood of the curve BC_{LM} shown schematically in case when the BCB curve $\mathcal{B}_m$ given in (19) related to BC_{LM} crosses an n-periodicity region of the skew tent map. The flip bifurcation at FB_n is subcritical in *a)* and superscritical in *b)*. The point p_{j+1} is a border-flip codimention-two blifurcation point.

g_n. Similar notations are used for the regions related to other attractors). Its boundary issuing from the point p_j is related to the fold BCB satisfying the condition

$$BC_n : \qquad T_L^{n-2} \circ T_M \circ T_L(w_c) = w_c.$$

Note that due to continuity of the map T at $w = w_c$ an equivalent condition of BC_n is $T_L^{n-2} \circ T_M^2(w_c) = w_c$. Crossing the boundary BC_n (from the right to the left in Fig.5) two n-cycles are born, an attracting cycle g_n and a repelling cycle $\widetilde{g}_n$. The cycle g_n has a periodic point w_n which satisfies $T_L^{n-1} \circ T_M \circ\ T_L(w_n) = w_n$, while the cycle $\widetilde{q}_n$ has a periodic point $\widetilde{w}_n$ satisfying $T_L^{n-2} \circ T_M^2(\widetilde{w}_n) = \widetilde{w}_n$.

The boundary of the n-periodicity region issuing from the point p_{j+1} is related to the flip bifurcation of g_n defined by the condition

$$FB_n : \qquad (T_L^{n-2} \circ T_M \circ\ T_L)'(w_n) = -1. \tag{29}$$

As already mentioned, one more bifurcation curve issues from p_{j+1}, namely, the curve BC_{2n} related to the border collision of a $2n$-cycle g_{2n} (as show in [8], it is tangent to the flip bifurcation curve). The curve BC_{2n} satisfies the condition

$$BC_{2n} : \qquad \left(T_L^{n-2} \circ T_M \circ T_L\right)^2 (w_c) = w_c. \tag{30}$$

Given that the arc $\mathcal{B}_m|_{p'_{j+1}}^{p'_{j+2}}$ belongs to the region related to a $2n$-cyclic chaotic intervals $Q_{n,2n}$ of the skew tent map, in the one-side neighborhood of the arc $BC_{LM}|_{p_{j+1}}^{p_{j+2}}$ there is a region related to $2n$-cyclic chaotic intervals $G_{n,2n}$ (see the dashed region in Fig.5). There are two possibilities: if the flip bifurcation FB_n is subcritical, as in Fig.5a, then in the region between FB_n and BC_{2n} an attracting n-cycle g_n coexists with a chaotic attractor $G_{n,2n}$, while if the flip bifurcation FB_n is supercritical, as in Fig.5b, the region between BC_{2n} and FB_n is related to an attracting $2n$-cycle g_{2n}. More precisely, in Fig.5a the curve BC_{2n} belongs to the stability region of g_n, and the bifurcation occurring at BC_{2n} is a fold BCB leading to a pair of repelling $2n$-cycles, g_{2n}, $\widetilde{g}_{2n}$, and to a chaotic attractor $G_{n,2n}$ coexisting with the n-cycle g_n (in fact, as we illustrate in Fig.8b, or Fig.9b, the cycle $\widetilde{g}_{2n}$ separates the basins of $G_{n,2n}$ and g_n, while the cycle g_{2n} belongs to $G_{n,2n}$). Then, moving from the right to the left the curve FB_n is crossed at which the repelling cycle $\widetilde{g}_{2n}$ merges with the attracting cycle g_n due to a subcritical flip bifurcation, so that after this bifurcation the attractor is $G_{n,2n.}$ In case of supercritical flip bifurcation, the crossing of the curve BC_{2n} leads from an attracting cycle g_{2n} to a chaotic attractor $G_{n,2n}$ (see Fig.5b).

Next, we can state that the one-side neighborhood of the arc $BC_{LM}|_{p_{j+2}}^{p_{j+3}}$ (see Fig.5) is related to n-cyclic chaotic intervals $G_{n,n}$ of the map T because the related arc $\mathcal{B}_m|_{p'_{j+2}}^{p'_{j+3}}$ belongs to the region of n-cyclic chaotic intervals $Q_{n,n}$ of the skew tent map. Its boundary issuing from the point p_{j+2} is related to the first homoclinic bifurcation of the cycle g_n, which satisfies the conditions

$$H_n : \quad \begin{cases} (T_L^{n-2} \circ T_M \circ T_L)^2(w_c) = w_n, \\ T_L^{n-2} \circ T_M \circ T_L(w_n) = w_n. \end{cases} \tag{31}$$

So, crossing the curve H_n we observe the merging bifurcation $G_{n,2n} \Rightarrow G_{n,n}$. See, for example, the curve H_3 in Fig.6 and the related merging bifurcation $G_{3,6} \overset{H_3}{\Rightarrow} G_{3,3}$ in Fig.9a. The boundary issuing from the point p_{j+3} corresponds to the first homoclinic bifurcation of the cycle $\widetilde{g}_n$ and satisfies the conditions

$$\widetilde{H}_n : \quad \begin{cases} T_L^{n-2} \circ T_M \circ T_L(w_c) = \widetilde{w}_n, \\ T_L^{n-2} \circ T_M^2(\widetilde{w}_n) = \widetilde{w}_n. \end{cases} \tag{32}$$

Thus, crossing the curve $\widetilde{H}_n$ an expansion bifurcation $G_{n,n} \Rightarrow G_1$ occurs. An example of the curve $\widetilde{H}_3$ is shown in Fig.6, and the related expansion bifurcation

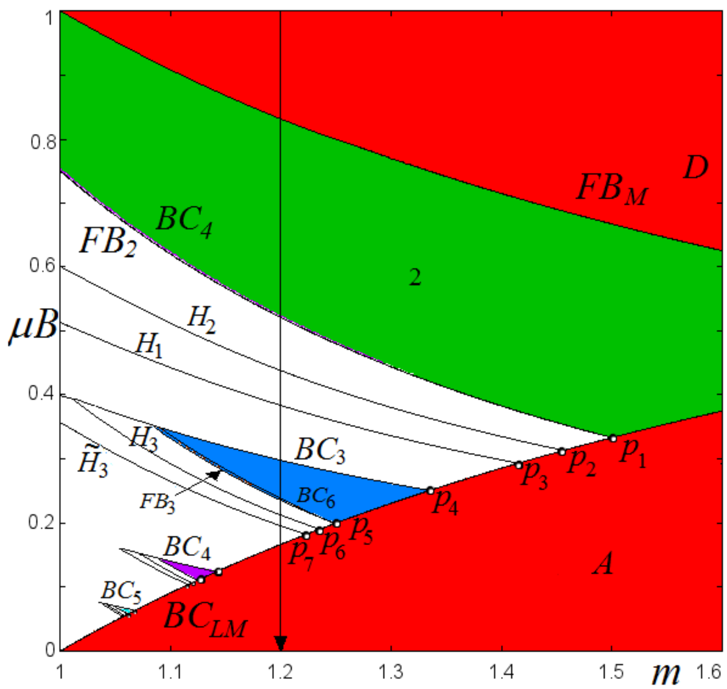

Figure 6: 2D bifurcation diagram in the $(m, \mu B)$-parameter plane at $\alpha = 0.5$. 1D bifurcation diagram for $m = 1.2$ and its enlargements are shown in Fig.s8 and 9.

$G_{3,3} \overset{\widetilde{H}_3}{\Rightarrow} G_1$ is illustrated in Fig.9a.

As we have seen, the curve $\mathcal{B}_m$ may not intersect the n-periodicity regions for $n \geq 3$, of the skew tent map (see Fig.2a). The description presented above can be easily adjusted to such a case. However, the 2-periodicity region is intersected for any m, and this case differs from the one described above. In fact, we know that from the border-flip point p_0 of the curve BC_{LM} the boundaries FB_M and BC_2 issue related to the flip bifurcation of the fixed point w_M^* and border collision of the 2-cycle g_2, as we show schematically in Fig.7. Differently from the generic case we have three possibilities as stated in Proposition 2 (see also Fig.3):

1) if the flip bifurcation is subcritical, that holds for $\alpha < 0.5$, then the curve BC_2 is related to a fold BCB leading to a pair of 2-cycles, an attracting one (g_2) and a repelling one ($\widetilde{g}_2$), in which case the region between BC_2 and FB_M

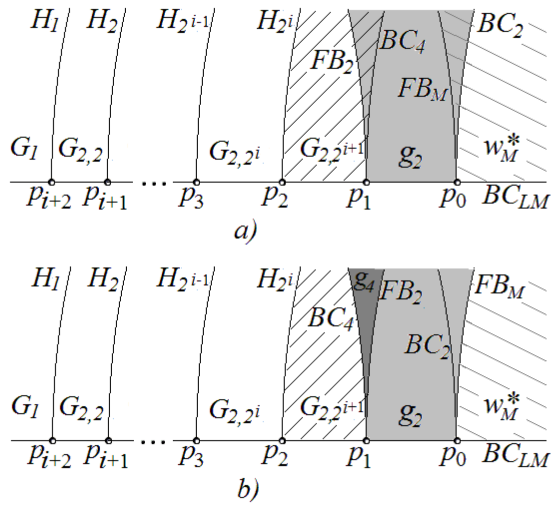

Figure 7: A neighborhood of the curve BC_{LM} shown schematically near the border-flip point p_0. The flip bifurcation at FB_M is subcritical in *a)* and supercritical in *b)*. The point p_1 is also a border-flip codimention-two blifurcation point.

is related to coexisting attractors, the fixed point w_M^* and the 2-cycle g_2 (see Fig.7a);

2) if the flip bifurcation is supercritical, that holds for $\alpha > 0.5$, then the curve BC_2 a is persistence border collision curve crossing which the 2-cycle g_2 born before due to supercritical flip bifurcation just changes its symbolic sequence, remaining attracting (see Fig.7b);

3) if the flip bifurcation is degenerate that holds for $\alpha = 0.5$, we have $FB_M = BC_2$, so that crossing this boundary one attracting cycle g_2 appears (with symbolic sequence LM).

Thus, in the one-side neighborhood of the arc $BC_{LM}|_{p_0}^{p_1}$ there is a region related to an attracting 2-cycle g_2 of the map T. From the border-flip point p_1 the boundaries FB_2 and BC_4 originate related to the flip bifurcation of g_2 and BCB of g_4. The next point p_2 corresponds to the intersection of $\mathcal{B}_m$ with the curve σ_{2^i} (25) for some $i \geq 1$. From p_2 a curve denoted H_{2^i} issues (see Fig.7), related to the first homoclinic bifurcation of the harmonic 2^i-cycle of the map T. For the skew tent map crossing the curve σ_{2^i} leads to the merging bifurcation $Q_{2,2^{i+1}} \Rightarrow Q_{2,2^i}$. Thus, in the one-side neighborhood of the arc $BC_{LM}|_{p_1}^{p_2}$ there is a region related to 2^{i+1}-cyclic chaotic intervals $G_{2,2^{i+1}}$, and crossing BC_4 leads to a chaotic attractor $G_{2,2^{i+1}}$. Similarly, the point p_3 is an issue point for the curve $H_{2^{i-1}}$ related to the first homoclinic bifurcation of the harmonic 2^{i-1}-cycle of the map T, and so on, up to the point p_{i+2} which is an issue point of the curve H_1 related to the first homoclinic bifurcation of the fixed point w_M^* (see Fig.7). For example, from the point p_{i+1} of the curve BC_{LM} related to the intersection of $\mathcal{B}_m$ with the curve σ_2 (see (25) for $i = 1$), the curve H_2 issues which corresponds to the first homoclinic bifurcation the cycle g_2, satisfying the conditions

$$H_2 : \quad \begin{cases} (T_M \circ T_L)^2(w_c) = w_2, \\ T_M \circ T_L(w_2) = w_2. \end{cases} \tag{33}$$

The crossing of this curve leads to the merging bifurcation $G_{2,4} \overset{H_2}{\Rightarrow} G_{2,2}$ (see, e.g., Fig.8a and the curve H_2 in Fig.6 issuing from the point p_2). From the point p_{i+2} the curve H_1 issues corresponding to the first homoclinic bifurcation of the fixed point w_M^*, satisfying the conditions

$$H_1 : \quad \begin{cases} T_L \circ T_M \circ T_L(w_c) = w_M^*, \\ T_M(w_M^*) = w_M^*. \end{cases} \tag{34}$$

The crossing of this curve leads to the merging bifurcation $G_{2,2} \overset{H_1}{\Rightarrow} G_1$ (see, e.g., Fig.8a and the corresponding curve H_1 is Fig.6 issuing from the point p_3).

The bifurcation structure described above is illustrated in Fig.6 in the $(m, \mu B)$-parameter plane at $\alpha = 0.5$. The curve BC_{LM} in such a case is defined by

$$BC_{LM}|_{\alpha=0.5} : \qquad \mu B = 1 - \frac{1}{m}$$

(note that for $\alpha = 0.5$ we have $B = \beta$). The curve $\mathcal{B}_m$ (19) in the (a_l, a_r)-parameter plane of the skew tent map represents a vertical line $a_l = 0.5$ where $\frac{-B}{m-1} < a_r < -1$ (see (20)):

$$\mathcal{B}_m|_{\alpha=0.5} : \quad a_l = 0.5, \quad a_r = -\frac{1}{m-1}. \tag{35}$$

Using the equations (21)-(25) we can obtain the points p'_j, $j = 0, ..., 15$, related to the intersection of $\mathcal{B}_m|_{\alpha=0.5}$ with the bifurcation curves of the skew tent map.

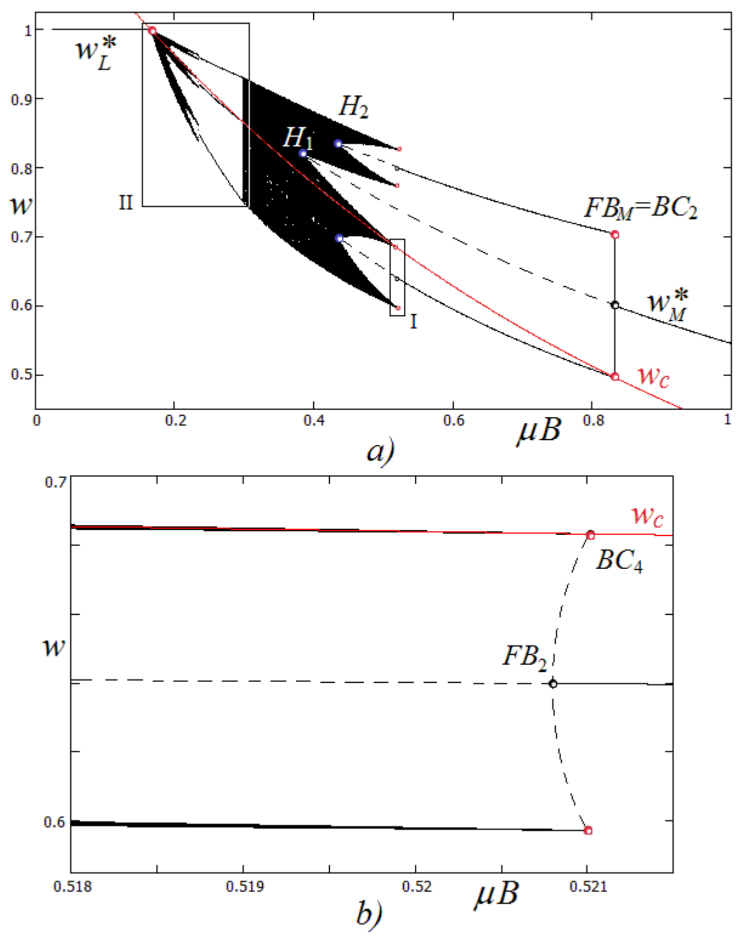

Figure 8: In *a)* 1D bifurcation diagram of the map T is shown for $\alpha = 0.5$, $m = 1.2$ and $\mu B \in [0, 1]$ related to the vertical line with an arrow in Fig.6. In *b)* the window I indicated in *a)* is shown enlarged.

Then, substituting the related values a_r into (35) we obtain the m-coordinates of the point p_j of the curve BC_{LM} (see Fig.6). The curves issuing from the points p_j in Fig.6 are obtained numerically using the related conditions (29)-(34).

To illustrate the bifurcations (29)-(34) occurring in the map T we present in Fig.8a 1D bifurcation diagram related to the vertical line with an arrow indicated in Fig.6. Enlargements of this diagram are shown in Fig.8b and Fig.9. The sequence of observed bifurcations for decreasing μB can be summarized as

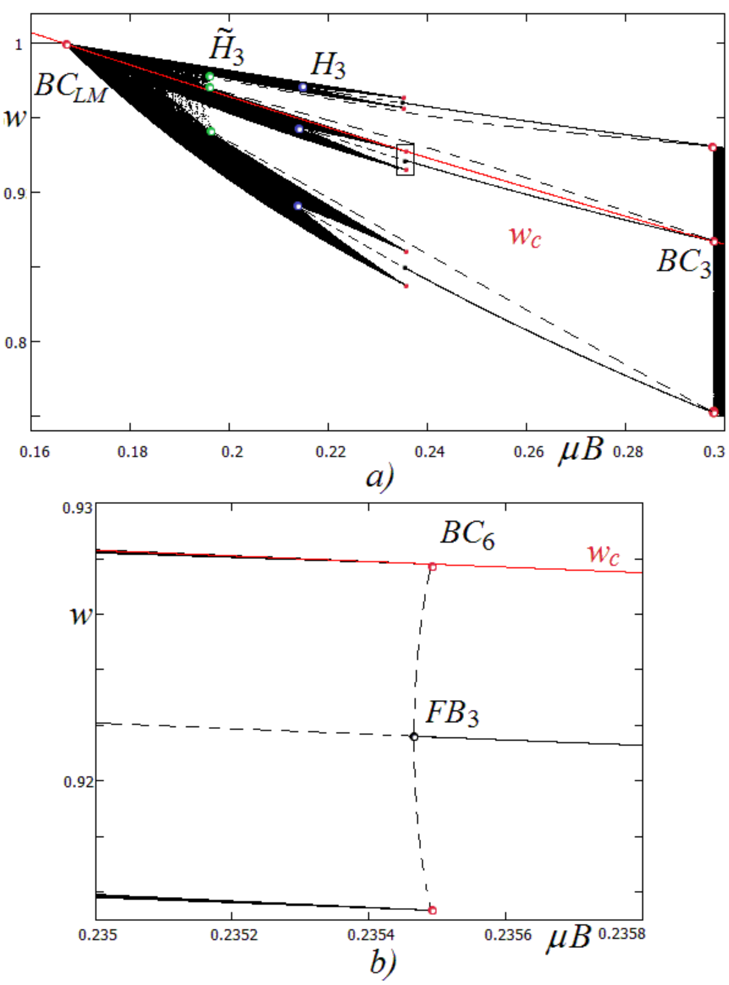

Figure 9: In *a)* an enlargement of window II indicated in Fig.8*a* is shown, and in *b)* the window indicated in *a)* is enlarged.

follows:

$$w_M^* \overset{FB_M=BC_2}{\Rightarrow} g_2 \overset{BC_4}{\Rightarrow} \{g_2, G_{2,4}\} \overset{FB_2}{\Rightarrow} G_{2,4} \overset{H_2}{\Rightarrow} G_{2,2} \overset{H_1}{\Rightarrow} G_1$$
$$\overset{BC_3}{\Rightarrow} g_3 \overset{BC_6}{\Rightarrow} \{g_3, G_{3,6}\} \overset{FB_3}{\Rightarrow} G_{3,6} \overset{H_3}{\Rightarrow} G_{3,3} \overset{\widetilde{H}_3}{\Rightarrow} G_1 \overset{BC_{LM}}{\Rightarrow} w_L^*$$

6. Conclusion

In the present paper we studied dynamics of a credit cycle model, first introduced in [20], under the additional assumption that the aggregate production function is Cobb-Douglas. In generic case this model is defined by a 4-parameter family of 1D piecewise smooth maps with upward, downward and flat branches. We considered the case when the flat branch is not involved into asymptotic dynamics that corresponds to the region E-I given in (12).

Bifurcation structure of the region E-I is described in detail, which is formed by the boundaries related to border collision bifurcations characteristic for non-smooth systems, as well as flip bifurcations and homoclinic bifurcations (causing merging and expansion of the chaotic attractors). These boundaries separate regions corresponding to different attractors of the map, namely, attracting cycles and chaotic attractors (cyclic chaotic intervals). In particular, possible results of a BCB of the fixed point are classified in Proposition 1 using skew tent map as a border collision normal form. The conditions are obtained under which this BCB leads directly to an attracting cycle of period n, or to an n-cyclic chaotic attractor, $n \geq 1$. The skew tent map helps also to describe the overall bifurcation structure of the region E-I in a neighborhood of the BCB boundary. Proposition 2 states that the flip bifurcation of the fixed point is supercritical for $\alpha > 0.5$, subcritical for $\alpha < 0.5$ and degenerate for $\alpha = 0.5$. It is shown that an attracting 2-cycle which appears due to the supercritical flip bifurcation soon after collides with the border point. In fact, a cascade of flip bifurcations characteristic for smooth unimodal maps cannot be realized in the considered map (that obviously is related to the absence of a smooth extremum in the map). Subcritical flip bifurcation is characterized by bistability related to coexistence of an attracting fixed point and attracting 2-cycle which is born, together with a repelling 2-cycle, due to a fold BCB before the flip bifurcation. From an economic point of view this implies corridor stability, i.e., the steady state of the economy is stable against small shocks but unstable against large shocks. Furthermore, when the steady state loses its stability as a parameter change causes such a subcritical flip bifurcation, the effect is catastrophic and irreversible in that restoring the stability of the steady state by reversing the parameter change is not enough for the economy to return to the steady state. Examples of an attracting cycle coexisting with a cyclic chaotic attractor are also presented.

It is important to emphasize that chaotic attractors of the considered map are robust, that is, they are persistent under parameter perturbations.

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References

[1] P. Aghion, A. Banerjee, T. Piketty, Dualism and Macroeconomic Volatility, The Quarterly Journal of Economics, November, 1999, 1359-1397.

[2] V. Avrutin, L.Gardini, M. Schanz, I. Sushko, F.Tramontana. Continuous and Discontinuous Piecewise-Smooth One-Dimensional Maps: Invariant Sets and Bifurcation structures. Springer, 2014 (in progress).

[3] C. Azariadis, B. Smith, Financial Intermediation and Regime Switching in Business Cycles, The American Economic Review 88 (3) (1998) 516-536.

[4] S. Banerjee, J. A. Yorke, and C, Grebogi, Robust chaos, Physical Review Letters, 80(14) (1998) 3049–3052.

[5] S. Banerjee, M.S. Karthik, G. Yuan, J.A. Yorke, Bifurcations in one-dimensional piecewise smooth maps — Theory and applications in switching circuits, IEEE Trans. Circuits Syst.-I: Fund. Th. Appl. 47 (2000) 389–394.

[6] B. Bernanke, M. Gertler, Agency Costs, Net Worth, and Business Fluctuations, The American Economic Review 79 (1) (1989), 14-31.

[7] G.I. Bischi, C. Chiarella, M. Kopel, and F. Szidarovszky, Nonlinear oligopolies: Stability and bifurcations, Heidelberg, Springer, 2009.

[8] A. Colombo, F. Dercole, Discontinuity Induced Bifurcations of Nonhyperbolic Cycles in Nonsmooth Systems, SIAM Journal on Imaging Sciences 3 (1) (2010) 62-83.

[9] R. Day, Complex Economic Dynamics, MIT Press, Cambridge, 1994.

[10] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, Piecewise-smooth Dynamical Systems: Theory and Applications, Applied Mathematical Sciences 163, Springer-Verlag, London, 2007.

[11] L. Gardini, I. Sushko and A. Naimzada, Growing Through Chaotic Intervals, Journal of Economic Theory, 143 (2008) 541–557.

[12] R. Goodwin, Non-linear Accelerator and the Persistence of Business Cycles, Econometrica, 1951.

[13] B.L. Hao, Elementary Symbolic Dynamics and Chaos in Dissipative Systems, World Scientific, Singapore, 1989.

[14] C. Hommes, A reconsideration of Hicks' nonlinear trade cycle model, Structural Change and Economic Dynamics 6 (1995) 435-459.

[15] C. Hommes, H. Nusse, Period three to period two bifurcation for piecewise linear models, J. Economics, 54, 2 (1991) 157–169.

[16] S. Ito, S. Tanaka, H. Nakada, On unimodal transformations and chaos II, Tokyo J. Math. 2 (1979) 241–259.

[17] C.P. Kindleberger, Manias, Panics, and Crashes: A History of Financial Crises, The Third Edition, New York, John Wiley & Sons, Inc., 1996.

[18] N. Kiyotaki, J. Moore, Credit Cycles, Journal of Political Economy 105 (2) (1997), 211-248.

[19] Y.L. Maistrenko, V.L. Maistrenko, L.O. Chua, Cycles of chaotic intervals in a time-delayed Chua's circuit, Int. J. Bifurcat. Chaos 3 (1993) 1557–1572.

[20] K. Matsuyama, Good and Bad Investment: An Inquiry into the Causes of Credit Cycles, Center for Mathematical Studies in Economics and Management, Science Discussion Paper No.1335, Northwestern University, 2001.

[21] K. Matsuyama, Credit Traps and Credit Cycles, American Economic Review, 97 (2007) 503–516.

[22] K. Matsuyama, The Good, The Bad, and The Ugly: An Inquiry into the Causes and Nature of Credit Cycles, Theoretical Economics 8 (2013) 623-651.

[23] N. Metropolis, M.L. Stein, P.R. Stein, On finite limit sets for transformations on the unit interval, J. Comb. Theory 15 (1973) 25–44.

[24] H.P. Minsky, The Financial Instability Hypothesis: Capitalistic Processes and the Behavior of the Economy, in C.P. Kindleberger and J.P. Laffargue, eds., Financial Crises: Theory, History, and Policy (Cambridge, Cambridge University Press, 1982), 13-29.

[25] H.E. Nusse, J.A. Yorke, Border-collision bifurcations including period two to period three for piecewise smooth systems, Physica D 57 (1992) 39–57.

[26] H.E. Nusse, J.A. Yorke, Border-collision bifurcation for piecewise smooth one-dimensional maps, Int. J. Bifurcation Chaos 5 (1995) 189–207.

[27] T. Puu, I. Sushko (Eds.), Oligopoly Dynamics, Models and Tools, Springer Verlag, New York, 2002.

[28] I. Sushko, A. Agliari, L. Gardini, Bifurcation Structure of Parameter Plane for a Family of Unimodal Piecewise Smooth Maps: Border-Collision Bifurcation Curves, in: G.I. Bischi and I. Sushko Eds., Dynamic Modeling in Economics and Finance in honor of Professor Carl Chiarella, Chaos, Solitons & Fractals 29 Issue 3 (2006) 756–770.

[29] I. Sushko, L. Gardini, Degenerate Bifurcations and Border Collisions in Piecewise Smooth 1D and 2D Maps, Int. J. Bif. and Chaos 20 (2010) 2045–2070.

[30] I. Sushko, L. Gardini, K. Matsuyama. Superstable Credit Cycles and U-sequence, Chaos, Solitons & Fractals (2013) (to appear).

[31] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, New York, 1996.

[32] Z.T. Zhusubaliyev, E. Mosekilde, Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems, World Scientific, Singapur, 2003.