J. Austral. Math. Soc. Ser. B 37(1995), 186–207

CHAOS IN AN ANHARMONIC OSCILLATOR

J. R. CHRISTIE and K. GOPALSAMY[1]

(Received 16 June 1993; revised 12 October 1993)

Abstract

Using Melnikov's method, the existence of chaotic behaviour in the sense of Smale in a particular time-periodically perturbed planar autonomous system of ordinary differential equations is established. Examples of planar autonomous differential systems with homoclinic orbits are provided, and an application to the dynamics of a one-dimensional anharmonic oscillator is given.

1. Introduction

There has been considerable interest recently in the study of problems related to chaotic behaviour of deterministic dynamical systems. Chaos refers to the unpredictable and apparently random motion of orbits of a dynamical system; the dynamical system may be described by a differential equation or by a map such as that of a difference equation.

Chaotic behaviour of some dynamical systems can be explained by the existence of transverse homoclinic points. For example, let F denote a diffeomorphism on a two dimensional manifold which has a saddle point, say p, and suppose the stable and unstable manifolds of p intersect transversely at a point $q \neq p$. Then, it has been shown (Smale [13, 14]) that there exists a set Λ in a neighbourhood of p, invariant with respect to some positive iterate F^m of F, such that the dynamics of F^m on Λ are topologically conjugate to the dynamics of the Bernoulli shift map on the space Σ of bi-infinite sequences of two symbols; the map originally constructed by Smale is commonly known as a horseshoe map. In this article, by chaotic behaviour of a map we mean the behaviour of the Bernoulli shift map on Σ. The Smale horseshoe map is known to be topologically conjugate to the Bernoulli shift map (for details see Guckenheimer and Holmes [5], Wiggins [16]). We will call the associated chaotic behaviour "horseshoe chaos". Examples of diffeomorphisms of the above type occur

[1]School of Info. Sci. and Tech., The Flinders University of SA, South Australia.

https://doi.org/10.1017/S0334270000007657 Published online by Cambridge University Press

as period maps (also known as Poincaré maps) of planar autonomous systems of ordinary differential equations subjected to time-periodic perturbations.

It was Poincaré [12] who first noticed the existence of transverse homoclinic points and conjectured the complex dynamical behaviour which we now call chaos. More recently, Melnikov [11] has provided an analytical sufficient condition for the existence of transverse homoclinic points of period maps resulting from time-periodic perturbations of planar autonomous systems with a saddle point and an associated homoclinic orbit. Well-known examples of the application of Melnikov's method to investigations of chaotic behaviour in planar autonomous systems analyse the dynamics of the simple pendulum (Holmes [7], Holmes and Marsden [8], Marsden [10]) and the Duffing oscillator (Guckenheimer and Holmes [5], Holmes [6], Lichtenberg and Lieberman [9], Wiggins [16]).

One of the essential requirements for the application of Melnikov's technique to planar systems is the explicit knowledge of a homoclinic orbit associated with a saddle point of the unperturbed differential system. This requirement has been quite a severe restriction for the application of Melnikov's method. The authors have been driven by curiosity to find planar systems for which the explicit knowledge of a homoclinic orbit is possible. The rest of the paper is organised as follows : in Section 2, we formulate a planar system and obtain a homoclinic orbit to a saddle point of this system; in Section 3, we parametrise some of the periodic orbits of the system; in Section 4, we demonstrate, by an application of the Melnikov technique, the existence of transverse homoclinic points for the Poincaré map of the time-periodically perturbed system, and we display computer simulations of the chaotic dynamics of the system for different choices of parameters. The integrals appearing in the calculation of the Melnikov function are listed in the Appendix. More examples of planar autonomous systems with homoclinic orbits are discussed in Section 5; the examples given follow from a generalisation of Section 2. Finally, in Section 6, we give an application to the dynamics of a one-dimensional anharmonic oscillator.

2. The homoclinic lemniscate

We consider the autonomous system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= a^2x - \mu y(x^2 + y^2), \\ \frac{dy}{dt} &= -a^2y + \mu x(x^2 + y^2) \end{aligned} \tag{2.1}$$

in which a and μ are positive real numbers. One can verify that $(0, 0)$, $(\frac{a}{\sqrt{2\mu}}, \frac{a}{\sqrt{2\mu}})$ and $(-\frac{a}{\sqrt{2\mu}}, -\frac{a}{\sqrt{2\mu}})$ are the equilibrium points of the system (2.1); we shall briefly consider the nature of these equilibrium points. The linear variational system corresponding to $(0, 0)$ is given by

$$\begin{aligned}\frac{dX}{dt} &= a^2 X,\\ \frac{dY}{dt} &= -a^2 Y.\end{aligned} \tag{2.2}$$

It can be easily verified that $(0, 0)$ is a saddle point for (2.2) and hence for (2.1); furthermore, the unstable and stable manifolds of the linear system (2.2) are respectively the x and y axes of the x, y-plane. For each of the other two equilibrium points, the corresponding linear variational system is

$$\begin{aligned}\frac{dX}{dt} &= -2a^2 Y,\\ \frac{dY}{dt} &= 2a^2 X.\end{aligned} \tag{2.3}$$

We can verify that $(\frac{a}{\sqrt{2\mu}}, \frac{a}{\sqrt{2\mu}})$ and $(-\frac{a}{\sqrt{2\mu}}, -\frac{a}{\sqrt{2\mu}})$ are both centres for (2.3) and hence for (2.1). We shall now find a Hamiltonian $H = H(x, y)$ for (2.1). By definition, H satisfies

$$\begin{aligned}\frac{dx}{dt} &= a^2 x - \mu y(x^2 + y^2) = \frac{\partial H}{\partial y},\\ \frac{dy}{dt} &= -a^2 y + \mu x(x^2 + y^2) = -\frac{\partial H}{\partial x}.\end{aligned} \tag{2.4}$$

One of the ways of determining H is to find the set of level curves, $H(x, y) = $ constant, which are solution curves of (2.1). We have from (2.1),

$$\frac{dy}{dx} = \frac{-a^2 y + \mu x(x^2 + y^2)}{a^2 x - \mu y(x^2 + y^2)}$$

or equivalently

$$[a^2 x - \mu y(x^2 + y^2)]dy + [a^2 y - \mu x(x^2 + y^2)]dx = 0$$

leading to

$$d\Big[a^2 xy - \mu(x^2 + y^2)^2/4\Big] = 0. \tag{2.5}$$

Hence, (2.1) is a Hamiltonian system with Hamiltonian H given by

$$H(x, y) = a^2 xy - \mu(x^2 + y^2)^2/4, \qquad (x, y) \in \mathbb{R}^2, \tag{2.6}$$

so the solution curves of (2.1) are given by the family of curves

$$\mu(x^2 + y^2)^2 = 4a^2 xy + \text{constant}.$$

We shall consider the solution curves approaching the origin, namely

$$\mu(x^2+y^2)^2 = 4a^2xy.$$

For convenience in the following, we shall choose $\mu = 2$; this entails no loss of generality. The consequent solution curves

$$(x^2+y^2)^2 = 2a^2xy \tag{2.7}$$

are together commonly known as Bernoulli's lemniscate, which is usually recognised by the polar equation

$$r^2 = a^2\sin(2\theta), \qquad \theta \in \mathbb{R}. \tag{2.8}$$

For our subsequent study of the dynamics of the system (2.1), we need an explicit solution of (2.1) in terms of the parameter t; furthermore, we want to explore the possibility of finding a solution $(x(t), y(t))$ such that

$$\lim_{t\to\pm\infty}(x(t), y(t)) = (0, 0) \tag{2.9}$$

and the convergence in (2.9) is exponential. To obtain $x(t)$ and $y(t)$ satisfying (2.9), we first of all rewrite the system (2.1) (with $\mu = 2$) in polar coordinates

$$\begin{aligned}\frac{dr}{dt} &= a^2r\cos(2\theta),\\ \frac{d\theta}{dt} &= 2r^2 - a^2\sin(2\theta).\end{aligned} \tag{2.10}$$

Substituting (2.8) in the second equation of (2.10) gives

$$\frac{d\theta}{dt} = a^2\sin(2\theta) \tag{2.11}$$

and after separating the variables and integrating we find that

$$\tan(\theta(t)) = ke^{2a^2t}, \qquad t \in \mathbb{R}, \tag{2.12}$$

where k is any positive constant. For convenience, we choose $k = 1$, so

$$y(t)/x(t) = \tan(\theta(t)) = e^{2a^2t}, \qquad t \in \mathbb{R}, \tag{2.13}$$

and hence

$$y(t) = e^{2a^2t}x(t), \qquad t \in \mathbb{R}. \tag{2.14}$$

Choosing $k = 1$ means that

$$(x(0), y(0)) = \left(\pm\frac{a}{\sqrt{2}}, \pm\frac{a}{\sqrt{2}}\right). \tag{2.15}$$

https://doi.org/10.1017/S0334270000007657 Published online by Cambridge University Press

Substituting (2.14) in (2.7), we obtain that

$$x_h{}^{\pm}(t) = \pm\frac{\sqrt{2}ae^{a^2t}}{1+e^{4a^2t}}, \qquad y_h{}^{\pm}(t) = \pm\frac{\sqrt{2}ae^{3a^2t}}{1+e^{4a^2t}}, \qquad t \in \mathbb{R}, \tag{2.16}$$

are explicit solutions of (2.1) for our choice of $\mu = 2$. It is easily seen that the solutions in (2.16) satisfy (2.9) and the convergence is exponential. Hence, the orbits of the solutions in (2.16) are both homoclinic orbits for (2.1). Thus, the saddle point (0,0) of (2.1) has a double separatrix (homoclinic) orbit lying in the first and third quadrants of the x,y-plane as shown in Figure 1.

3. Parametrisation of periodic orbits

We first of all examine the phase space structure of (2.1) (with $\mu = 2$). Computer simulations indicate that all solutions of (2.1) (with $\mu = 2$) are bounded and the following argument provides a proof of this. From (2.6), we deduce that the solution curves of (2.1) are

$$a^2xy - \tfrac{1}{2}(x^2+y^2)^2 = \sigma, \tag{3.1}$$

where σ is a constant such that $\sigma \le \frac{a^4}{8}$, since standard techniques of several variable calculus show that

$$H(x,y) \le a^4/8, \qquad (x,y) \in \mathbb{R}^2.$$

We shall prove that the solution curves are bounded.
Writing (3.1) in polar coordinates, we have

$$\tfrac{1}{2}a^2r^2\sin(2\theta) - \tfrac{1}{2}r^4 = \sigma. \tag{3.2}$$

If $\limsup\limits_{t\to T\le\infty} r(t) = \infty$, then there is a sequence $\{t_n\} \to T$ as $n \to \infty$ such that

$$\tfrac{1}{2}a^2r^2(t_n)\sin(2\theta(t_n)) - \tfrac{1}{2}r^4(t_n) = \sigma,$$

where the left hand side tends to $-\infty$ in the limit as $n \to \infty$; this contradicts the right hand side and hence $\limsup\limits_{t\to T\le\infty} r(t) < \infty$, so all solutions of (2.1) (with $\mu = 2$) are bounded.

A property of all n-degree-of-freedom integrable Hamiltonian systems is that their bounded motions lie on sets homeomorphic to n-dimensional tori or on homoclinic or heteroclinic orbits (see Abraham and Marsden [1], Arnold [2]). Since all one-degree-of-freedom Hamiltonian systems are integrable, their bounded motions lie on either periodic, homoclinic or heteroclinic orbits. As all solutions of (2.1) (with $\mu = 2$) are bounded, we can deduce that all orbits apart from those which are homoclinic

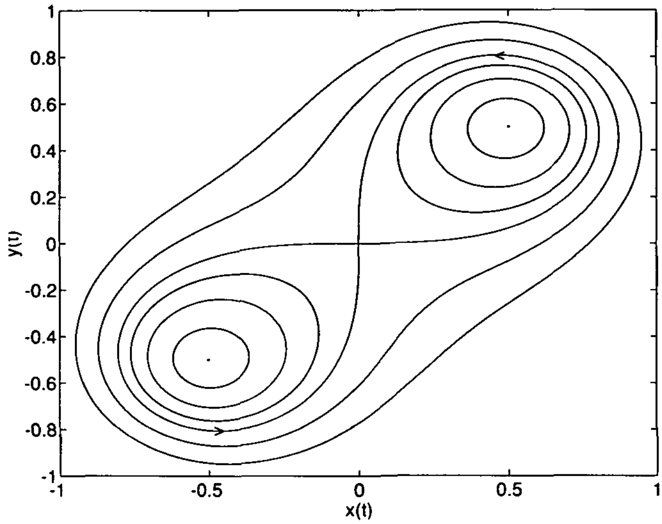

FIGURE 1. Phase space of (2.1) with $a = 1$ and $\mu = 2$.

are periodic. Hence, the interior of each homoclinic orbit is filled with a continuous family of periodic orbits surrounding a centre, as shown in Figure 1. We note that all solution curves are ovals of Cassini.

For $\sigma = \frac{a^4}{8}$, the two centres $(-\frac{a}{2}, -\frac{a}{2})$ and $(\frac{a}{2}, \frac{a}{2})$ are solutions. For $0 < \sigma < \frac{a^4}{8}$, the solution curves are the periodic orbits inside the double homoclinic loop. For $\sigma = 0$, the solution curves are the double homoclinic loop. For $\sigma < 0$, the solution curves are the periodic orbits outside the double homoclinic loop. We would like to parametrise the inner periodic orbits; hence, we consider the case where $0 < \sigma < \frac{a^4}{8}$. We rearrange the polar system (2.10) to obtain the equations

$$\begin{aligned} r^2\left(\frac{dr}{dt}\right)^2 &= a^4r^4 - (r^4 + 2\sigma)^2, \\ \left(\frac{d\theta}{dt}\right)^2 &= a^4 - 8\sigma - a^4\sin^2\left[2\left(\theta - \frac{\pi}{4}\right)\right]. \end{aligned} \tag{3.3}$$

If we use the transformation $u = r^2$, the first equation of (3.3) reduces to

$$\left(\frac{du}{dt}\right)^2 = 4({u_2}^2 - u^2)(u^2 - {u_1}^2), \tag{3.4}$$

where ${u_1}^2 = \frac{1}{2}(a^4 - 4\sigma - a^2\sqrt{a^4 - 8\sigma})$ and ${u_2}^2 = \frac{1}{2}(a^4 - 4\sigma + a^2\sqrt{a^4 - 8\sigma})$, and

https://doi.org/10.1017/S0334270000007657 Published online by Cambridge University Press

(3.4) has solution of the form

$$u^2 = {u_1}^2 \sin^2 \chi + {u_2}^2 \cos^2 \chi, \tag{3.5}$$

where

$$\frac{d\chi}{dt} = 2u = 2\sqrt{{u_2}^2 - ({u_2}^2 - {u_1}^2)\sin^2 \chi}, \tag{3.6}$$

and if we choose $\chi(0) = 0$, then (3.6) gives

$$\int_0^{\chi} \frac{d\phi}{\sqrt{1 - {k_1}^2 \sin^2 \phi}} = 2u_2 t, \tag{3.7}$$

in which $k_1 = \frac{\sqrt{{u_2}^2 - {u_1}^2}}{u_2}$. Equation (3.7) implies that

$$\sin \chi = \operatorname{sn}(2u_2 t, k_1), \tag{3.8}$$

where sn denotes a Jacobi elliptic function (see Byrd and Friedman [3]). We note that $0 < k_1 < 1$. Using (3.5) and (3.8), we have

$$r(t) = ({u_2}^2 + ({u_1}^2 - {u_2}^2)\operatorname{sn}^2(2u_2 t, k_1))^{\frac{1}{4}}, \qquad t \in \mathbb{R}. \tag{3.9}$$

If we choose $\theta(0) = \frac{\pi}{4}$, then the second equation of (3.3) gives

$$\int_0^{2(\theta - \frac{\pi}{4})} \frac{d\phi}{\sqrt{1 - {k_2}^2 \sin^2 \phi}} = 2\sqrt{a^4 - 8\sigma}\ t, \tag{3.10}$$

where $k_2 = \frac{a^2}{\sqrt{a^4 - 8\sigma}}$. Equation (3.10) implies that

$$\sin\left[2\left(\theta - \frac{\pi}{4}\right)\right] = \operatorname{sn}(2\sqrt{a^4 - 8\sigma}\ t, k_2). \tag{3.11}$$

Now $k_2 > 1$ but we want the modulus of the sn term in (3.11) to belong to the interval (0,1). We apply the reciprocal modulus transformation (see Byrd and Friedman [3]) to the right hand side to get

$$\sin\left[2\left(\theta - \frac{\pi}{4}\right)\right] = \frac{1}{k_2} \operatorname{sn}\left(2k_2\sqrt{a^4 - 8\sigma}\ t, \frac{1}{k_2}\right), \tag{3.12}$$

which after some simplification gives

$$\sin^2 \theta = \frac{\sqrt{a^4 - 8\sigma}}{2a^2} \operatorname{sn}\left(2a^2 t, \frac{1}{k_2}\right) + \frac{1}{2}. \tag{3.13}$$

https://doi.org/10.1017/S0334270000007657 Published online by Cambridge University Press

Hence

$$\theta(t) = \arcsin\left(\pm\sqrt{\frac{\sqrt{a^4-8\sigma}}{2a^2}\,\mathrm{sn}\left(2a^2t, \frac{1}{k_2}\right)+\frac{1}{2}}\right), \qquad t \in \mathbb{R}. \tag{3.14}$$

When we now find expressions for $x(t)$ and $y(t)$, we would like the Jacobi elliptic functions in (3.8) and (3.13) to have the same modulus. To do this, we use Gauss' transformation (see Byrd and Friedman [3]) on the right hand side of (3.8) to get

$$\mathrm{sn}(2u_2t, k_1) = \frac{2u_2}{u_1+u_2}\frac{\mathrm{sn}^2(a^2t, \frac{1}{k_2})}{1+\frac{1}{k_2}\mathrm{sn}(a^2t, \frac{1}{k_2})}. \tag{3.15}$$

By using properties of the Jacobi elliptic functions we eventually get a continuous family of periodic solutions in the first quadrant given in Cartesian coordinates by

$$\begin{aligned} x^k(t) &= \frac{a\sqrt{k+1}}{2}\frac{\mathrm{dn}(a^2t,k)-k\,\mathrm{sn}(a^2t,k)\mathrm{cn}(a^2t,k)}{k\,\mathrm{sn}^2(a^2t,k)+1}, \\ y^k(t) &= \frac{a\sqrt{k+1}}{2}\frac{\mathrm{dn}(a^2t,k)-k\,\mathrm{sn}(a^2t,k)\mathrm{cn}(a^2t,k)}{k\,\mathrm{sn}^2(a^2t,k)+1}, \qquad t \in \mathbb{R}, \end{aligned} \tag{3.16}$$

where $k = \frac{\sqrt{a^4-8\sigma}}{a^2} \in (0, 1)$ and cn and dn denote Jacobi elliptic functions. Using properties of the Jacobi elliptic functions, we have from (2.16) that

$$\lim_{k\to 1^-}(x^k(t), y^k(t)) = (x_h^+(t), y_h^+(t)), \quad t \in \mathbb{R}, \tag{3.17}$$

and the period of the solution $(x^k(t), y^k(t))$ is

$$T(k) = (2/a^2)K(k), \tag{3.18}$$

where $K(k)$ is the complete elliptic integral of the first kind. Therefore

$$\lim_{k\to 1^-} T(k) = \infty. \tag{3.19}$$

For the purpose of applying Melnikov's method in the next section, we do not need to parametrise the periodic orbits outside the double homoclinic loop.

4. Horseshoe chaos and Melnikov's method

Chaotic behaviour is a form of complex behaviour in deterministic dynamical systems; such behaviour involves aperiodic solutions, Smale horseshoes, strange attractors and fractal limit sets. A dynamical system displaying horseshoe chaos possesses a compact invariant set containing a countably infinite number of periodic orbits of all periods and uncountably infinite aperiodic orbits emanating from certain portions of the phase space exhibiting sensitivity to initial conditions, and a dense orbit. The majority of the cases in which chaotic behaviour has been published in the literature are based on numerical simulation or intuitive arguments. A technique due to Melnikov [11] provides an analytical procedure for the determination of the presence of a horseshoe in a class of time-periodically perturbed nonlinear dynamical systems. The greatest advantage of the Melnikov technique is that only the homoclinic solution of the unperturbed system is involved in the calculations, so we do not have to solve the perturbed system; while this is an advantage, the explicit knowledge of a homoclinic solution is unobtainable in most interesting cases for applications in such areas as mechanics, optics and population dynamics, especially when the unperturbed system is non-Hamiltonian.

We consider systems of the form

$$\frac{dx}{dt} = f_1(x, y) + \epsilon\, g_1(x, y, t),$$
$$\frac{dy}{dt} = f_2(x, y) + \epsilon\, g_2(x, y, t), \quad (x, y, t) \in \mathbb{R}^3, \tag{4.1}$$

where f_1, f_2, g_1 and g_2 are at least C^2 on the region of interest, and $0 \le \epsilon \ll 1$ is a perturbation parameter. We assume that g_1 and g_2 are periodic in t with period $\frac{2\pi}{\omega}$, for some $\omega > 0$. To apply Melnikov's method, we also need the following assumptions (Wiggins [16]) :

(1) The unperturbed system ($\epsilon = 0$) possesses a saddle point, p_0, connected to itself by a homoclinic orbit $q_0(t) \equiv (x_0(t), y_0(t))$, $t \in \mathbb{R}$.

(2) Let $\Gamma_{p_0} = \{q \in \mathbb{R}^2 : q = q_0(t), t \in \mathbb{R}\} \cup \{p_0\} = W^s(p_0) \cap W^u(p_0) \cup \{p_0\}$. The interior of Γ_{p_0} is filled with a continuous family of periodic orbits $q^\alpha(t)$ with period T^α, $\alpha \in (0, 1)$. We assume that $\lim_{\alpha \to 1^-} q^\alpha(t) = q_0(t)$, and $\lim_{\alpha \to 1^-} T^\alpha = \infty$.

We have shown in Sections 2 and 3 that these two assumptions are satisfied for the system (2.1) (with $\mu = 2$). Let us consider a time-periodic perturbation of the system (2.1) in the form of (4.1) where

$$\begin{aligned} f_1(x, y) &= a^2 x - 2y(x^2 + y^2), \\ f_2(x, y) &= -a^2 y + 2x(x^2 + y^2), \\ g_1(x, y, t) &= b \sin(\omega t) + cx, \\ g_2(x, y, t) &\equiv 0, \end{aligned} \tag{4.2}$$

in which b and c are real numbers. It is convenient to suspend the system (4.1) in $\mathbb{R}^3$ to the form

$$\frac{dx}{dt} = f_1(x, y) + \epsilon\, g_1(x, y, \theta),$$

$$\frac{dy}{dt} = f_2(x, y) + \epsilon\, g_2(x, y, \theta),$$
$$\frac{d\theta}{dt} = \omega, \tag{4.3}$$

where $\theta \in [0, 2\pi)$ with 0 and 2π identified, that is, $\theta \in S^1$ so that $(x, y, \theta) \in \mathbb{R}\times\mathbb{R}\times S^1$ where $S^1 = \mathbb{R} \pmod{2\pi}$. Now (4.3) is an autonomous system which is periodic in θ with period 2π. Define the global cross-section at time $t = t_0$ as

$$\Sigma^{t_0} = \{(x, y, \theta) \in \mathbb{R} \times \mathbb{R} \times [0, 2\pi) : \theta = \omega t_0 \pmod{2\pi}\}.$$

Consider the map $P_\epsilon^{t_0} : \Sigma^{t_0} \to \Sigma^{t_0}$ defined by

$$P_\epsilon^{t_0} : (x(t_0), y(t_0), \theta(t_0)) \to \left(x\left(t_0 + \frac{2\pi}{\omega}\right), y\left(t_0 + \frac{2\pi}{\omega}\right), \theta\left(t_0 + \frac{2\pi}{\omega}\right)\right). \tag{4.4}$$

As $\theta(t_0 + \frac{2\pi}{\omega}) = \theta(t_0)$, we have to consider only the map $\tilde{P}_\epsilon^{t_0} : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$\tilde{P}_\epsilon^{t_0} : (x(t_0), y(t_0)) \to \left(x\left(t_0 + \frac{2\pi}{\omega}\right), y\left(t_0 + \frac{2\pi}{\omega}\right)\right), \tag{4.5}$$

for some fixed but arbitrary t_0. This map $\tilde{P}_\epsilon^{t_0}$ is known as the period map or Poincaré map and it captures the same dynamics as (4.1). It is known (Guckenheimer and Holmes [5, page 186]) that for sufficiently small ϵ, the saddle point $(0, 0)$ of the unperturbed Poincaré map gets perturbed to a saddle point $(x_\epsilon^{t_0}, y_\epsilon^{t_0})$ of the Poincaré map of (4.3); furthermore when $\epsilon \neq 0$, the homoclinic manifold splits, leading to the stable and unstable manifolds of the new saddle point of the Poincaré map. We now examine whether the stable and unstable manifolds of the perturbed saddle point can intersect transversely at some point of the phase space on the cross-section Σ^{t_0} by calculating the distance between them. The importance in evaluating the distance between the stable and unstable manifolds arises from the fact that if they intersect transversely once, then they intersect each other transversely infinitely many times. This leads to the formation of a Smale horseshoe in the dynamics of some positive iterate of the Poincaré map.

Let $(x_0(t), y_0(t))$ be a homoclinic solution to the saddle point $(0, 0)$ of the unperturbed system (2.1). A measure of the distance between the stable and unstable manifolds of the Poincaré map of (4.3) at the point $(x_0(0), y_0(0), \omega t_0)$ on Σ^{t_0} is given by (for details see Guckenheimer and Holmes [5], Wiggins [16])

$$M(t_0) = \int_{-\infty}^{\infty} \begin{bmatrix} f_1(x_0(t-t_0), y_0(t-t_0)) \\ f_2(x_0(t-t_0), y_0(t-t_0)) \end{bmatrix} \wedge \begin{bmatrix} g_1(x_0(t-t_0), y_0(t-t_0), t) \\ g_2(x_0(t-t_0), y_0(t-t_0), t) \end{bmatrix} dt, \tag{4.6}$$

where $\wedge$ denotes the wedge product

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \wedge \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1.$$

M is called the Melnikov function and the Melnikov method calculates $M(t_0)$ which is an approximation to a scaled version of the actual distance between the two manifolds. Consequently, if the Melnikov function has a simple zero, that is, if there exists a $\tilde{t_0} \in \mathbb{R}$ such that

$$M(\tilde{t_0}) = 0, \quad \frac{dM}{dt_0}(\tilde{t_0}) \neq 0, \tag{4.7}$$

then it follows that the Poincaré map has a transverse homoclinic point to the saddle point $(x_\epsilon^{t_0}, y_\epsilon^{t_0})$. It is known then by the Smale-Birkhoff homoclinic theorem (Guckenheimer and Holmes [5], Wiggins [16]) that there exists an invariant Cantor set on which the dynamics of some positive iterate of the Poincaré map are topologically conjugate to those of Bernoulli's shift map on the space of bi-infinite sequences of two symbols. Since the dynamics of the shift map are " chaotic", we can conclude that the Poincaré map of (4.3) and hence the time-periodically perturbed system (4.1) possess chaotic behaviour. Substituting (4.2) in (4.6), we obtain

$$M(t_0) = -\int_{-\infty}^{\infty} f_2(x_0(t-t_0), y_0(t-t_0))[b\sin(\omega t) + cx_0(t-t_0)]\,dt, \tag{4.8}$$

where $(x_0(t), y_0(t))$ is one of the loops of the homoclinic lemniscate ($\mu = 2$) given from (2.16) by

$$x_0(t) = \frac{\sqrt{2}ae^{a^2t}}{1+e^{4a^2t}}, \qquad y_0(t) = \frac{\sqrt{2}ae^{3a^2t}}{1+e^{4a^2t}}, \qquad t \in \mathbb{R}. \tag{4.9}$$

From (2.1), (4.2) and (4.8), we derive, after making the change of variable $s = t - t_0$, that, for $t_0 \in \mathbb{R}$,

$$\begin{aligned} M(t_0) = &-b\int_{-\infty}^{\infty} \frac{dy_0}{ds}(s)\sin[\omega(s+t_0)]\,ds + a^2c\int_{-\infty}^{\infty} x_0(s)y_0(s)\,ds \\ &-2c\int_{-\infty}^{\infty} {x_0}^4(s)\,ds - 2c\int_{-\infty}^{\infty} {x_0}^2(s){y_0}^2(s)\,ds, \end{aligned} \tag{4.10}$$

and using integration by parts on the first integral, we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dy_0}{ds}(s)\sin[\omega(s+t_0)]\,ds &= y_0(s)\sin[\omega(s+t_0)]\Big|_{-\infty}^{\infty} \\ &\qquad -\omega\int_{-\infty}^{\infty} y_0(s)\cos[\omega(s+t_0)]\,ds \\ &= -\omega\int_{-\infty}^{\infty} y_0(s)\cos[\omega(s+t_0)]\,ds, \end{aligned}$$

where we have used (2.9). From (4.9) and (4.10) we now have

$$M(t_0) = \sqrt{2}ab\,\omega\cos(\omega t_0)\int_{-\infty}^{\infty} \frac{e^{3a^2s}}{1+e^{4a^2s}}\cos(\omega s)\,ds$$

$$-\sqrt{2}ab\,\omega\sin(\omega t_0)\int_{-\infty}^{\infty}\frac{e^{3a^2s}}{1+e^{4a^2s}}\sin(\omega s)\,ds$$
$$+2a^4c\int_{-\infty}^{\infty}\frac{e^{4a^2s}}{(1+e^{4a^2s})^2}\,ds-8a^4c\int_{-\infty}^{\infty}\frac{e^{4a^2s}}{(1+e^{4a^2s})^4}\,ds$$
$$-8a^4c\int_{-\infty}^{\infty}\frac{e^{8a^2s}}{(1+e^{4a^2s})^4}\,ds. \tag{4.11}$$

The integrals in (4.11) are evaluated using the calculus of residues and are listed in the Appendix; we have from (4.11) and the integrals of the Appendix that

$$\begin{aligned} M(t_0) &= K[A\cos(\omega t_0)+B\sin(\omega t_0)]+2a^4c\frac{1}{4a^2}-8a^4c\frac{1}{12a^2}-8a^4c\frac{1}{24a^2} \\ &= K\sqrt{A^2+B^2}\sin(\omega t_0+\alpha)-\frac{1}{2}a^2c, \qquad t_0\in\mathbb{R}, \end{aligned} \tag{4.12}$$

where

$$K=\frac{\pi b\,\omega}{2a}\frac{\exp(\frac{\pi\omega}{4a^2})}{1+\exp(\frac{\pi\omega}{a^2})}, \qquad A=1+\exp\Big(\frac{\pi\omega}{2a^2}\Big),$$
$$B=1-\exp\Big(\frac{\pi\omega}{2a^2}\Big), \qquad \alpha=\arctan\Big(\frac{A}{B}\Big),$$

and this simplifies to

$$M(t_0)=\frac{\pi b\,\omega}{\sqrt{2}a}\frac{\exp(\frac{\pi\omega}{4a^2})}{\sqrt{1+\exp(\frac{\pi\omega}{a^2})}}\sin(\omega t_0+\alpha)-\frac{1}{2}a^2c, \qquad t_0\in\mathbb{R}. \tag{4.13}$$

From (4.13), the Melnikov function has infinitely many zeros provided

$$\sin(\omega t_0+\alpha)=\frac{a^3c}{\sqrt{2}\pi b\,\omega}\frac{\sqrt{1+\exp(\frac{\pi\omega}{a^2})}}{\exp(\frac{\pi\omega}{4a^2})}. \tag{4.14}$$

The zeros of the Melnikov function will be simple if $\frac{dM}{dt_0}\neq 0$ at the zeros of M. We have from (4.13) that

$$\frac{dM}{dt_0}(t_0)=\frac{\pi b\,\omega^2}{\sqrt{2}a}\frac{\exp(\frac{\pi\omega}{4a^2})}{\sqrt{1+\exp(\frac{\pi\omega}{a^2})}}\cos(\omega t_0+\alpha), \qquad t_0\in\mathbb{R}. \tag{4.15}$$

Thus a sufficient condition for $\frac{dM}{dt_0}(t_0)\neq 0$ when $M(t_0)=0$ is that $\sin^2(\omega t_0+\alpha)<1$, and using (4.14), this reduces after some simplification to

$$a^6c^2\exp\Big(\frac{\pi\omega}{a^2}\Big)-2\pi^2b^2\omega^2\exp\Big(\frac{\pi\omega}{2a^2}\Big)+a^6c^2<0. \tag{4.16}$$

It is found from (4.16) that the Melnikov function has simple zeroes for $c = 0$, and if $c \neq 0$, then c has to be small enough to satisfy (4.16). The existence of transverse homoclinic points for the Poincaré map of (4.2) – (4.3) for sufficiently small ϵ follows from the result due to Melnikov [11]. The "chaotic behaviour" (or horseshoe chaos) of the perturbed system will now follow from the Smale-Birkhoff homoclinic theorem. Consider the case when $a = b = \omega = 1$. The inequality (4.16) now becomes

$$c^2 e^{\pi} - 2\pi^2 e^{\frac{\pi}{2}} + c^2 < 0$$

which simplifies approximately to

$$|c| < 1.98. \tag{4.17}$$

We now present some computer simulations in Figures 2-7.

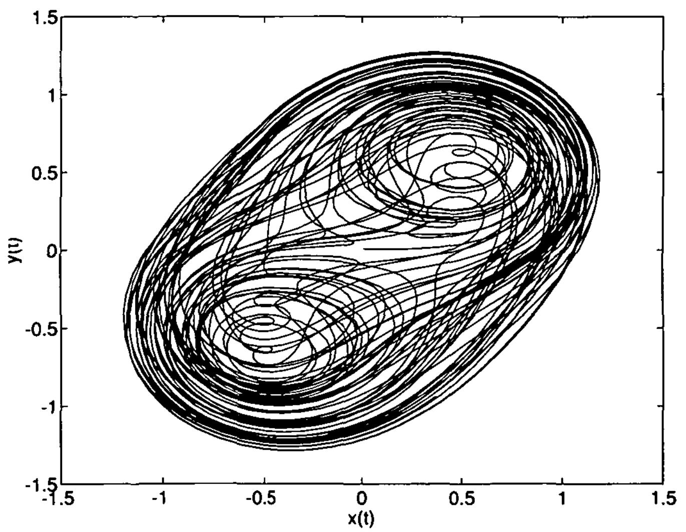

FIGURE 2. An orbit of (4.1)-(4.2) with $a = b = \omega = 1$, $c = -0.05$ and $\epsilon = 0.5$.

https://doi.org/10.1017/S0334270000007657 Published online by Cambridge University Press

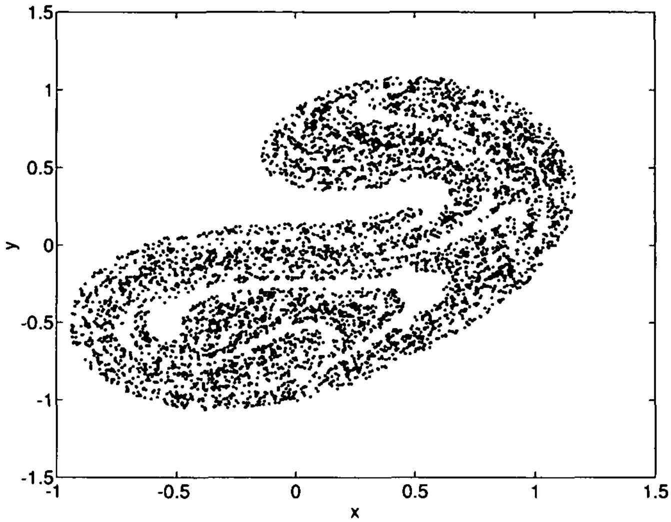

FIGURE 3. An attractor of (4.1)-(4.2) with $a = b = \omega = 1,\ c = -0.05$ and $\epsilon = 0.5$.

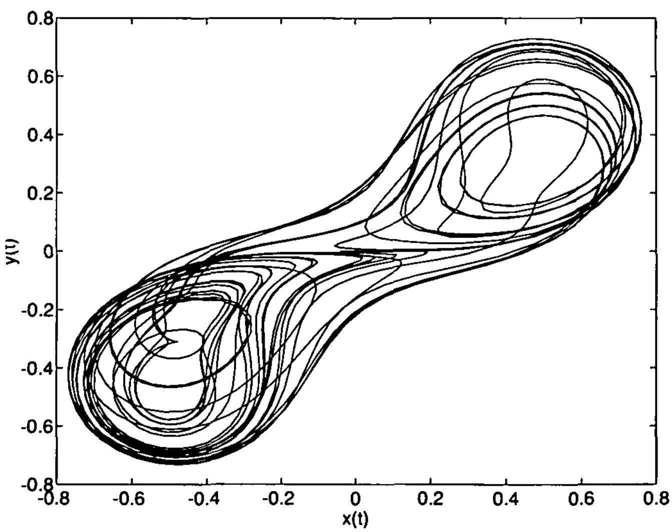

FIGURE 4. An orbit of (4.1)-(4.2) with $a = b = \omega = 1,\ c = -1.9$ and $\epsilon = 0.2$.

https://doi.org/10.1017/S0334270000007657 Published online by Cambridge University Press

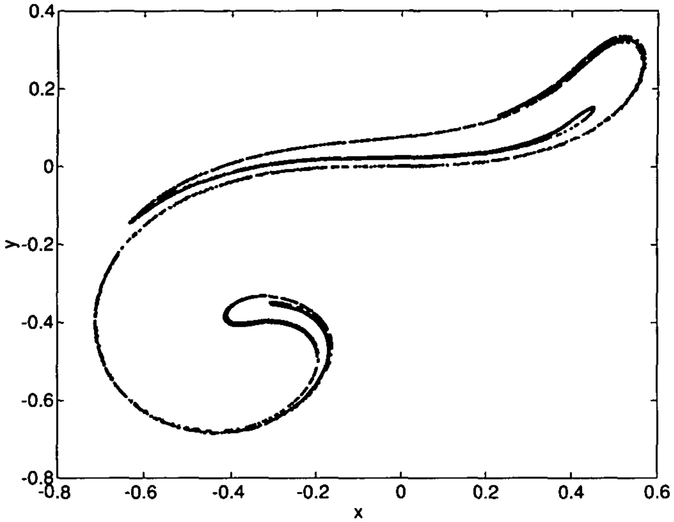

FIGURE 5. An attractor of (4.1)-(4.2) with $a = b = \omega = 1$, $c = -1.9$ and $\epsilon = 0.2$.

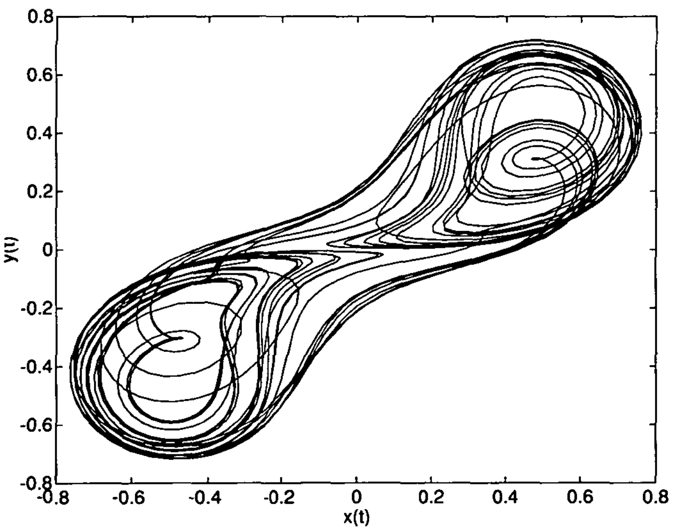

FIGURE 6. An orbit of (4.1)-(4.2) with $a = b = \omega = 1$, $c = -2$ and $\epsilon = 0.2$.

https://doi.org/10.1017/S0334270000007657 Published online by Cambridge University Press

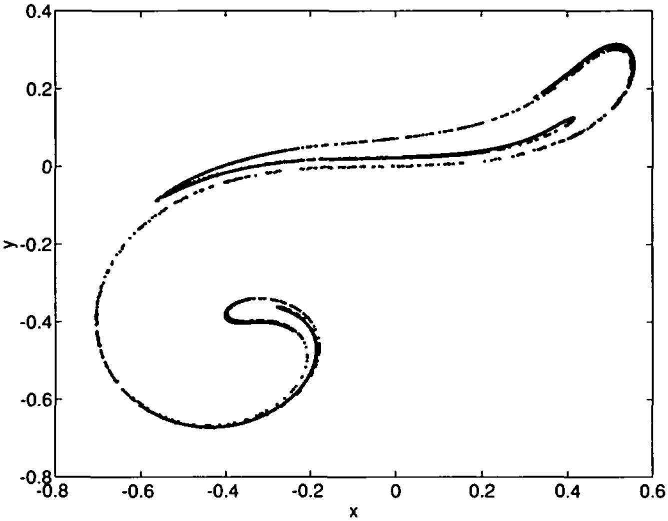

FIGURE 7. An attractor of (4.1)-(4.2) with $a = b = \omega = 1,\ c = -2$ and $\epsilon = 0.2$.

In Figures 2-5, the values of c satisfy the Melnikov condition (4.17) and the computer simulations demonstrate the chaotic behaviour for sufficiently small ϵ. However, in the case $c = -2$ in Figures 6 and 7, the Melnikov condition (4.17) is not satisfied but the computer simulations still indicate the presence of chaos; this is not contradictory because Melnikov's method only provides a sufficient condition for the existence of horseshoe chaos.

5. A class of homoclinic orbits

We now present a class of homoclinic orbits for systems of planar autonomous ordinary differential equations in the nonnegative quadrant of the x,y-plane. We begin by considering the system of differential equations

$$\frac{dx}{dt} = ax - \mu mny^{n-1}(x^n + y^n)^{m-1},$$
$$\frac{dy}{dt} = -ay + \mu mnx^{n-1}(x^n + y^n)^{m-1}, \tag{5.1}$$

where m, n are such that $m \geq 1, n \geq 1$ and $mn > 2$; and $a > 0, \mu > 0$. The equilibrium points of the system (5.1) in the nonnegative quadrant are $(0, 0)$ and $\left(\left[2^{1-m}a/(\mu mn)\right]^{1/(mn-2)}, \left[2^{1-m}a/(\mu mn)\right]^{1/(mn-2)}\right)$; the variational system corresponding to (0, 0) is

$$\begin{aligned}\frac{dX}{dt} &= aX,\\ \frac{dY}{dt} &= -aY,\end{aligned}\tag{5.2}$$

from which we observe that (0, 0) is a saddle point for (5.1); the unstable and stable manifolds of the linear system (5.2) are respectively the x and y axes of the x,y-plane. For the other equilibrium point, the corresponding variational system is

$$\begin{aligned}\frac{dX}{dt} &= \frac{a}{2}(2-mn+n)X + \frac{a}{2}(2-mn-n)Y,\\ \frac{dY}{dt} &= -\frac{a}{2}(2-mn-n)X - \frac{a}{2}(2-mn+n)Y,\end{aligned}\tag{5.3}$$

and one can show that this equilibrium point is a centre for (5.1). The nonlinear system (5.1) has a Hamiltonian H such that

$$\begin{aligned}\frac{dx}{dt} &= ax - \mu mny^{n-1}(x^n+y^n)^{m-1} = \frac{\partial H}{\partial y},\\ \frac{dy}{dt} &= -ay + \mu mnx^{n-1}(x^n+y^n)^{m-1} = -\frac{\partial H}{\partial x},\end{aligned}\tag{5.4}$$

where

$$H(x, y) = axy - \mu(x^n+y^n)^m, \qquad (x, y) \in \mathbb{R}^2.\tag{5.5}$$

The set of level curves $H(x, y) =$ constant are solution curves of (5.1), since we have from (5.1)

$$\frac{dy}{dx} = \frac{-ay + \mu mnx^{n-1}(x^n+y^n)^{m-1}}{ax - \mu mny^{n-1}(x^n+y^n)^{m-1}},$$

which gives

$$[ax - \mu mny^{n-1}(x^n+y^n)^{m-1}]dy + [ay - \mu mnx^{n-1}(x^n+y^n)^{m-1}]dx = 0,$$

simplifying to

$$d[axy - \mu(x^n+y^n)^m] = 0.\tag{5.6}$$

Thus the solution curves of (5.1) are given by the family of curves

$$H(x, y) = axy - \mu(x^n+y^n)^m = h,$$

where h is a constant. In particular, when $h = 0$, we obtain the solution curve approaching the origin, namely

$$\mu(x^n+y^n)^m = axy.$$

Since a and μ are both positive constants, without loss of generality we let $\mu = 1$, so that the solution curve approaching the origin is

$$(x^n + y^n)^m = axy, \qquad m \geq 1, n \geq 1, mn > 2. \tag{5.7}$$

To obtain a homoclinic orbit for (5.1), we need to find a solution $(x(t), y(t))$ of (5.1) such that

$$\lim_{t\to\pm\infty}(x(t), y(t)) = (0, 0). \tag{5.8}$$

To find $x(t)$ and $y(t)$ satisfying (5.8), we proceed as in Section 2; we differentiate the equation $\tan(\theta(t)) = y(t)/x(t)$ with respect to time to obtain

$$r^2\frac{d\theta}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt}, \tag{5.9}$$

and substituting the equations of (5.1) (with $\mu = 1$) and using (5.7) we get

$$\frac{d\theta}{dt} = \frac{1}{2}a(mn-2)\sin(2\theta) \tag{5.10}$$

and after separating the variables and integrating we find that

$$\tan(\theta(t)) = ke^{a(mn-2)t}, \qquad t \in \mathbb{R}, \tag{5.11}$$

where k is any positive constant. For convenience, we choose $k = 1$, so

$$\frac{y(t)}{x(t)} = \tan(\theta(t)) = e^{a(mn-2)t}, \qquad t \in \mathbb{R}, \tag{5.12}$$

and hence

$$y(t) = e^{a(mn-2)t}x(t), \qquad t \in \mathbb{R}. \tag{5.13}$$

Choosing $k = 1$ means that

$$(x(0), y(0)) = \left(\left[\frac{a}{2^m}\right]^{\frac{1}{mn-2}}, \left[\frac{a}{2^m}\right]^{\frac{1}{mn-2}}\right). \tag{5.14}$$

Substituting (5.13) in (5.7), we obtain that

$$x(t) = \frac{a^{\frac{1}{mn-2}}e^{at}}{[1+e^{an(mn-2)t}]^{\frac{m}{mn-2}}}, \qquad y(t) = \frac{a^{\frac{1}{mn-2}}e^{a(mn-1)t}}{[1+e^{an(mn-2)t}]^{\frac{m}{mn-2}}}, \qquad t \in \mathbb{R}, \tag{5.15}$$

is a solution of (5.1) for $\mu = 1$. One can easily verify, on using $mn > 2$, that the solution in (5.15) satisfies (5.8) and hence the orbit of the solution in (5.15) is a homoclinic orbit for (5.1). Thus, the saddle point (0,0) of (5.1) has a homoclinic orbit lying in the nonnegative quadrant of the x, y-plane. By choosing particular values of a, m and n, we obtain particular homoclinic orbits. Consider the following examples.

EXAMPLE 1. If $m = 2$, $n = 2$ and a is replaced by $2a^2$, then (5.1) becomes

$$\frac{dx}{dt} = 2a^2x - 4y(x^2 + y^2),$$
$$\frac{dy}{dt} = -2a^2y + 4x(x^2 + y^2)$$

and (5.7) is

$$(x^2 + y^2)^2 = 2a^2xy,$$

which we recognise from Section 2 as Bernoulli's lemniscate. From (5.15), a homoclinic parametrisation of this curve (in the nonnegative quadrant) is

$$x(t) = \frac{\sqrt{2}ae^{2a^2t}}{1 + e^{8a^2t}}, \qquad y(t) = \frac{\sqrt{2}ae^{6a^2t}}{1 + e^{8a^2t}}, \qquad t \in \mathbb{R}.$$

EXAMPLE 2. If $m = 1$, $n = 3$ and a is replaced by $3a$, then (5.1) becomes

$$\frac{dx}{dt} = 3ax - 3y^2,$$
$$\frac{dy}{dt} = -3ay + 3x^2$$

and (5.7) is

$$x^3 + y^3 = 3axy,$$

which is known as the folium of Descartes. From (5.15), a homoclinic parametrisation of this curve (in the nonnegative quadrant) is

$$x(t) = \frac{3ae^{3at}}{1 + e^{9at}}, \qquad y(t) = \frac{3ae^{6at}}{1 + e^{9at}}, \qquad t \in \mathbb{R}.$$

EXAMPLE 3. If $m = \frac{3}{2}$, $n = 2$ and a is replaced by $2a$, then (5.1) becomes

$$\frac{dx}{dt} = 2ax - 3y(x^2 + y^2)^{\frac{1}{2}},$$
$$\frac{dy}{dt} = -2ay + 3x(x^2 + y^2)^{\frac{1}{2}}$$

and (5.7) is

$$(x^2 + y^2)^{\frac{3}{2}} = 2axy,$$

which corresponds to two leaves of the four-leaved rose which is usually recognised by the polar equation

$$r^2 = a^2\sin^2(2\theta), \qquad \theta \in \mathbb{R}.$$

From (5.15), a homoclinic parametrisation of this curve (in the nonnegative quadrant) is

$$x(t) = \frac{2ae^{2at}}{(1+e^{4at})^{\frac{3}{2}}}, \qquad y(t) = \frac{2ae^{4at}}{(1+e^{4at})^{\frac{3}{2}}}, \qquad t \in \mathbb{R}.$$

We conclude this section with the remark that we have presented a class of infinitely many homoclinic orbits, each one of which is a solution curve of a planar autonomous system of ordinary differential equations.

6. An application to a one-dimensional oscillator

We consider the dynamics of an anharmonic oscillator parametrically driven at twice the resonant frequency ω_0. Such a system has recently been applied in optics in relation to the squeezing of light (DiFilippo et al. [4]). Very recently, Wielinga and Milburn [15] have considered an equivalent model in the context of quantum-mechanical tunnelling. In this section, we study the potential for a one-dimensional oscillator with a small ($|\alpha|z^2 \ll 1$) quartic anharmonic correction whose frequency is modulated at $2\omega_0$ by a weak ($\epsilon_1 \ll 1$) parametric drive :

$$U(z,t) = \frac{1}{2}m{\omega_0}^2 z^2 \left(1 + \epsilon_1 \sin(2\omega_0 t) + \frac{1}{2}\alpha z^2\right), \tag{6.1}$$

where m denotes the mass of a particle undergoing the oscillatory motion. Neglecting the higher order harmonics, we consider the oscillations with frequency ω_0 together with a dynamic phase as follows :

$$\begin{aligned} z(t) &= r(t)\cos(\omega_0 t - \theta(t)) \\ &= C(t)\cos(\omega_0 t) + S(t)\sin(\omega_0 t), \end{aligned} \tag{6.2}$$

where $C(t) = r(t)\cos(\theta(t))$ and $S(t) = r(t)\sin(\theta(t))$. Using the equation of motion

$$m\frac{d^2 z}{dt^2} = -\frac{\partial U}{\partial z}, \tag{6.3}$$

we can derive that

$$\begin{aligned} \frac{d^2 C}{dt^2} &= 2\omega_0\left(-\frac{\epsilon_1\omega_0}{4}S - \frac{3\alpha\omega_0}{8}C(C^2+S^2) - \frac{dS}{dt}\right), \\ \frac{d^2 S}{dt^2} &= -2\omega_0\left(\frac{\epsilon_1\omega_0}{4}C + \frac{3\alpha\omega_0}{8}S(C^2+S^2) - \frac{dC}{dt}\right). \end{aligned} \tag{6.4}$$

https://doi.org/10.1017/S0334270000007657 Published online by Cambridge University Press

We assume that $C(t)$ and $S(t)$ are slowly varying in the sense that $\frac{d^2C}{dt^2}$ and $\frac{d^2S}{dt^2}$ can be neglected in the system (6.4). With this approximation, (6.4) becomes

$$\begin{aligned}\frac{dC}{dt} &= \frac{\epsilon_1\omega_0}{4}C + \frac{3\alpha\omega_0}{8}S(C^2+S^2),\\ \frac{dS}{dt} &= -\frac{\epsilon_1\omega_0}{4}S - \frac{3\alpha\omega_0}{8}C(C^2+S^2).\end{aligned} \tag{6.5}$$

If we assume that $\epsilon_1 > 0$ and $\alpha < 0$, the system (6.5) is identical to (2.1) with $a^2 = \epsilon_1\omega_0/4$ and $\mu = -3\alpha\omega_0/8$. As in Section 2, we choose $\mu = 2$ for convenience.

Suppose that we apply to the system described by (6.1) an external potential of the form $\epsilon_2 z \sin(\omega_0 t)\sin(\omega t)$, where $0 \le \epsilon_2 \ll 1$ is a perturbation parameter and $\omega > 0$ is the frequency of the perturbation. The perturbed potential is now

$$U(z,t) = \frac{1}{2}m{\omega_0}^2 z^2\left(1 + \epsilon_1\sin(2\omega_0 t) + \frac{1}{2}\alpha z^2\right) + \epsilon_2 z\sin(\omega_0 t)\sin(\omega t), \tag{6.6}$$

and with our approximation the perturbed system is governed by the equations

$$\begin{aligned}\frac{dC}{dt} &= \frac{\epsilon_1\omega_0}{4}C + \frac{3\alpha\omega_0}{8}S(C^2+S^2) + \frac{\epsilon_2}{2m\omega_0}\sin(\omega t),\\ \frac{dS}{dt} &= -\frac{\epsilon_1\omega_0}{4}S - \frac{3\alpha\omega_0}{8}C(C^2+S^2).\end{aligned} \tag{6.7}$$

The perturbed system (6.7) is identical to (4.1) – (4.2) with $a^2 = \epsilon_1\omega_0/4, \alpha = -16/(3\omega_0), b = 1, c = 0$ and $\epsilon = \epsilon_2/(2m\omega_0)$. From (4.16), the Melnikov function for the system has simple zeros, so horseshoe chaos exists for sufficiently small ϵ.

Appendix

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{3a^2s}}{1+e^{4a^2s}}\cos(\omega s)ds &= \frac{\pi\exp(\frac{\pi\omega}{4a^2})[\exp(\frac{\pi\omega}{2a^2})+1]}{2\sqrt{2}a^2[\exp(\frac{\pi\omega}{a^2})+1]},\\ \int_{-\infty}^{\infty}\frac{e^{3a^2s}}{1+e^{4a^2s}}\sin(\omega s)ds &= \frac{\pi\exp(\frac{\pi\omega}{4a^2})[\exp(\frac{\pi\omega}{2a^2})-1]}{2\sqrt{2}a^2[\exp(\frac{\pi\omega}{a^2})+1]},\\ \int_{-\infty}^{\infty}\frac{e^{4a^2s}}{(1+e^{4a^2s})^2}ds &= \frac{1}{4a^2},\\ \int_{-\infty}^{\infty}\frac{e^{4a^2s}}{(1+e^{4a^2s})^4}ds &= \frac{1}{12a^2},\\ \int_{-\infty}^{\infty}\frac{e^{8a^2s}}{(1+e^{4a^2s})^4}ds &= \frac{1}{24a^2}.\end{aligned}$$

Acknowledgements

The first author is the holder of a Research Science Cadetship from the Defence Science and Technology Organisation (DSTO), Australia. The authors acknowledge the assistance of T.W. Sag of Flinders University with the computer simulations, and thank C.J. Coleman of DSTO and R.R. Huilgol of Flinders University for their suggestions. The authors also wish to thank the referees for their helpful comments.

References

[1] R. H. Abraham and J. E. Marsden, *Foundations of mechanics* (Benjamin/Cummings, Reading, Massachusetts, 1978).

[2] V. I. Arnold, *Mathematical methods of classical mechanics* (Springer-Verlag, New York, 1978).

[3] P. F. Byrd and M. D. Friedman, *Handbook of elliptic integrals for engineers and scientists* (Springer-Verlag, New York, 1971).

[4] F. DiFilippo, V. Natarajan, K. R. Boyce and D. E. Pritchard, "Classical amplitude squeezing for precision measurements", *Phys. Rev. Lett.* **68** (1992) 2859–2862.

[5] J. Guckenheimer and P. J. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields* (Springer-Verlag, New York, 1983).

[6] P. J. Holmes, "A nonlinear oscillator with a strange attractor", *Phil. Trans. Roy. Soc. A* **292** (1979) 419–448.

[7] P. J. Holmes, "Nonlinear oscillations and the Smale horseshoe map", *Proc. Symp. Appl. Math.* **39** (1989) 25–40.

[8] P. J. Holmes and J. E. Marsden, "Horseshoes in perturbations of Hamiltonians with two degrees of freedom", *Comm. Math. Phys.* **82** (1982) 523–544.

[9] A. J. Lichtenberg and M. A. Lieberman, *Regular and stochastic motion* (Springer-Verlag, New York, 1982).

[10] J. E. Marsden, "Chaos in dynamical systems by the Poincaré-Melnikov-Arnold method", in *Chaos in nonlinear dynamical systems* (ed. J. Chandra), (SIAM, Philadelphia, 1984) 19–31.

[11] V. K. Melnikov, "On the stability of the center for time periodic perturbations", *Trans. Moscow Math. Soc.* **12** (1963) 1–57.

[12] H. Poincaré, "Sur les équations de la dynamique et le problème des trois corps", *Acta Math.* **13** (1890) 1–270.

[13] S. Smale, "Diffeomorphisms with many periodic points", in *Differential and combinatorial topology* (ed. S.S̃. Cairns), (Princeton University Press, Princeton, 1963) 63–80.

[14] S. Smale, "Differentiable dynamical systems", *Bull. Amer. Math. Soc.* **73** (1967) 747–817.

[15] B. Wielinga and G. J. Milburn, "Quantum tunneling in a Kerr medium with parametric pumping", *Phys. Rev. A* **48** (1993) 2494–2496.

[16] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos* (Springer-Verlag, New York, 1990).