



Chaos in fractional-order autonomous nonlinear systems

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Abstract

We numerically investigate chaotic behavior in autonomous nonlinear models of fractional order. Linear transfer function approximations of the fractional integrator block are calculated for a set of fractional orders in $(0, 1]$, based on frequency domain arguments, and the resulting equivalent models are studied. Two chaotic models are considered in this study; an electronic chaotic oscillator, and a mechanical chaotic “jerk” model. In both models, numerical simulations are used to demonstrate that for different types of model nonlinearities, and using the proper control parameters, chaotic attractors are obtained with system orders as low as 2.1. Consequently, we present a conjecture that third-order chaotic nonlinear systems can still produce chaotic behavior with a total system order of $2 + \varepsilon$, $1 > \varepsilon > 0$, using the appropriate control parameters. The effect of fractional order on the chaotic range of the control parameters is studied. It is demonstrated that as the order is decreased, the chaotic range of the control parameter is affected by contraction and translation. Robustness against model order reduction is demonstrated.

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1. Introduction

Chaotic systems have been a focal point of renewed interest for many researchers in the past few decades. Such nonlinear systems can occur in various natural and man-made systems, and are known to have great sensitivity to initial conditions. Thus, two trajectories starting at arbitrarily nearby initial conditions in such systems could evolve in drastically different fashions, and soon become totally uncorrelated. At first glance, chaotic time trajectories look very much like noise. In fact, chaotic signals and noise have similar broad-band frequency spectrum characteristics. However, there is a fundamental difference between noise and chaos, *determinism*: whereas chaos can be classified as deterministic but unpredictable, noise is neither deterministic nor predictable. This unpredictability of chaotic time signals has been utilized for secure communication applications. Basically, the useful signal is encapsulated in a chaotic envelope (produced by a chaotic oscillator) at the transmitter end, and is transmitted over the communication channel as a chaotic signal. At the receiver end, the information-bearing signal is recovered using various techniques, e.g. synchronization [1–3].

According to the Poincaré–Bendixon theorem [4], an integer order chaotic nonlinear system must have a minimum order of 3 for chaos to appear. However, in fractional-order nonlinear systems, it is not the case. For example, it has been shown that Chua’s circuit of order as low as 2.7 can produce a chaotic attractor [5]. Nonautonomous Duffing systems of fractional order have been addressed in [6], where it is shown that a sinusoidally driven Duffing system of order less than 2 can still behave in a chaotic manner.

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The underlying theory behind fractional-order systems is deeply rooted in fractional calculus [7]. Basically, these systems are characterized by characteristic equations of fractional order. From a state-space point of view, the fractional order of the resulting characteristic equation results from a fractional-order integrator placed, for example, in place of the output integrator in the system block diagram realization. From a circuit realization standpoint, a fractional-order circuit is obtained using fractional capacitors. Recently, a solid-state implementation of fractal capacitors has been reported [8]. A fractional capacitor ‘ C ’ of order ‘ m ’ has an Laplace domain impedance given by $Z(s) = 1/(Cs)^m$. Clearly, this device is characterized by a phase shift of $m\pi/2$, for sinusoidal excitation, and becomes 90° as expected when the order becomes unity. Moreover, the fractional capacitor is not lossless, as can be seen from its impedance. As more advances are brought about in the way of solid-state implementations of such devices, we are likely to see a flurry of fractional-order integrated circuits. Thus, the need to understand and be able to analyze such circuits becomes vital.

The analysis of fractional-order systems is by no means trivial. Therefore, the course of numerical simulations is often adopted in order to study the behavior of these systems. For example, fractional-order nonchaotic Wien bridge

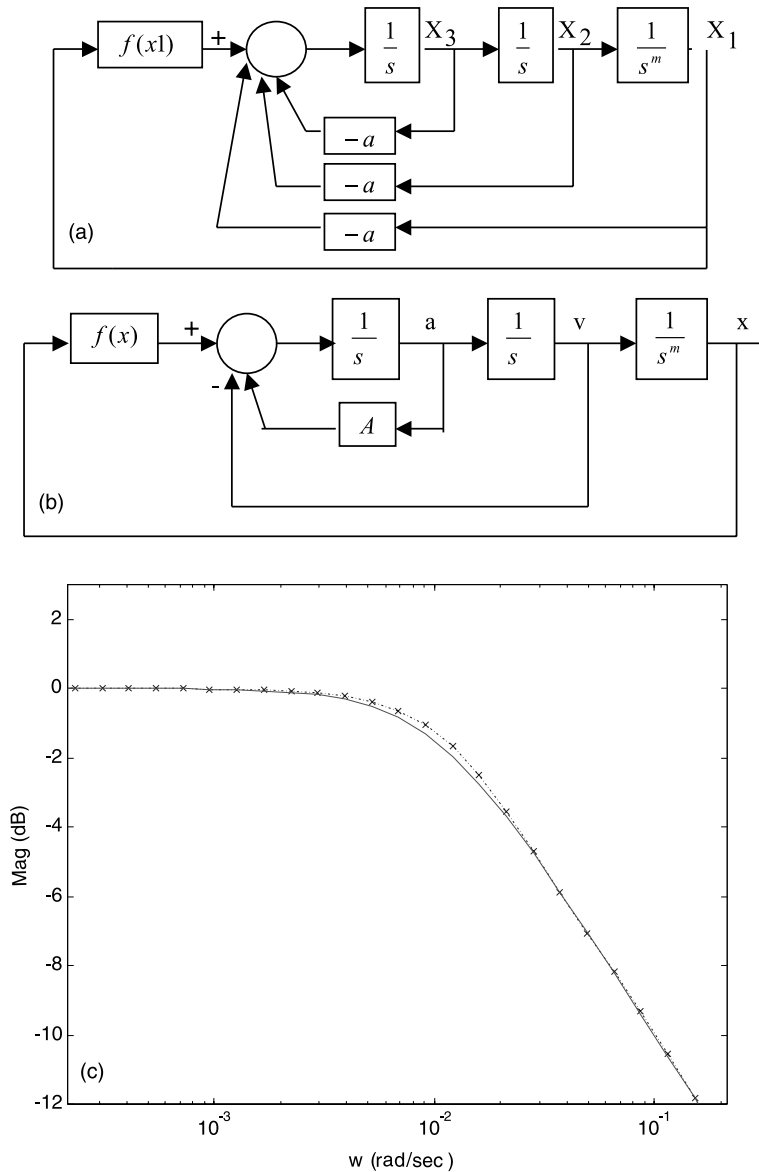


Fig. 1. (a) Block diagram of fractional chaotic oscillator. (b) Block diagram of fractional chaotic jerk model. (c) Fractional integrator frequency responses: (—) actual, (—x) approximation; $m = 0.5$, and corner frequency of 0.01 rad/s.

oscillators have recently been studied in [9], where it is shown that limit cycles can be attained for any fractional order, with the proper value of the amplifier gain which, of course, is a function of the fractional order. In our study we will

Table 1
Linear transfer function approximations of fractional integrator of order 'm', with maximum discrepancy $y = 2$ dB

m	N	$H(s)$
0.1	2	$\frac{1584.8932(s + 0.1668)(s + 27.83)}{(s + 0.1)(s + 16.68)(s + 2783)}$
0.2	2	$\frac{79.4328(s + 0.05623)(s + 1)(s + 17.78)}{(s + 0.03162)(s + 0.5623)(s + 10)(s + 177.8)}$
0.3	4	$\frac{39.8107(s + 0.0416)(s + 0.3728)(s + 3.34)(s + 29.94)}{(s + 0.02154)(s + 0.1931)(s + 1.73)(s + 15.51)(s + 138.9)}$
0.4	5	$\frac{35.4813(s + 0.03831)(s + 0.261)(s + 1.778)(s + 12.12)(s + 82.54)}{(s + 0.01778)(s + 0.1212)(s + 0.8254)(s + 5.623)(s + 38.31)(s + 261)}$
0.5	5	$\frac{15.8489(s + 0.03981)(s + 0.2512)(s + 1.585)(s + 10)(s + 63.1)}{(s + 0.01585)(s + 0.1)(s + 0.631)(s + 3.981)(s + 25.12)(s + 158.5)}$
0.6	5	$\frac{10.7978(s + 0.04642)(s + 0.3162)(s + 2.154)(s + 14.68)(s + 100)}{(s + 0.01468)(s + 0.1)(s + 0.6813)(s + 4.642)(s + 31.62)(s + 215.4)}$
0.7	5	$\frac{9.3633(s + 0.06449)(s + 0.578)(s + 5.179)(s + 46.42)(s + 416)}{(s + 0.01389)(s + 0.1245)(s + 1.116)(s + 10)(s + 89.62)(s + 803.1)}$
0.8	4	$\frac{5.3088(s + 0.1334)(s + 2.371)(s + 42.17)(s + 749.9)}{(s + 0.01334)(s + 0.2371)(s + 4.217)(s + 74.99)(s + 1334)}$
0.9	2	$\frac{2.2675(s + 1.292)(s + 215.4)}{(s + 0.01292)(s + 2.154)(s + 359.4)}$

Table 2
Linear transfer function approximations of fractional integrator of order 'm', with maximum discrepancy $y = 3$ dB

m	N	$H(s)$
0.1	2	$\frac{501.14(s + 0.6811)}{(s + 0.3162)(s + 681.1)}$
0.2	2	$\frac{141.2538(s + 0.1334)(s + 10)}{(s + 0.05623)(s + 4.217)(s + 316.2)}$
0.3	4	$\frac{125.8925(s + 0.08483)(s + 2.276)(s + 61.05)}{(s + 0.03162)(s + 0.8483)(s + 22.76)(s + 610.5)}$
0.4	5	$\frac{26.6073(s + 0.07499)(s + 1.334)(s + 23.71)}{(s + 0.02371)(s + 0.4217)(s + 7.499)(s + 133.4)}$
0.5	5	$\frac{50.1187(s + 0.07943)(s + 1.259)(s + 19.95)(s + 316.2)}{(s + 0.01995)(s + 0.3162)(s + 5.012)(s + 79.43)(s + 1259)}$
0.6	5	$\frac{28.1838(s + 0.1)(s + 1.778)(s + 31.62)(s + 562.3)}{(s + 0.01778)(s + 0.3162)(s + 5.623)(s + 100)(s + 1778)}$
0.7	5	$\frac{7.9433(s + 0.1638)(s + 4.394)(s + 117.9)}{(s + 0.01638)(s + 0.4394)(s + 11.79)(s + 100)(s + 316.2)}$
0.8	4	$\frac{8.1752(s + 0.487)(s + 36.52)(s + 2738)}{(s + 0.0154)(s + 1.155)(s + 86.6)(s + 6494)}$
0.9	2	$\frac{4.2987(s + 14.68)(s + 31620)}{(s + 0.01468)(s + 31.62)(s + 68130)}$

focus on two types of autonomous chaotic systems that are of practical interest: an electronic chaotic oscillator model [10], and a mechanical chaotic “jerk” model [11,12]. We will present simulation results of the chaotic behaviors produced from these two interesting systems as they acquire fractional orders.

One way to study fractional-order systems is through linear approximations. By utilizing frequency domain techniques based on Bode diagrams, one can obtain a linear approximation for the fractional-order integrator, the order of which depends on the desired bandwidth and discrepancy between the actual and the approximate magnitude Bode diagrams [13]. In this paper, we use this approach to study the behaviors of the fractional-order versions of our chaotic oscillator and jerk models. We derive the approximate linear transfer functions for the fractional integrator of order that varies from 0.1 to 0.9, and study the resulting behavior of the entire system for each case under the effect of different types of nonlinearities. We demonstrate that chaotic attractors are obtained for total system order as low as 2.1.

In the following, we will consider the state space equations of the systems under study. The output integrator block is assumed to have a fractional order ‘ m ’ in $(0, 1]$. In the Laplace domain, a fractional integrator of order ‘ m ’ can be represented by the transfer function:

$$F(s) = \frac{1}{s^m} \tag{1}$$

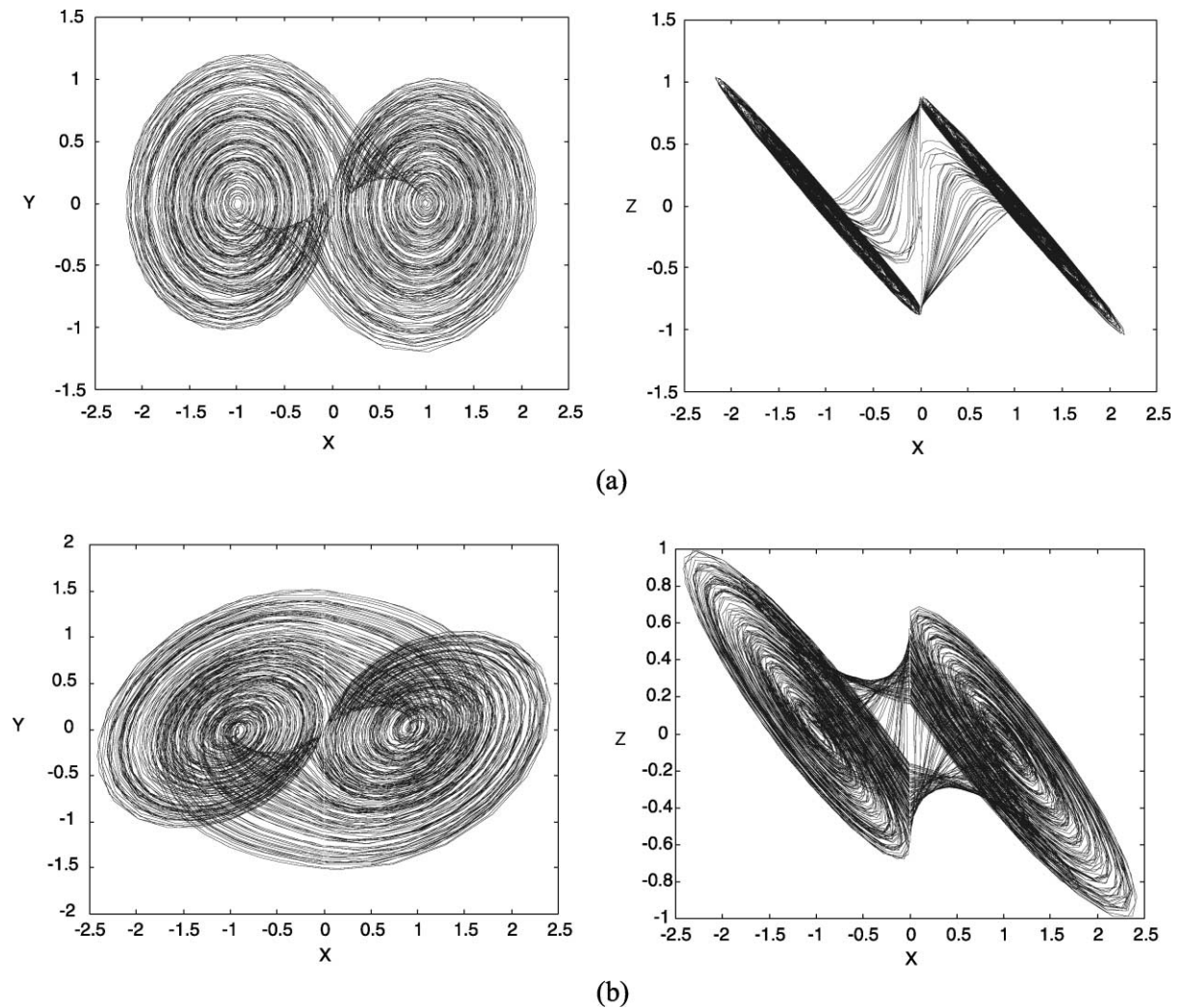


Fig. 2. (a) Integer order chaotic oscillator, $a = 0.8$, $f(x_1) = \text{sgn}(x_1)$. (b) Fractional oscillator, $m = 0.9$, $a = 0.4$, $f(x_1) = \text{sgn}(x_1)$, $y = 2$ dB.

The transfer function in (1) has a Bode diagram characterized by a slope of $-20m$ dB/decade. We will follow the algorithm in [13] to calculate linear transfer function approximations of (1). This approximation is based on approximating the $-20m$ dB/decade line with a number of zig-zag lines connected together with alternate slopes of 0 dB/decade and -20 dB/decade. According to [13], if the discrepancy between the actual and approximate lines is specified as ‘ y ’ dB over a frequency range of ω_{\max} , and for a corner frequency p_r , then Eq. (1) can be approximated as

$$F(s) = \frac{1}{s^m} \approx \frac{1}{\left(1 + \frac{s}{p_r}\right)^m} \approx \frac{\prod_{i=0}^{N-1} \left(1 + \frac{s}{z_i}\right)}{\prod_{i=0}^N \left(1 + \frac{s}{p_i}\right)} \tag{2}$$

In other words, the fractional integrator is approximated by a linear transfer function of order $N + 1$, where N is given by

$$N = 1 + \text{Integer} \left(\frac{\log \left(\frac{\omega_{\max}}{p_0} \right)}{\log(ab)} \right) \tag{3}$$

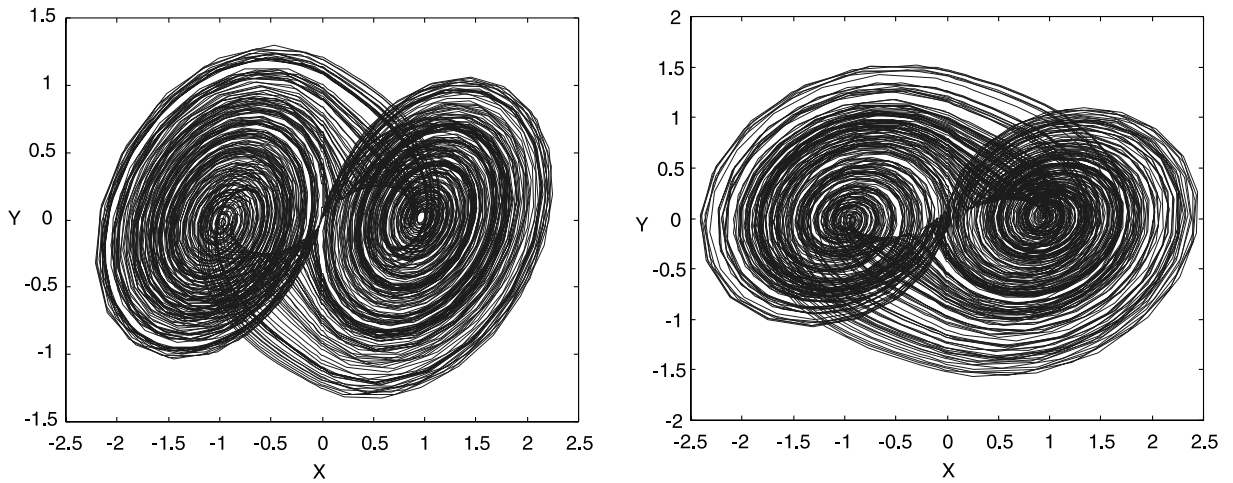


Fig. 3. Fractional oscillator, $m = 0.9$, $a = 0.5$, $f(x_1) = \text{sgn}(x_1)$: (left) $y = 2$ dB, (right) 3 dB.

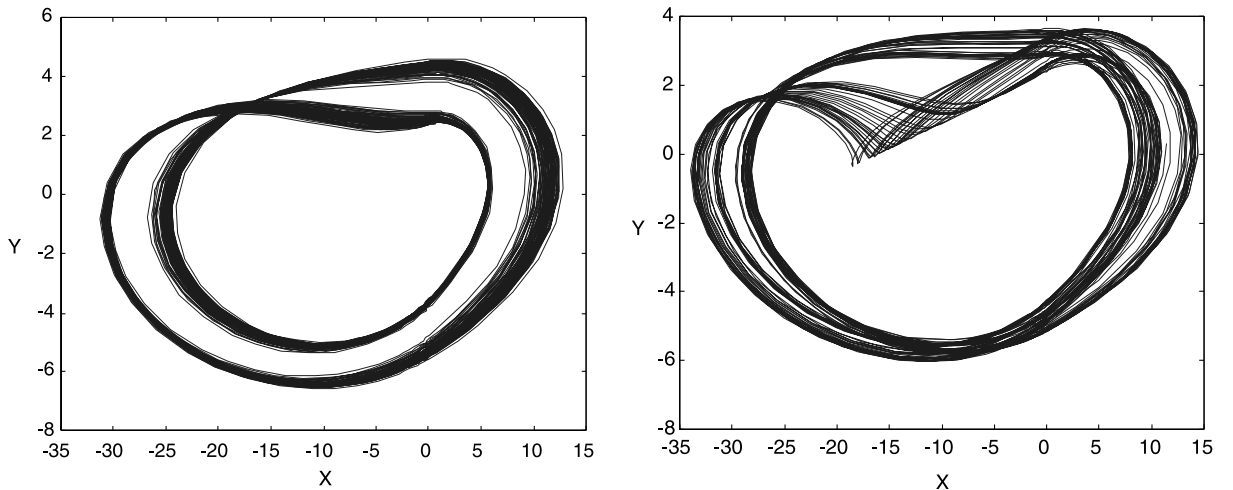


Fig. 4. Fractional oscillator, $m = 0.9$, $a = 0.05$, nonlinearity 9b: (left) $y = 2$ dB, (right) 3 dB.

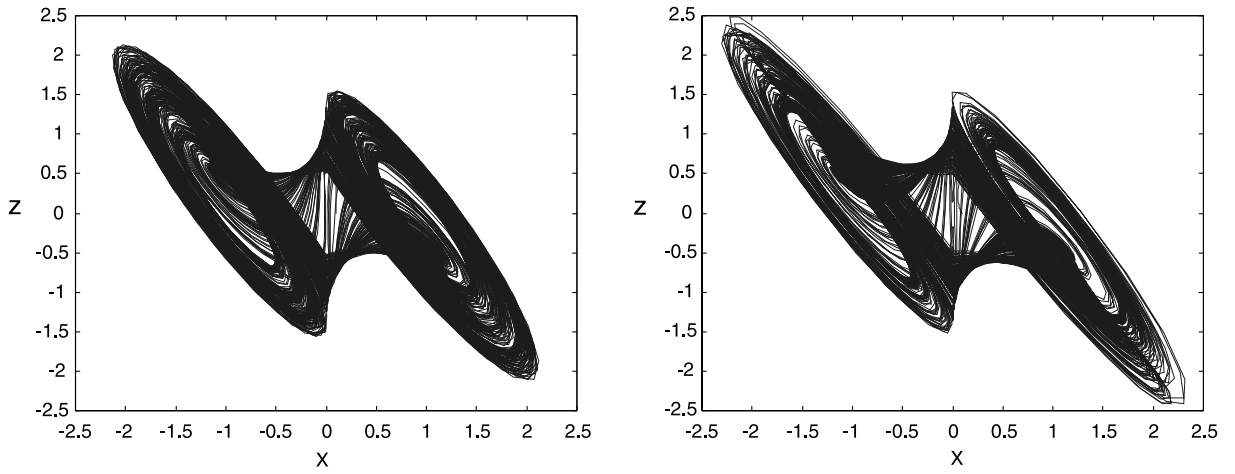


Fig. 5. Fractional oscillator, $m = 0.9$, $a = 0.97$, nonlinearity 9c: (left) $y = 2$ dB, (right) 3 dB.

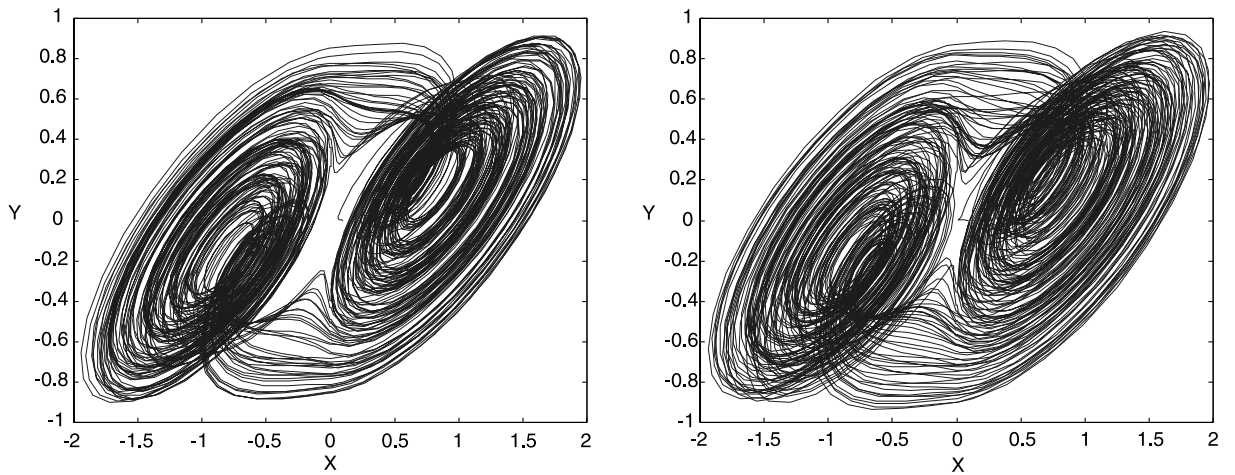


Fig. 6. Fractional oscillator, $m = 0.5$, $a = 0.2$, $f(x_1) = \text{sgn}(x_1)$: (left) $y = 2$ dB, (right) 3 dB.

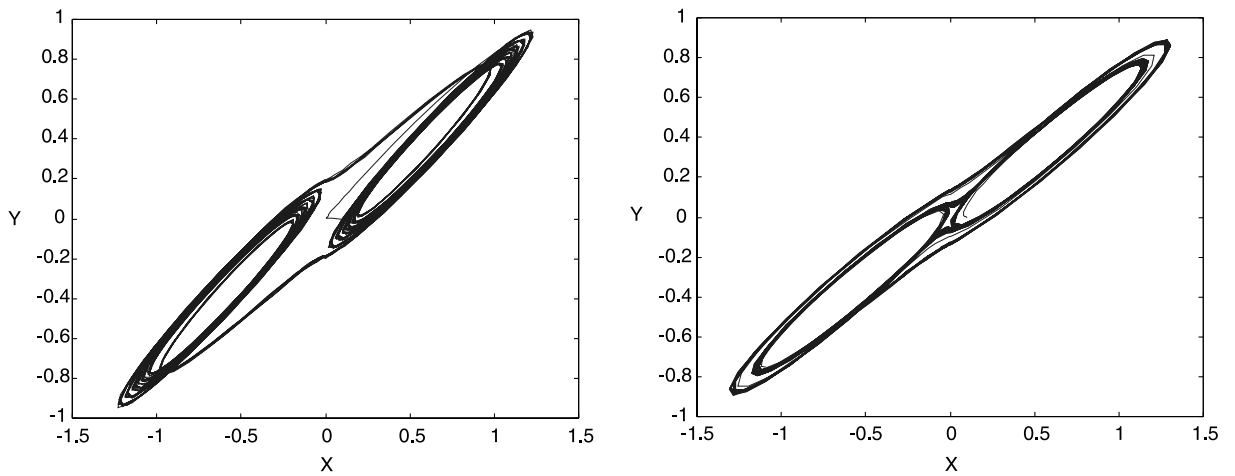


Fig. 7. Fractional oscillator, $m = 0.1$, $a = 0.02$, $f(x_1) = \text{sgn}(x_1)$: (left) $y = 2$ dB, (right) 3 dB.

and p_0 , a , and b are given by

$$p_0 = p_T 10^{(y/20m)} \tag{4}$$

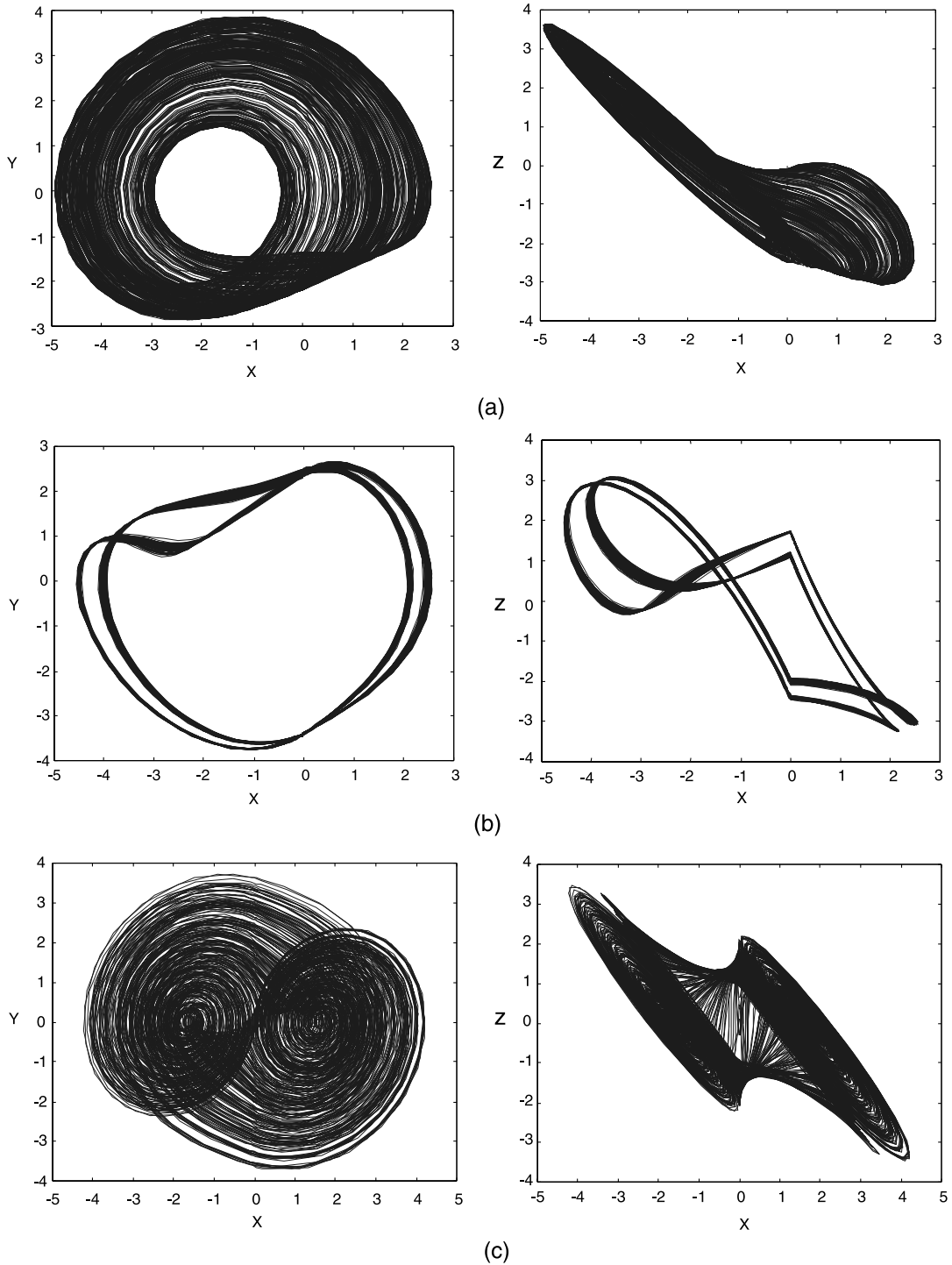


Fig. 8. Jerk model, $a = 0.6$, nonlinearity 9a (a), 9b (b) and 9c (c).

$$a = 10^{[\gamma/10(1-m)]} \tag{5}$$

$$b = 10^{(\gamma/10m)} \tag{6}$$

Clearly, the order of the approximating transfer function depends on the fractional order ‘ m ’ as will be verified shortly. We will choose arbitrarily $\omega_{\max} = 100$ and $p_T = 0.01$ to calculate the approximating transfer functions that will be used in the simulations.

2. Nonlinear models

2.1. Fractional chaotic oscillator

We consider the one-parameter, third-order chaotic oscillator of canonical structure reported in [10] as being the simplest possible structure for a chaotic oscillator. This model is shown in block diagram in Fig. 1a. The output in-

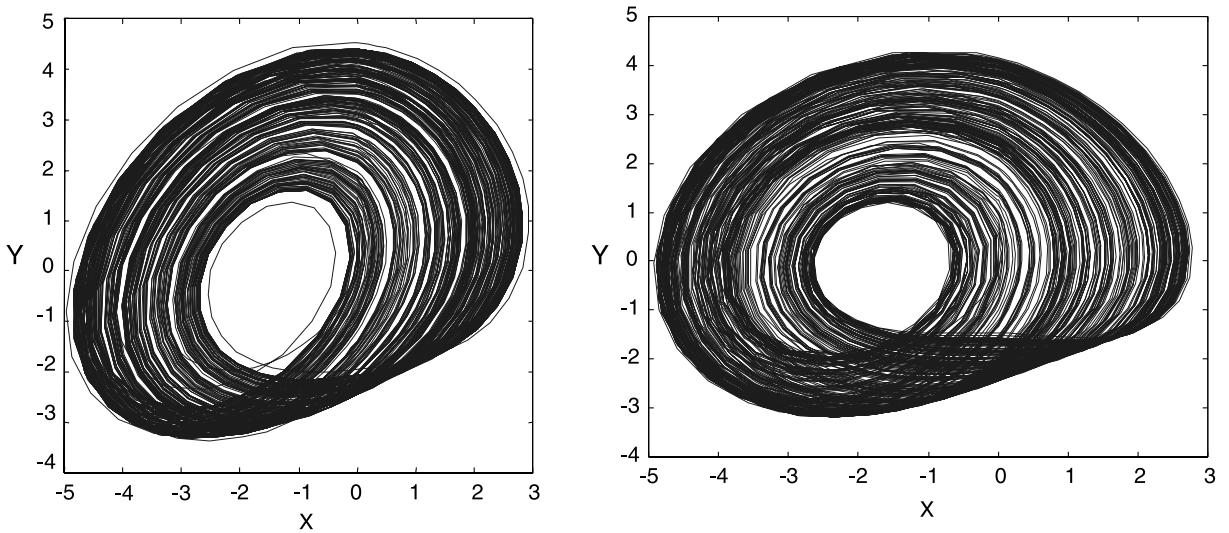


Fig. 9. Fractional jerk, $m = 0.9$, nonlinearity 9a: (left) $a = 0.4$, $\gamma = 2$ dB, (right) $a = 0.5$, $\gamma = 3$ dB.

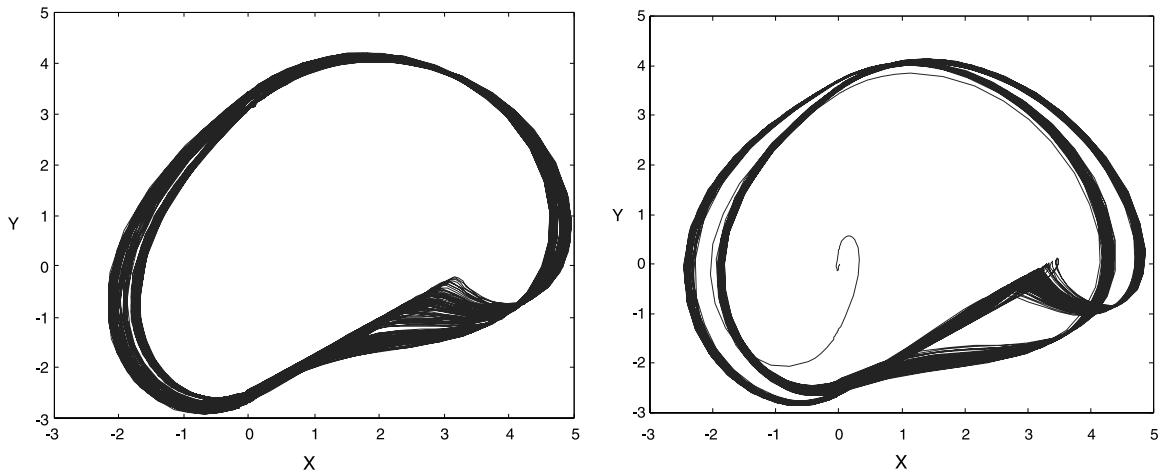


Fig. 10. Fractional jerk, $m = 0.9$, nonlinearity 9b: (left) $a = 0.3$, $\gamma = 2$ dB, (right) $a = 0.49$, $\gamma = 3$ dB.

tegrator of this model is then replaced with a fractional integrator. The resulting fractional chaotic oscillator model is then given in state space as

$$\begin{aligned} \frac{dx_1^m}{dt^m} &= x_2 \\ \frac{dx_2}{dt} &= x_3 \\ \frac{dx_3}{dt} &= -a(x_1 + x_2 + x_3 + f(x_1)) \end{aligned} \tag{7}$$

where ‘ a ’ is the control parameter for this oscillator, and $f(x_1) = \text{sgn}(x_1)$ is the model nonlinearity. In the integer order case, this oscillator is known to give a double-scroll-like chaotic attractor in the range $1.0 > a > 0.49$.

2.2. Fractional jerk model

This model, shown in Fig. 1b, is used to determine the time derivative of acceleration of an object, referred to as jerk [3,11,12]. In state space, it’s given as follows:

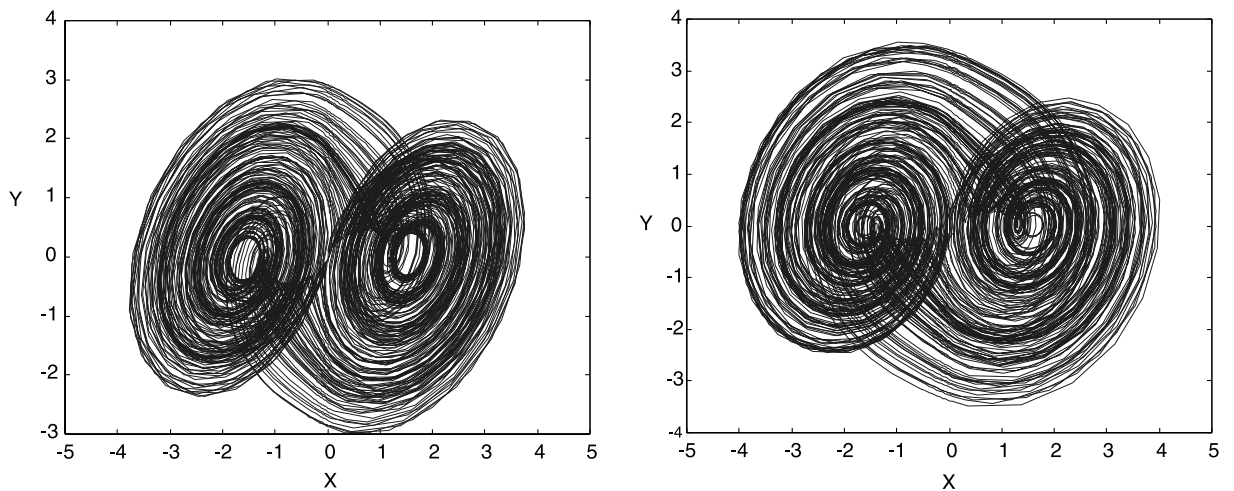


Fig. 11. Fractional jerk, $m = 0.9$, $a = 0.6$, nonlinearity 9c: (left) $y = 2$ dB, (right) $y = 3$ dB.

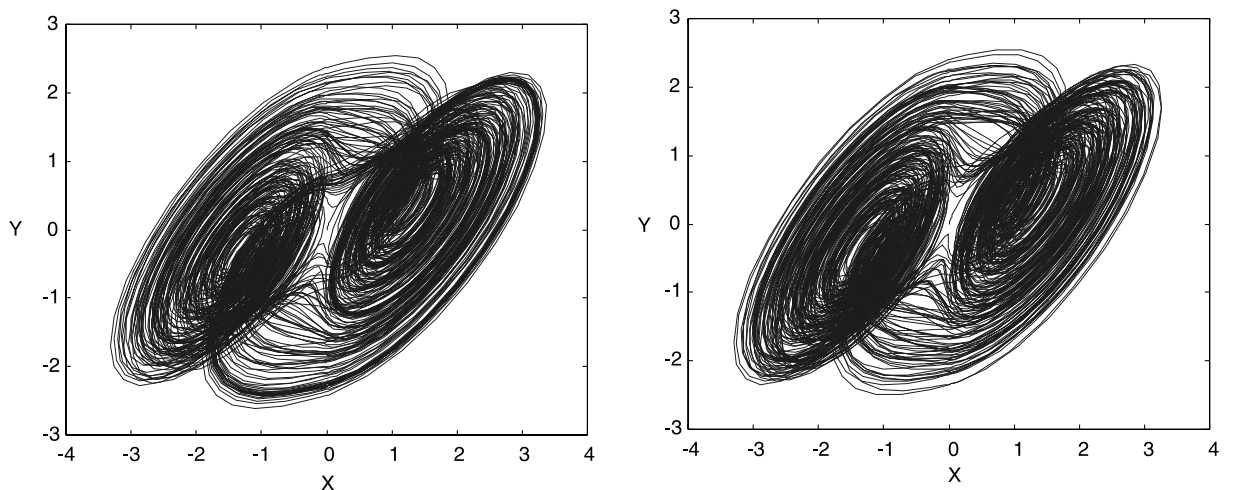


Fig. 12. Fractional jerk, $m = 0.5$, $a = 0.3$, nonlinearity 9c: (left) $y = 2$ dB, (right) $y = 3$ dB.

$$\begin{aligned}
 \frac{dx^m}{dt^m} &= v \\
 \frac{dv}{dt} &= a \\
 \frac{da}{dt} &= -Aa - v + f(x)
 \end{aligned}
 \tag{8}$$

where x , v , and a are, respectively, the position, velocity, and acceleration of the object, $f(x)$ is the model nonlinearity, and ‘ A ’ is the control parameter. In its integer form, i.e. $m = 1$, this model is known to give chaos for $A \approx 0.6$ and different types of nonlinearities, as shown below:

$$f(x) = \left\{ \begin{array}{ll} |x| - 2 & \text{(a)} \\ 1.2x - 4.5 \operatorname{sgn}(x) & \text{(b)} \\ -1.2x + 2 \operatorname{sgn}(x) & \text{(c)} \end{array} \right\}
 \tag{9}$$

We will study the behavior of the two models above for the nonlinearities given in (9).

3. Transfer function approximations

Using the algorithm in [11], Table 1 gives the resulting approximating transfer functions, $H(s)$ for different fractional orders, in increments of 0.1, for the m th order output integrator, assuming $\omega_{\max} = 100$ and $p_T = 0.01$. The maximum error, ‘ γ ’, assumed in the calculation is 2 dB. We also calculate in Table 2 another set of approximate transfer functions assuming $\gamma = 3$ dB, in order to study the sensitivity of the system behavior to approximation errors. Notice that the order of the approximate transfer functions increases as the desired error decreases.

The chaotic systems can then be represented by block diagrams as shown in Fig. 1a and b.

In Fig. 1c, we show the magnitude Bode diagrams for a fractional integrator of order $m = 0.5$, and its linear approximating transfer function (from Table 1). It can be seen that within the bandwidth of interest the two diagrams are in good agreement.

4. Simulation results

In our simulations, we have visually inspected the bifurcation diagrams to identify chaos. We have also utilized the CDA® [14] software package to confirm our findings by looking at Lyapunov exponents and power spectra in some cases.

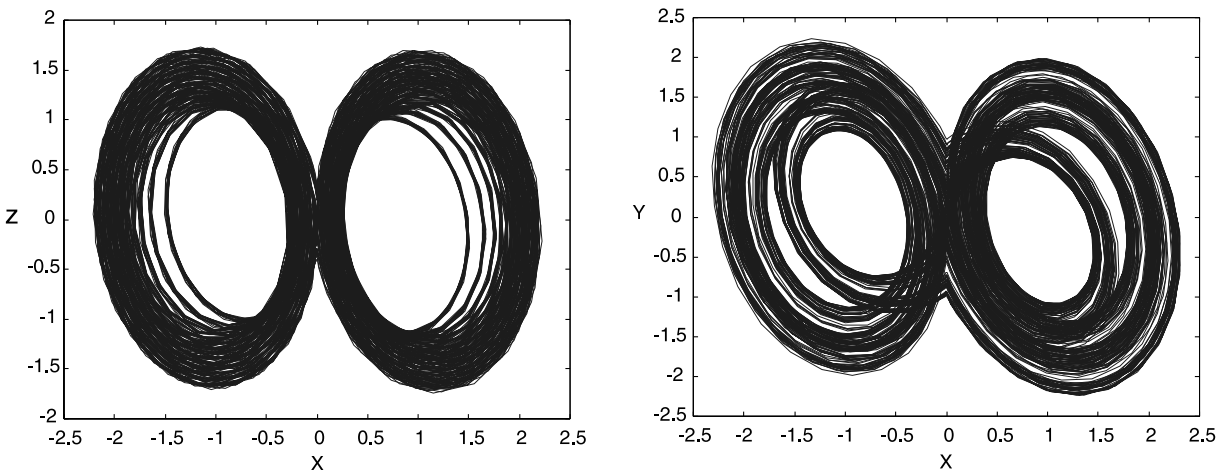


Fig. 13. Fractional jerk, $m = 0.1$, $a = 0.05$, nonlinearity 9c: (left) $\gamma = 2$ dB, (right) $\gamma = 3$ dB.

The integer order canonical chaotic oscillator with nonlinearity given by $f(x_1) = \text{sgn}(x_1)$ is known to give a double-scroll-like chaotic attractor for control parameter values in the range $1.0 > a > 0.49$. This is shown in Fig. 2a which was obtained for $a = 0.8$. In Fig. 2b, we show that the same type of chaotic attractor is obtained for $a = 0.4$ and $m = 0.9$ which amounts to a total system order of 2.9. Notice, however, that this value of the control parameter was not in the chaotic range of the integer order model at all. In fact, the integer order system gives an unstable response at $a = 0.4$. In Fig. 3, we show the behavior obtained from the chaotic oscillator for the two error values (2 and 3 dB) at $a = 0.5$, $m = 0.9$, and the nonlinearity $f(x_1) = \text{sgn}(x_1)$. As can be seen from the figure, the double-scroll-like attractor is

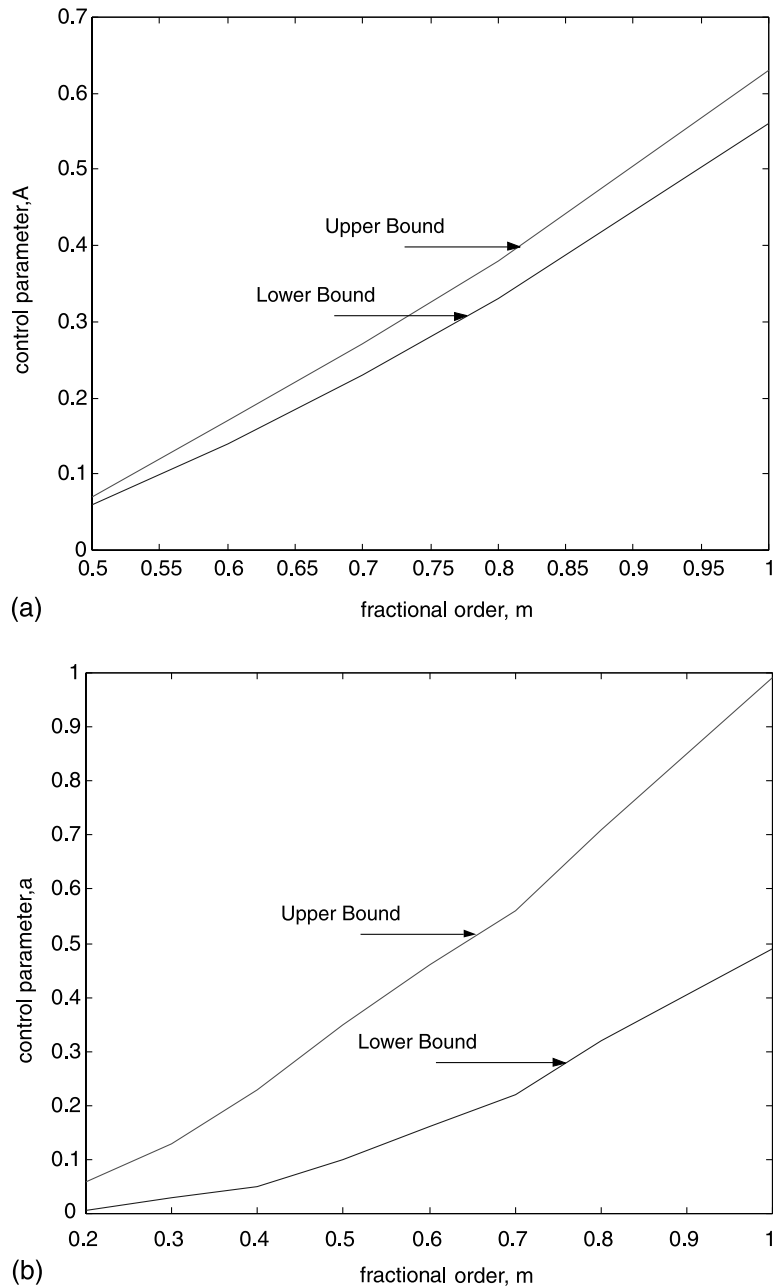


Fig. 14. (a) Chaotic range versus fractional order for fractional jerk model. (b) Chaotic range versus fractional order for fractional oscillator model.

preserved despite the increase in modeling error. The same model is simulated in Fig. 4 using nonlinearity 9b and $a = 0.05$, and in Fig. 5 with $a = 0.97$ and nonlinearity 9c. We further reduced the total system order down to 0.5, Fig. 6, and 2.1, Fig. 7, and in both cases we obtained the same type of attractor with $a = 0.2$ and 0.02 , respectively, and using the nonlinearity $f(x_1) = \text{sgn}(x_1)$. These figures demonstrate clearly that indeed the chaotic behavior of the oscillator is not destroyed by order reduction. Moreover, reasonable modeling errors seem to be tolerable and not to affect the chaotic behavior of the system. Also, it can be noticed that the effective chaotic range of the control parameter shifts downward away from that corresponding to the integer order model. For example, with $m = 0.5$, we obtain chaotic behavior, a double-scroll-like attractor, in the range of 'a' between 0.1 and 0.3, as opposed to 0.49 to 1 for $m = 1$. Also, $a = 1$ for the integer order oscillator gives a limit cycle, whereas with $m = 0.9$ the system is chaotic.

As for the fractional jerk model, we have noticed that it was more sensitive to modeling error than the oscillator model. We have carried out similar simulations using different fractional-order values and different nonlinearities. The baseline plots, obtained from the integer order jerk model, are shown in Fig. 8. In Figs. 9–11, we show simulation results of the fractional jerk model of total order of 2.9 and using the nonlinearities 9a, 9b, and 9c. The control parameter value had to be somewhat tuned as the error was changed from 2 to 3 dB. Qualitatively speaking, the chaotic behavior is preserved in each case. In Figs. 12 and 13, we show simulations of the fractional jerk model of total orders 2.5 and 2.1, using nonlinearity 9c, and control values of 0.3 and 0.05, respectively. Notice, once again, the reduction in the effective chaotic range of the control parameter, A , as 'm' decreases. In Fig. 14, we plot the bounds of the effective chaotic range of the control parameter for the two models versus the value of 'm' assuming nonlinearity 9a as an example. Clearly, as the fractional order increases the effective chaotic range increases somewhat proportionally. It is worth mentioning that the chaotic range for the fractional jerk model is narrow to begin with. Consequently, as the fractional order decreases below $m = 0.5$, the chaotic range shrinks significantly, and becomes much less pronounced than that for the chaotic oscillator as shown in the figure.

Finally, we have conducted several numerical simulations on fractional systems with total system order less than 2, as would be obtained by replacing all three integrators by fractional ones. However, no evidence of chaos has been shown. From the above results, we present the following conjecture:

Conjecture. *A third-order chaotic autonomous nonlinear system, with the appropriate nonlinearity and control parameters, is chaotic for any fractional order $2 + \varepsilon$, $1 > \varepsilon > 0$.*

5. Conclusion

We have demonstrated via numerical simulations that chaotic autonomous nonlinear systems can still exhibit chaotic behaviors when the order becomes fractional. Linear approximation techniques which are based on frequency domain arguments have been used to obtain approximate linear models of the given systems. The resulting order of the approximating function of the fractional integrator depends on the desired bandwidth, and error between actual and approximate Bode magnitude plots of the fractional integrator. Both electronic chaotic oscillators and chaotic jerk models were studied. The effective chaotic range of the control parameter for either model shrinks and moves downward as the total system order decreases and approaches 2, i.e. the chaotic range is affected by contraction and translation processes as the fractional order decreases. The chaotic behaviors of the fractional-order models exhibit some degree of qualitative robustness against model order reduction as would be caused by modeling errors.

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