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CHAOS-REVEALING MULTIPLICATIVE REPRESENTATION OF QUANTUM EIGENSTATES

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ABSTRACT

The quantization of the two-dimensional toric and spherical phase spaces is considered in analytic coherent state representations. Every pure quantum state admits there a finite multiplicative parametrization by the zeros of its Husimi function. For eigenstates of quantized systems, this description explicitly reflects the nature of the underlying classical dynamics: in the semiclassical regime, the distribution of the zeros in the phase space becomes one-dimensional for integrable systems, and highly spread out (conceivably uniform) for chaotic systems. This multiplicative representation thereby acquires a special relevance for semiclassical analysis in chaotic systems.

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It is still a basic open problem in semiclassical mechanics to describe the *individual* eigenfunctions of a quantum system when the corresponding classical dynamics is chaotic. The simplest models are found among area-preserving maps on compact phase spaces of dimension two, of which several have been quantized. However, as far as *rigorously proven* chaotic maps are concerned (e.g. the cat and baker's maps), the *torus* T^2 is the only phase space occurring to date. (See recent reviews by Eckhardt 1988, Voros 1989, and references therein.)

This letter is inspired by the successes of phase space representations for quantum eigenstates (Wigner: Hannay and Berry 1980, Husimi: Leboeuf and Saraceno 1990). From *analytic* coherent state representations for the torus and the sphere, we extract *finite multiplicative* parametrizations of the (pure) quantum states, which reveal distinctive patterns of semiclassical behaviour for eigenstates of integrable and chaotic systems.

The torus phase space is a periodically repeated unit square in suitable (q,p) coordinates. As its quantum Hilbert space, we can take the space $\mathcal{H}_N$ of wave functions $|\psi\rangle\rangle$ periodic both in position and momentum representations. This space has a *finite* dimension N, with the consistency condition

$$2\pi N\hbar = \text{Area } (=1). \tag{1}$$

Usual representations of this quantum mechanics are *discrete* and labelled by the finite integer N; $N \longrightarrow \infty$ gives the classical limit (Hannay and Berry 1980). We prefer to use a *continuous*, actually *analytic* coherent-state representation of $\mathcal{H}_N$, built out from the standard (Weyl group) coherent states on the plane R^2 (Klauder and Skagerstam 1985, Perelomov 1986).

A non-normalized coherent state of the plane, centered at a point (q,p), is

$$|z\rangle = \mathrm{e}^{\bar{z}a^\dagger}|0\rangle, \qquad z = 2^{-1/2}(q - ip) \; ; \tag{2}$$

it has the kernel

$$\langle z \mid q'\rangle = (\pi\hbar)^{-1/4}\mathrm{e}^{-\left[\frac{z^2+q'^2}{2} - \sqrt{2}\; zq'\right]\Big/\hbar} . \tag{3}$$

The coherent state decomposition of a wave function $|\psi\rangle$ over the whole line,

$$\psi(z) = \langle z \mid \psi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}q' \,\langle z \mid q'\rangle\, \psi\,(q') \;\; , \tag{4}$$

is an entire function $\psi(z)$ of order 2, the Bargmann transform of $|\psi\rangle$ (Bargmann 1961, 1967).

The adaptation to the torus is based on two observations:

a) a state $|\psi\rangle\rangle$ of $\mathcal{H}_N$ admits a natural position representation on the infinite coordinate line as a periodicized sum,

$$\psi(q) = \sum_{\nu=-\infty}^{+\infty} \sum_{n=0}^{N-1} \psi_n \delta(q - n/N - \nu), \tag{5}$$

where ψ_n are the discrete position components: $\psi_n = \langle\langle n|\psi\rangle\rangle$, the eigenstate $|n\rangle\rangle$ being "localized" at $q = \frac{n}{N}$ mod 1.

b) The original Bargmann transformation directly operates on the representation (5), which lies in the distribution space $\mathcal{F}^{-2}$ (in the notation of Bargmann 1967). There is consequently no need to invent a different transformation for $\mathcal{H}_N$. With Eqs.(1) and (3), Eq.(4) reduces to

$$\psi(z) = (2N)^{1/4} \sum_{n=0}^{N-1} \theta_3 \left(-i\pi N \left(\sqrt{2}\ z - n/N\right) \middle| iN\right) \mathrm{e}^{-\pi N\left[z^2+(n/N)^2-2\sqrt{2}\ z(n/N)\right]} \psi_n \quad (6)$$

where θ_3 is the Jacobi theta function (Whittaker and Watson 1965),

$$\theta_3(v \mid \tau) = \sum_{\nu=-\infty}^{+\infty} \mathrm{e}^{i\pi\tau\nu^2+2i\nu v}. \quad (7)$$

We may also interpret Eq.(6) as a scalar product in $\mathcal{H}_N$, $\psi(z) = \langle\langle z \mid \psi\rangle\rangle = \sum_{n=0}^{N-1} \langle\langle z|n\rangle\rangle\psi_n$, thereby defining the $\mathcal{H}_N$-coherent state $\langle\langle z|$, which obeys *quasi-period* relations for ν, μ integers:

$$\langle\langle z + 2^{-1/2}(\nu - i\mu)| = \mathrm{e}^{\pi N\left[i\mu\nu+\left(\nu^2+\mu^2\right)/2+\sqrt{2}(\nu+i\mu)z\right]} \langle\langle z|. \quad (8)$$

We now exploit the *analytic* properties of the representation (4). For $|\psi\rangle\rangle$ in $\mathcal{H}_N$, $\psi(z)$ is a function analytic in the fundamental square $\left[0, 2^{-1/2}\right] \times \left[0, 2^{-1/2}\right]$ with its boundary Γ included, and satisfying the continuation conditions drawn from Eq.(8),

$$\psi\left(z + 1/\sqrt{2}\right) = \mathrm{e}^{\pi N\left(\frac{1}{2}+\sqrt{2}\ z\right)} \psi(z)\ , \quad (9a)$$

$$\psi\left(z + i/\sqrt{2}\right) = \mathrm{e}^{\pi N\left(\frac{1}{2}-i\sqrt{2}\ z\right)} \psi(z)\ . \quad (9b)$$

Such a function has the following general properties (*compactness* of phase space enters now).

1) $$(2\pi i)^{-1} \oint_\Gamma \frac{\psi'}{\psi} \mathrm{d}z = N\ : \quad (10)$$

like a "polynomial of degree N", $\psi(z)$ has exactly N zeros in the square (zeros will be counted with their multiplicities).

2) $$(2\pi i)^{-1} \oint_\Gamma z\frac{\psi'}{\psi} \mathrm{d}z = 2^{-3/2} N(1+i) \qquad \mathrm{mod} \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right) \quad (11)$$

giving one constraint among the N zeros,

$$\sum_{j=1}^{N} z_j = 2^{-3/2} N(1+i) \qquad \mathrm{mod} \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right) \quad (12)$$

3) Of particular importance is the case $N = 1$, connected with the lattice representations of quantum mechanics (reviewed by Perelomov 1986). The Bargmann transform of the unique state in $\mathcal{H}_1$,

$$\psi_1(z) = 2^{1/4} \mathrm{e}^{-\pi z^2} \theta_3(-i\pi\sqrt{2}\ z \mid i). \tag{13}$$

is "the monomial". Its single zero is constrained by Eq.(12) to lie at the center z_0 of the square,

$$z_0 = 2^{-3/2}(1+i)\ . \tag{14}$$

4) The Weierstrass-Hadamard factorization allows the reconstruction of entire functions from their zeros. Here, it reduces to a multiplication formula from elliptic function theory

$$\psi(z) = Z\ \mathrm{e}^{\left[2\pi \sum_1^N (\bar{z}_j - \bar{z}_0)\right] z} \prod_1^N \psi_1\left(z + z_0 - z_j\right)\ . \tag{15}$$

Consequently, ignoring the complex factor Z (required by normalization), *we can represent each quantum state of $\mathcal{H}_N$ as an $(N-1)$-parameter family of points in the fundamental square, the complex zeros $\{z_j\}$*. A similar scheme has been recently used in a different context, the quantized Hall effect (Arovas et al. 1988).

For the phase space of a spin, the Bloch sphere S^2, a similar treatment has also long existed. If $S = \frac{1}{2}, 1, \ldots$ is the spin value, the Bargmann space $\mathcal{H}$ consists of ordinary polynomials of degree $2S$, with dim $\mathcal{H} = 2S + 1$ (Klauder and Skagerstam 1985). These have $2S$ unconstrained zeros, and Eq.(15) is replaced by the elementary factorization for polynomials. More generally, the factorization approach can be used with many spaces of entire functions. In higher dimensions, however, one will run into all the difficulties of analytic function theory in several variables.

Consider now the *Husimi representation* of a state $|\psi\rangle$ over the whole plane (see Kurchan et al. 1989 for the spherical case),

$$W_\psi\left(z, \bar{z}\right) = \frac{|\langle z \mid \psi\rangle|^2}{\langle z \mid z\rangle} = \mathrm{e}^{-|z|^2/\hbar} |\langle z|\psi\rangle|^2. \tag{16}$$

For ψ in $\mathcal{H}_N$, W_ψ becomes doubly-periodic by Eqs.(9), thereby defining a *Husimi distribution over the torus*. This W_ψ is not only positive as usual, but also precisely vanishing at the N zeros of $\psi(z)$. (Hence, by positivity, an impure state will give strictly fewer zeros, and generically none). Moreover, W_ψ factorizes via Eq.(15), as

$$W_\psi(z) = C \prod_1^N W_{\psi_1}\left(z + z_0 - z_j\right), \tag{17}$$

where W_{ψ_1} is the Husimi function of the "monomial" ψ_1 with $N = 1$ ($C > 0$ sets the normalization).

We have thus arrived at the conclusion that by itself, the set of zeros of W_ψ encodes the full quantum information, which is retrieved via Eq.(15). We therefore suggest that *semiclassical analysis itself can be directly based upon the zeros* of the Husimi function (rather than on the high density regions of W_ψ, or scars). Here, as in the WKB method, $\log\psi$ (or ψ'/ψ, or $\log W_\psi$) appear to be the relevant functions.

The most immediate semiclassical property of the zeros is a global one: their number is precisely N (in the square), and $N \longrightarrow \infty$. Further semiclassical properties must then translate themselves upon the asymptotic distribution of the zeros as $N \longrightarrow \infty$. We now confirm this idea by showing a sample of eigenstates in the Husimi representation, for various systems on the torus and the sphere.

Each Husimi function W_ψ is plotted twice: the top plot, on a *linear* density scale, stresses in dark the peaks of W_ψ (classical features, scars), while the bottom plot, on a *logarithmic* density scale with an adjusted contrast factor, stresses as white spots the zeros of W_ψ (quantum features).

Fig.1a shows the "monomial" ψ_1 ($N = 1$), given by Eq.(13), which is also the elementary factor in Eq.(15), while Fig.1b shows the coherent state $|0\rangle\rangle$ for $N = 16$.

Fig.2 shows eigenstates of *Hamiltonians*, this being the classically integrable case.

The Harper Hamiltonian is $H = 2 - \cos 2\pi p - \cos 2\pi q$ (on the torus). Its ground state looks very similar to the coherent state $|0\rangle\rangle$ having the same N (cf. Fig.1b). Fig.2a shows the 15-th state for $N = 31$, lying just below the classical *separatrix* energy (coordinates are shifted). The zeros are neatly aligned on four straight lines. In fact, all (nondegenerate) states of the system have this property.

The Lipkin Hamiltonian on the sphere (Lipkin et al. 1965) is $H = \cos\theta + \frac{1}{2}\chi \sin^2\theta \cos 2\varphi$; Fig.2b shows the 2nd state of the quantized model having $S = 15$ and $\chi = 0.5$. Again, *the zeros lie along curves*, which depend on the classical energy but seemingly not on $\dim \mathcal{H}$.

This is a property we have observed in all integrable examples studied.

Eigenfunctions of *chaotic maps* are now displayed on Figs.3-4.

Fig.3 shows, on the torus, two states of the quantized baker's map with $N = 64$ (Balazs and Voros 1989, Saraceno 1990), chosen for their distinctive features *on the linear scale*: on Fig.3a, an "almost ergodic" state; on Fig.3b, in shifted coordinates, a state strongly scarred by the unique classical fixed point at the origin. Unlike the former, the latter is highly concentrated and reminiscent of the integrable separatrix state of Fig.2a. However, the pictures on the logarithmic scale display a totally different connection: both baker states have their *zeros highly spread out* on the square, as opposed to the concentration along curves of integrable examples. This holds for all baker states; we have only found, for $N = 2^k$, one isolated (arithmetical ?) exception of a state with a dominantly linear distribution of zeros.

The same property holds for the cat map (also defined on the torus). One "irregular"(typical) state, and a very regular one (there are a few of them), are shown on Figs.4a-b, for the quantized cat map with $N = 64$ (Hannay and Berry 1980).

Thus, irrespectively of the superficial features of the Husimi distribution (irregular or regular, heavily scarred or not), *the zeros seem to fill, like a gas, the whole phase space area left out by the high density regions.* We view this behavior as a *characteristic quantal signature of classical chaos.*

Finally, we have examined a kicked spin map on the sphere, which classically exhibits the generic regular-to-chaotic transition as a coupling constant $\beta = \mu B$ is increased: for $\beta = 0.2$ the phase space has a mixed structure, while for $\beta = 1$ the classical motion looks totally chaotic (Nakamura et al. 1986). In the quantized version ($S = 30$), this transition seems to reflect itself in the organization of the zeros. For the eigenfunction of Fig.5a ($\beta = 0.2$) some zeros are aligned and others spread out, while in Fig.5b ($\beta = 1$) the zeros diffuse over the whole sphere. All of this clearly requires further study.

The linear concentration of zeros for integrable systems has a semiclassical explanation. As $\hbar \longrightarrow 0$, an eigenstate admits a finite WKB expression, $\psi(z) \sim \sum_{k=1}^{K} A_k \exp\left[S_k(z)/\hbar\right]$, where $\{S_k\}$ are branches of the classical complex action in the z variable (Kurchan et al. 1989, Voros 1989). The vanishing of such ψ generically requires $\mathrm{Re}\ S_j = \mathrm{Re}\ S_k > \mathrm{Re}\ S_\ell$ for two branches j, k (ℓ running over all other branches involved). Each such equality defines a classical curve (a sort of anti-Stokes line), upon which the zeros are selected by a further condition, $\mathrm{Im}\,(S_j - S_k) = (2m\pi + \mathrm{const})\hbar$, which makes them regularly distributed with spacing $2\pi\hbar\left[\mathrm{Im}\left(S_j' - S_k'\right)\right]^{-1} = O\left(N^{-1}\right)$. Conversely, the product formula (15) over a distribution of zeros of this type will become equivalent to a WKB expression in the large N limit.

By contrast, the observed distribution of zeros in all chaotic examples is roughly *bidimensional*, implying an average spacing $O\left(N^{-1/2}\right)$. This is the signature of an *altogether different* semiclassical regime.

We risk an explanation connected with the *ergodicity* of the classical motion. This, and the correspondence principle, suggest that the quantum phase space distributions must tend to the microcanonical (i.e. uniform) density in the classical limit (see Eckhardt 1988). With the Wigner distribution, this can only happen after some smoothing (its wild oscillations do not tend to vanish as $\hbar \longrightarrow 0$). The Husimi function is precisely a Wigner function smoothed over widths $O\left(\sqrt{\hbar}\right) = O\left(N^{-1/2}\right)$: can *it* tend pointwise to the flat density? Now, the mere presence of the *zeros*, and their *proliferation* as $N \longrightarrow \infty$, demonstrate that this smoothing is not sufficient. However, the spreading of those zeros should make the Husimi distribution more amenable to uniformization upon further smoothing. Indeed, for identical N we have observed a *much lower global contrast* (before adjustment) in chaotic Husimi functions than in integrable ones.

In conclusion, the zeros of Husimi functions offer new opportunities to semiclassical analysis in chaotic systems. First, we have yet to understand the dynamical significance of the zeros, and also to unveil their possible limit distributions (could these be uniform ? or fractal ?). Two approaches suggest themselves next: use Eq.(15) as a basis for semiclassical descriptions, thus generalizing the WKB method; or apply the techniques of random matrix theory to the set of zeros, treating it as if it were the spectrum of a non-Hermitian operator.

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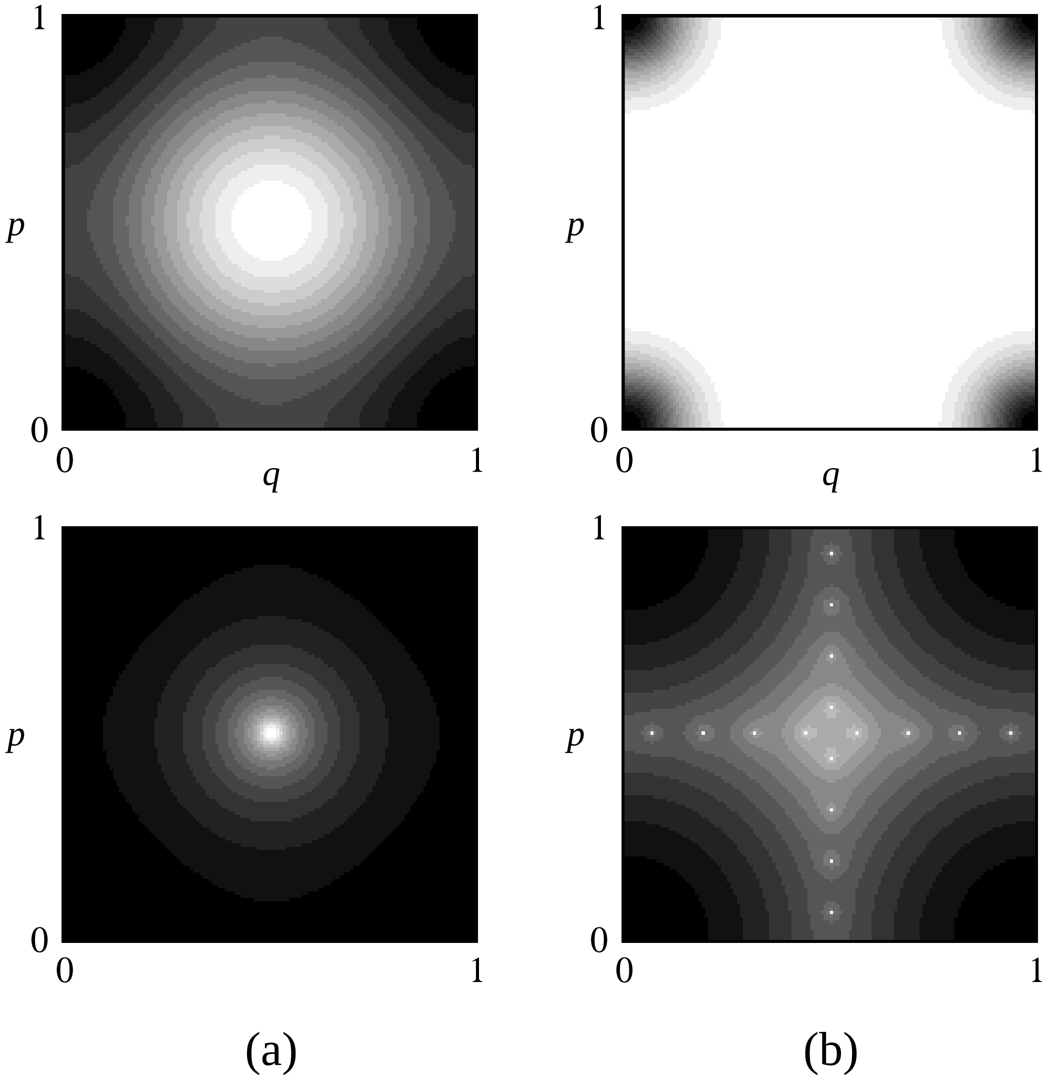
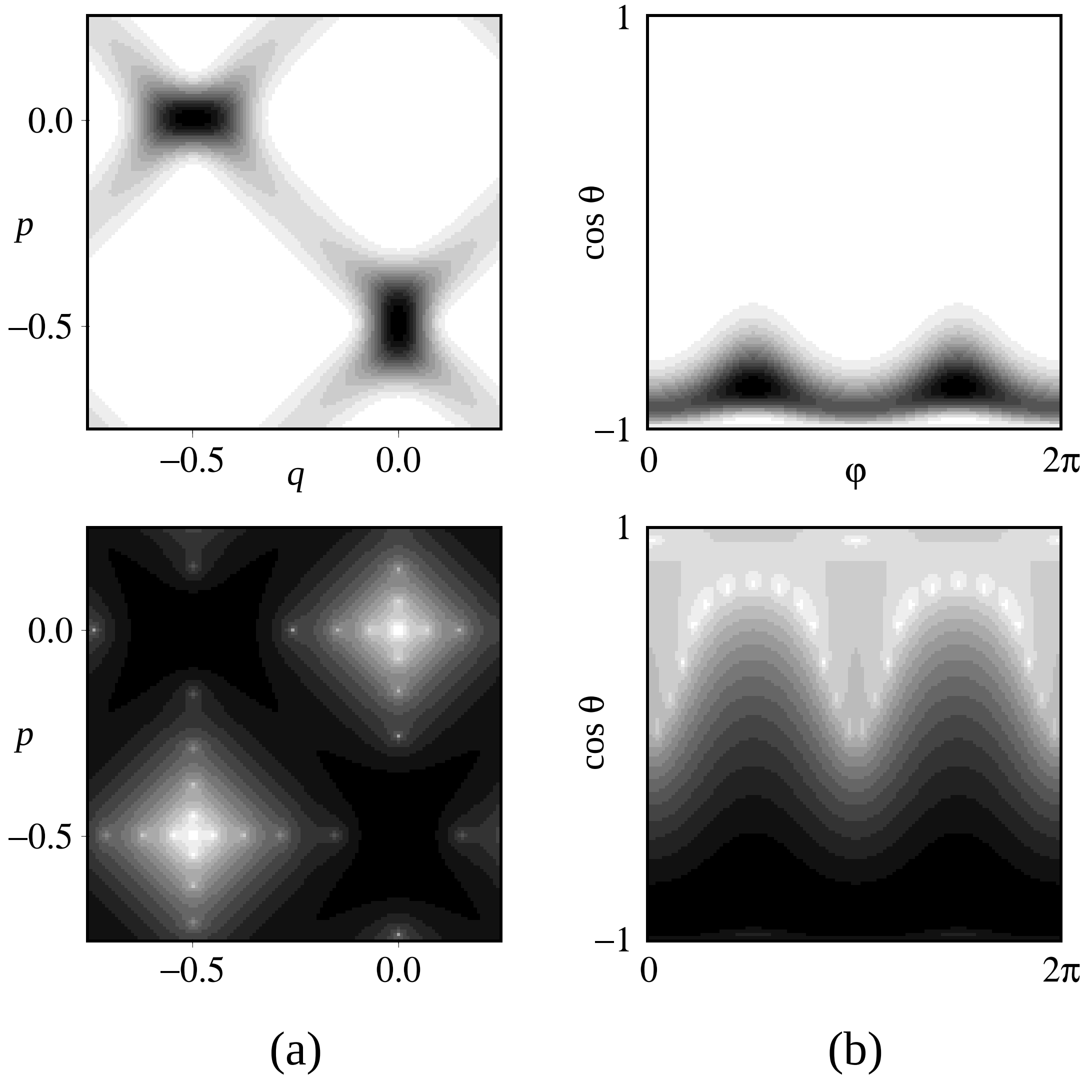


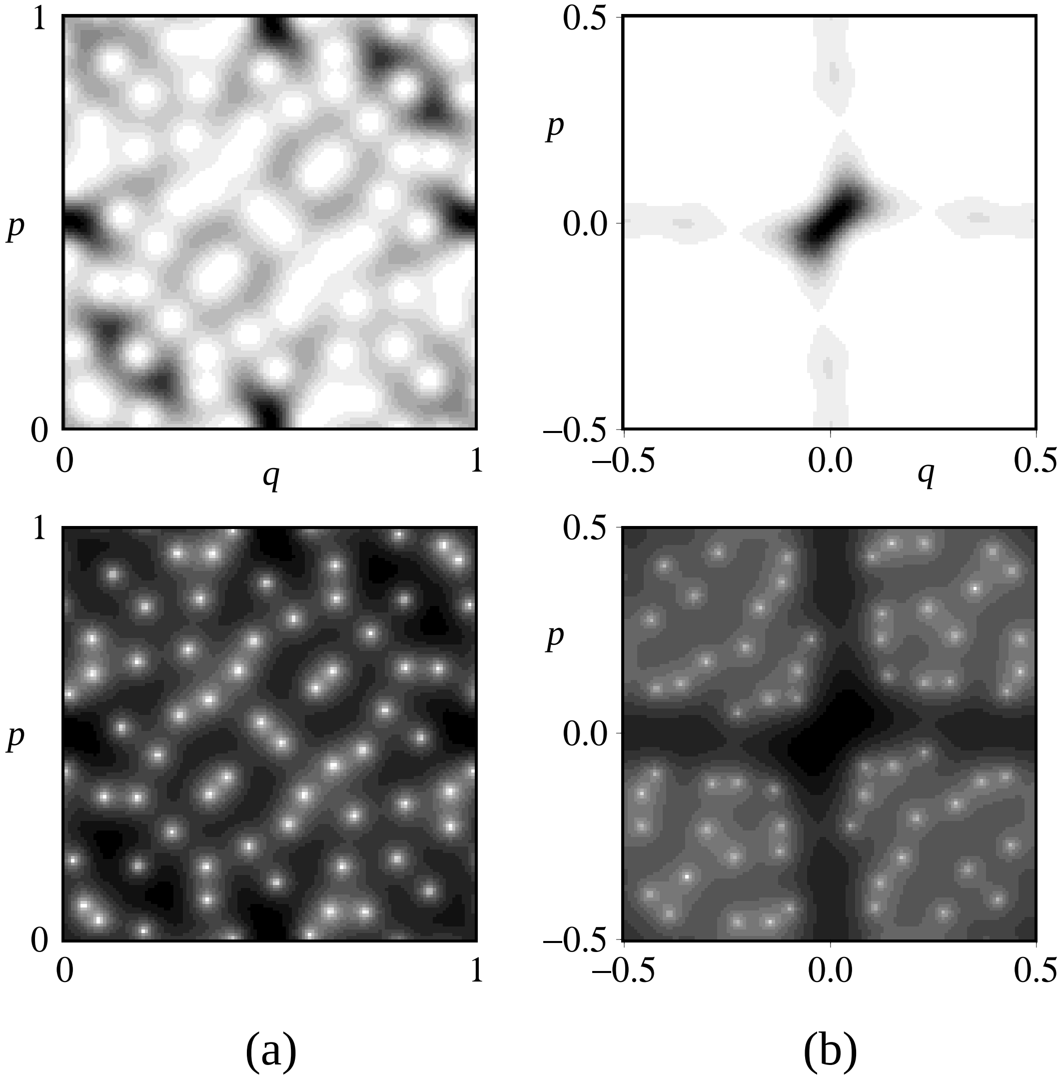
Husimi function plots. Top half: linear density scale; bottom half: a logarithmic density scale stressing the location of the zeros.

Fig.1 : a) The monomial ψ_1 $(N = 1)$; b) The coherent state $|0\rangle\rangle$ $(N = 16)$.



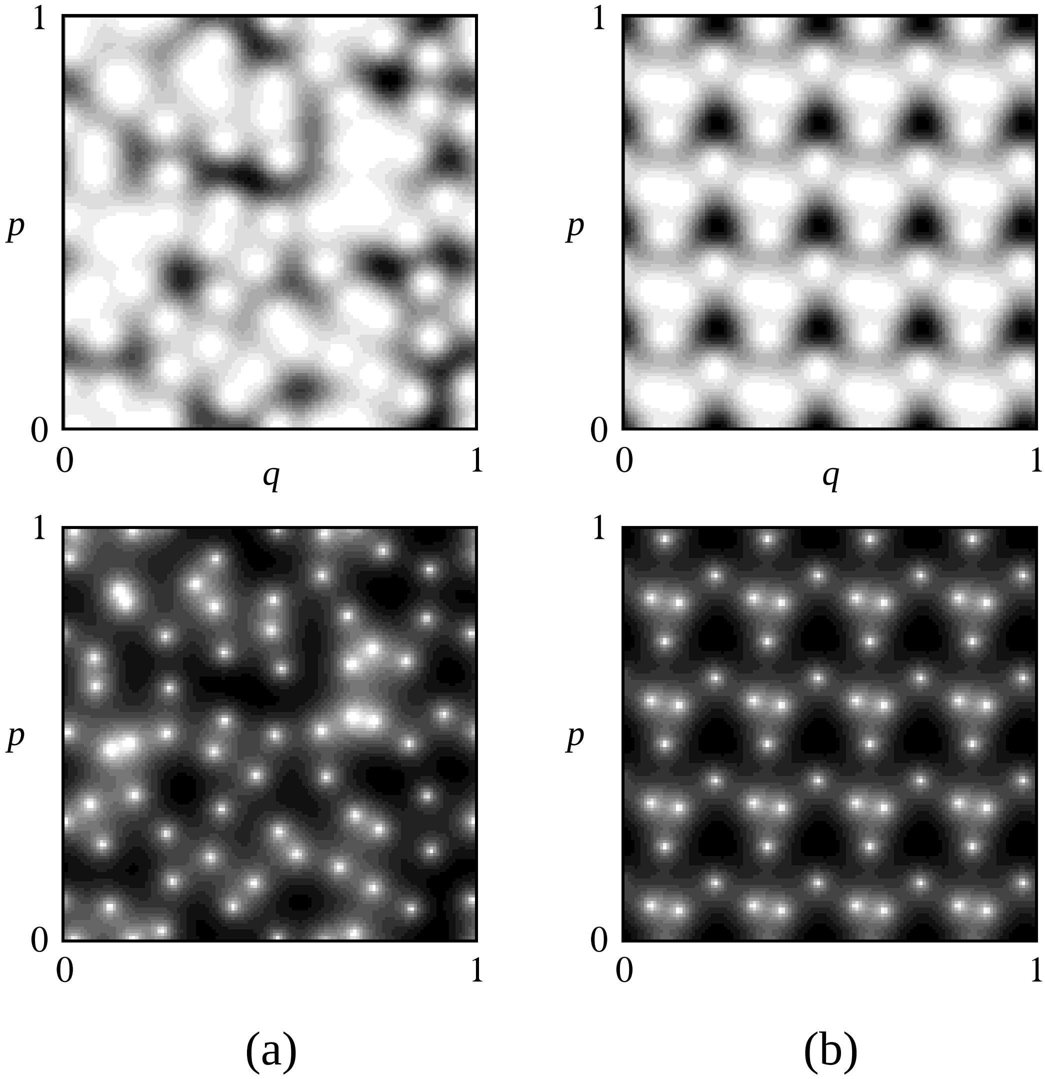
Husimi function plots. Top half: linear density scale; bottom half: a logarithmic density scale stressing the location of the zeros.

Fig.2 : Eigenstates of integrable systems. a) Harper's equation ($N = 31$); b) Lipkin model ($S = 15$).



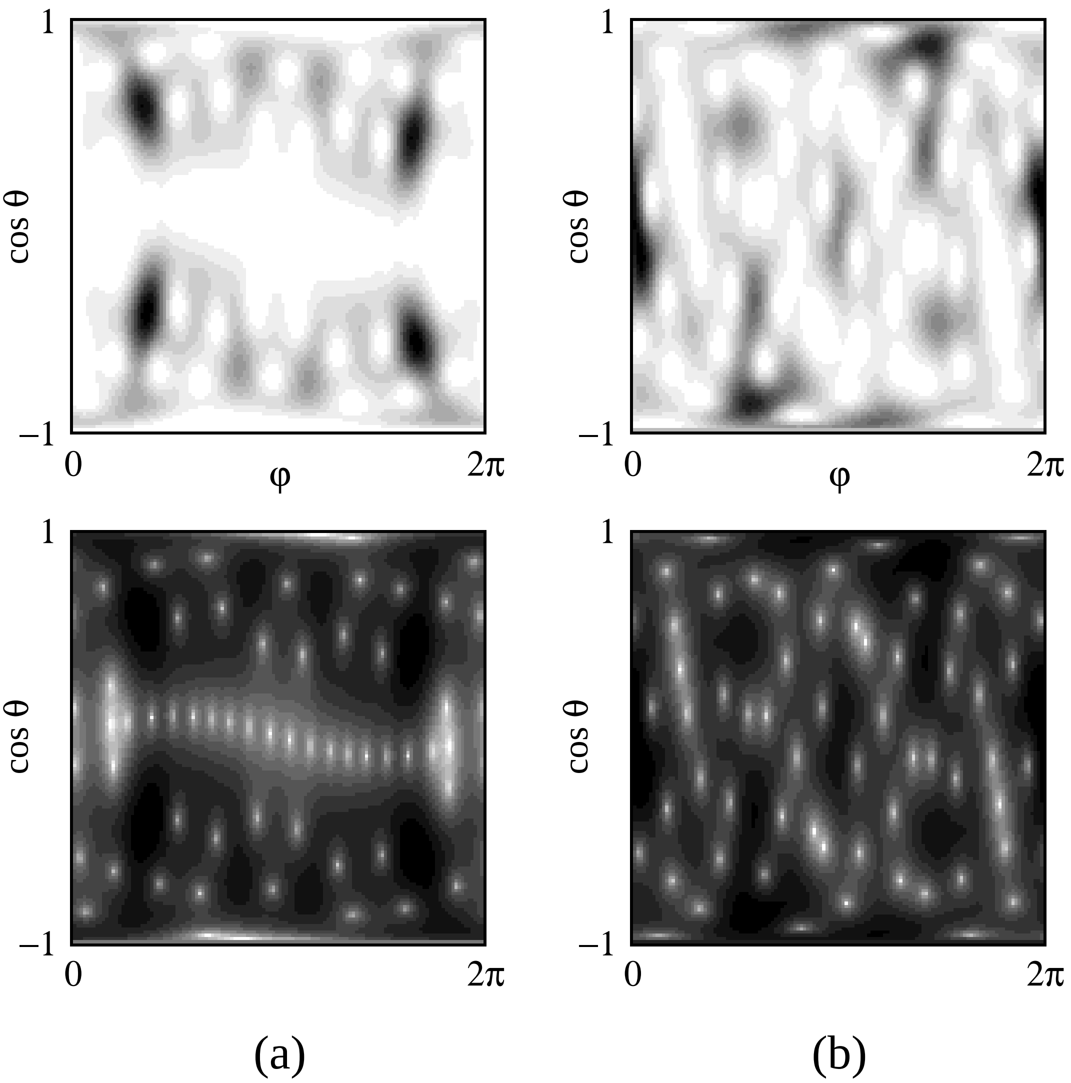
Husimi function plots. Top half: linear density scale; bottom half: a logarithmic density scale stressing the location of the zeros.

Fig.3 : Two eigenstates of the baker's map ($N = 64$).



Husimi function plots. Top half: linear density scale; bottom half: a logarithmic density scale stressing the location of the zeros.

Fig.4 : Two eigenstates of the cat map ($N = 64$).



Husimi function plots. Top half: linear density scale; bottom half: a logarithmic density scale stressing the location of the zeros.

Fig.5 : Eigenstates of a kicked spin model ($S = 30$). a) Mixed classical regime; b) Chaotic classical regime.