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Chaos synchronization of general complex dynamical networks

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Abstract

Recently, it has been demonstrated that many large-scale complex dynamical networks display a collective synchronization motion. Here, we introduce a time-varying complex dynamical network model and further investigate its synchronization phenomenon. Based on this new complex network model, two network chaos synchronization theorems are proved. We show that the chaos synchronization of a time-varying complex network is determined by means of the inner coupled link matrix, the eigenvalues and the corresponding eigenvectors of the coupled configuration matrix, rather than the conventional eigenvalues of the coupled configuration matrix for a uniform network. Especially, we do not assume that the coupled configuration matrix is symmetric and its off-diagonal elements are nonnegative, which in a way generalizes the related results existing in the literature.

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Keywords: Complex dynamical networks; Chaos synchronization; Time varying

1. Introduction

Complex networks exist in all fields of sciences and humanities, and have been intensively studied over the past decades [1–3]. Among these are computer networks,

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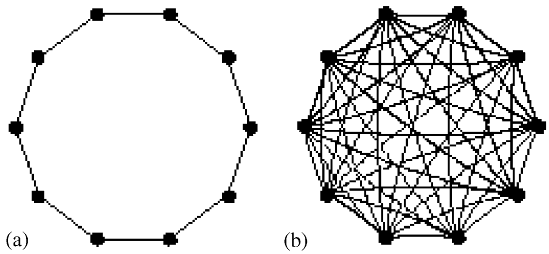

Fig. 1. (a,b) Regular graphs (source: Strogatz, Nature, Vol. 410, 2001).

the World Wide Web, telephone call graphs, food webs, neural networks, electrical power grids, coauthorship and citation networks of scientists, cellular and metabolic networks, etc. [2]. In general, a complex network is a large set of interconnected nodes, where a node is a fundamental unit with detailed contents. Many properties of real-world complex networks can be well understood by considering the network's interactions. It is now known that basic properties of very large networks are mainly determined by the way the connections among the nodes are made. Therefore, much recent research work on complex networks has focused on the structural topology of these networks.

Traditionally, a network is represented by a graph in mathematical terms. A graph is a pair of sets, $G = \{P, E\}$, where P is a set of N nodes (or points or vertices) $P_1, P_2, \ldots, P_N$, and E are a set of links (or edges or lines) each of which connects two elements of P [3]. Chains, grids, lattices and fully connected graphs have been formulated as completely regular networks (Fig. 1). Those simple architectures allow us to focus on the complexity caused by the nonlinear dynamics of nodes, without considering any additional complexity in the network structure itself. On the other hand, one may take a complementary approach, setting dynamics aside and turning networks to have more complex architectures. In doing so, as the opposite end of the spectrum from the regular network, the random graph is a natural choice.

Over the past four decades, complex networks have been studied in depth as a branch of pure mathematics: random graphs. The theory of random graphs was introduced by Paul Erdős and Alfréd Rényi after Erdős discovered that probabilistic methods were often useful to tackle problems in graph theory [3,4]. In fact, the random-graph theory studies properties of the probability space associated with graphs of N nodes as $N \to \infty$. Fig. 2 shows a typical random graph. Although many real complex networks are neither completely regular nor completely random, the ER random graph models have been serving as idealized coupling architectures for ecosystems, gene networks and the spread of infectious diseases and computer virus over the past decades [2]. Despite the fact that the position of an edge is random, a typical random graph is rather homogeneous: the majority of the nodes have about the same number of edges. The random networks show negligible local clustering and a small average distance between two connected nodes.

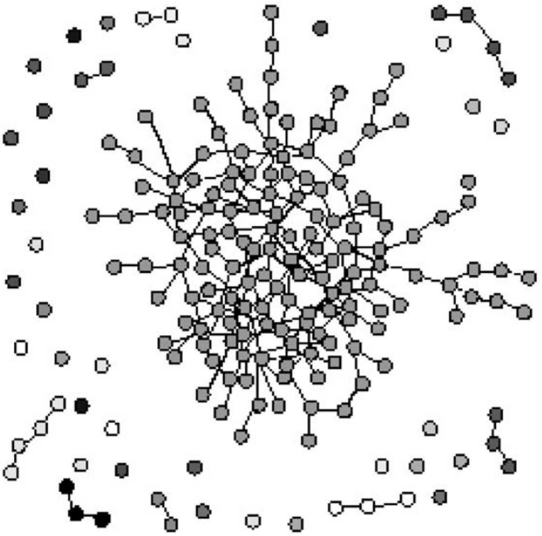

Fig. 2. Random graph network (source: Strogatz, Nature, Vol. 410, 2001).

A regular network is clustered, but does not exhibit the small-world effect [5]. However, the random graph shows the small-world effect, but does not show clustering. Recently, Watts and Strogatz [5] developed an interesting model of small-world networks. Shortly after, Newman and Watts [6] in some sense further improved the original WS model. Most of the recent work on small-world models were performed using the variant of the WS model: the NW model. Fig. 3 displays a typical small-world network. The small-world networks have intermediate connectivity properties but exhibit a high degree of clustering as in regular networks and a small average distance between two connected nodes as in random networks. It is demonstrated that, like power grids, neural networks, social networks and collaboration graphs of film actors, can be modelled using small-world networks. The spread of an epidemic is much faster in a small-world network than in a regular network and almost close to that of a random network [1,7,8]. Moreover, random graph model and the WS model are both exponential networks, which are homogeneous in nature.

However, in many real networks, some nodes are more highly connected than the others, such as in the Internet, the WWW, and the metabolic network. That is, they are scale-free: the degree distributions of these networks follow a power-law form $P(k) \sim k^{-\gamma}$ for a large integer k, where $P(k)$ is the probability that a node in the network is connected to k other nodes, and γ is a positive real number [2]. The origin of such typical power-law degree distributions observed in networks was first addressed by Barabási and Albert, who argued that the scale-free nature of real networks is rooted in two general mechanisms: growth and preferential attachment [2,9]. Furthermore, a scale-free network is inhomogeneous in nature, and most nodes have very few connections but a small number of particular nodes have many connections, as shown by the example in Fig. 4.

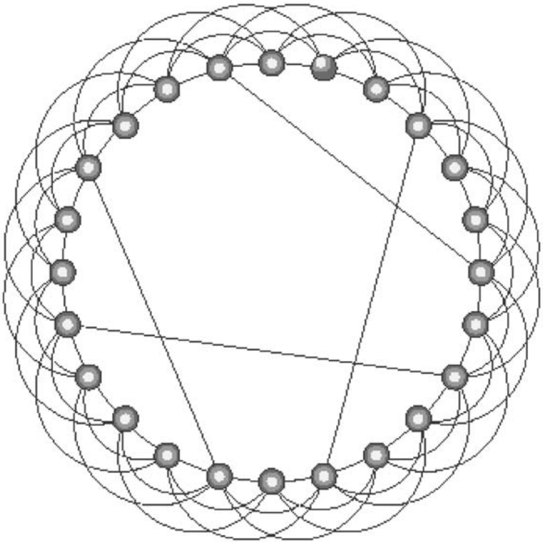

Fig. 3. Small-world network (source: Strogatz, Nature, Vol. 410, 2001).

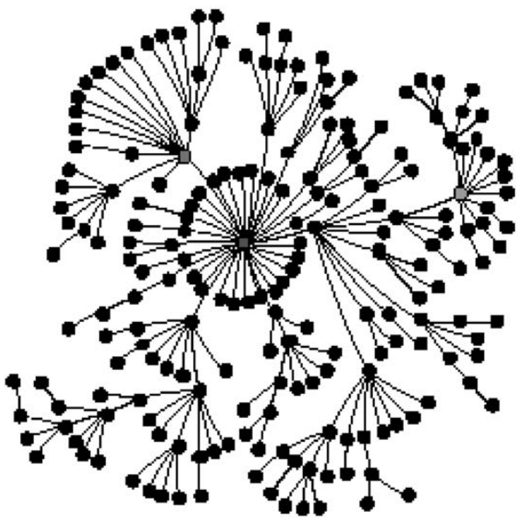

Fig. 4. Scale-free network (source: Strogatz, Nature, Vol. 410, 2001).

Undoubtedly, many real systems in nature, such as biological, technological and social systems, can be described by various models of complex networks. On the other hand, one can also extend the existing network models by introducing dynamical elements into the network nodes. Nonlinear dynamics of complex networks have been intensively studied in the last few years, in various fields such as biology, physics, and engineering. Collective motions of complex dynamical networks have been the subject of considerable recent interest within science and technology communities. Particularly, one of the interesting and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes [10–12]. In fact, synchronization is a basic motion in nature. Moreover, the synchronization of coupled oscillators can well explain many natural phenomena [13–18]. It has been demonstrated that many real-world problems have close relationship with network synchronization, such as the spread of an epidemic or computer virus. Furthermore, network synchronization has many applications in various fields, such as the synchronous information exchange in the Internet and the WWW, and the synchronous transfer of digital or analog signals in the communication networks. Recently, synchronization in networks of coupled chaotic systems has also received a great deal of attention [13,14,16]. Most of the existing works have been focused on completely regular networks, such as the continuous-time cellular neural networks (CNN) and the discrete-time coupled map lattices (CML); while some studies address the synchronization of randomly coupled networks [11,12,15,19]. However, many real complex networks, such as metabolic networks, the WWW, and food webs, are neither completely regular nor completely random. Very recently, Wang and Chen [11] presented a simple scale-free dynamical network model and further investigated its synchronization. Also, they have studied the synchronization issue in small-world dynamical networks [12].

It is noticed that the model of Wang and Chen is a uniform network with the same coupling strength for the whole network [11,12]. However, most real complex networks have different coupling strengths c_{ij} for different links and c_{ij} is not always nonnegative. Moreover, some real-world complex dynamical networks may be directed networks, such as the WWW, whose coupling configuration matrix $C(t)$ is not symmetric. Furthermore, real complex networks are more likely to be time-varying evolving networks. Therefore, in this paper, we introduce a time-varying complex dynamical network model and then further investigate the synchronization phenomena in such networks. Here, we do not assume that $C(t)$ is symmetric and its off-diagonal elements are nonnegative. Based on this network model, we present two network synchronization theorems. Especially, our results show that the synchronization of a time-varying network is determined by means of the inner coupled link matrix $A(t)$, and the eigenvalues and the corresponding eigenvectors of the coupled configuration matrix $C(t)$, rather than the conventional eigenvalues of the coupled configuration matrix C for the uniform networks [10,12].

This paper is organized as follows: a general time-varying complex dynamical network is introduced in Section 2. In Section 3, two chaos synchronization criteria for the new time-varying complex dynamical network are proposed. Several useful corollaries are deduced and an example is analyzed in Section 4. Conclusions are given in Section 5.

2. A general complex dynamical network model

Here, we consider a general complex dynamical network consisting of N identical linearly and diffusively coupled nodes, with each node being an n-dimensional dynamical system. This dynamical network is described by

$$\dot{x}_i = f(x_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij}(t) A(t)(x_j - x_i), \quad i = 1, 2, \ldots, N , \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^{\mathrm{T}} \in \mathbf{R}^n$ is a state vector representing the state variables of node i, $A(t) = (a_{ij}(t))_{n \times n} \in \mathbf{R}^{n \times n}$ is a coupling link matrix between node i and node $j (i \neq j)$ for all $1 \leqslant i,\ j \leqslant N$ at time t, $C(t) = (c_{ij}(t))_{N \times N}$ is the coupling configuration matrix representing the coupling strength and topological structure of the network at time t, in which $c_{ij}(t)$ is defined as follows: if there is a connection from node i to node j $(j \neq i)$, then the coupling strength $c_{ij}(t) \neq 0$; otherwise, $c_{ij}(t) = 0$ $(j \neq i)$, and the diagonal elements of matrix $C(t)$ are defined by

$$c_{ii}(t) = -\sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij}(t), \quad i = 1, 2, \ldots, N . \tag{2}$$

Thus, the time-varying network (1) can be rewritten in a compact form as

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} c_{ij}(t) A(t) x_j, \quad i = 1, 2, \ldots, N . \tag{3}$$

As a special case, the coupling linking matrix $A(t)$ can be a constant $0-1$ matrix in the form $\Gamma = \mathrm{diag}\{r_1, r_2, \ldots, r_n\}$ and the coupling configuration matrix $C(t) = c\left(c_{ij}\right)_{N \times N}$ for all time t, where c is a constant and c_{ij} satisfies the following condition: if there is a connection between node i and node $j (i \neq j)$, then $c_{ij} = c_{ji} = 1$; otherwise, $c_{ij} = c_{ji} = 0$ $(i \neq j)$. Then, in this case, the time-varying network (3) reduces to the model of Wang and Chen [11,12]:

$$\dot{x}_i = f(x_i) + c \sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij} \Gamma x_j, \quad i = 1, 2, \ldots, N . \tag{4}$$

Hereafter, suppose that the network (3) is connected in the sense that there are no isolate clusters. Thus, the coupling configuration matrix $C(t)$ is an irreducible matrix at any time t.

Most real-world complex dynamical networks are time-varying evolving networks. It implies that the coupling configuration matrix $C(t) = (c_{ij}(t))_{N \times N}$ and the inner coupling matrix $A(t) = (a_{ij}(t))_{N \times N}$ are functions of time t. Moreover, real-world complex dynamical networks may be directed networks, such as the WWW, whose coupling configuration matrix $C(t)$ is not symmetric. Here, we do not assume symmetry of $C(t)$ in model (3). Also, we do not assume that all the off-diagonal elements of $C(t)$ are nonnegative.

3. Chaos synchronization of complex dynamical networks

3.1. Mathematical preliminaries

In this section, we present a rigorous mathematical definition for network synchronization and introduce several necessary mathematical lemmas.

Assume that each node of network (3) is an n-dimensional dynamical system $\dot{x} = f(x)$. In the following, we consider the chaos synchronization of dynamical network (3).

Definition 1. Let $x_i(t;t_0,x_1^0,\ldots,x_N^0)$ $(i=1,2,\ldots,N)$ be a solution for the dynamical network

$$\dot{x}_i = f(x_i) + g_i(x_1,x_2,\ldots,x_N), \quad i=1,2,\ldots,N, \tag{5}$$

where $f : D \to \mathbf{R}^n$ and $g_i : D \times \cdots \times D \to \mathbf{R}^n$ $(i=1,2,\ldots,N)$ are continuously differentiable, $D \subseteq \mathbf{R}^n$. If there is a nonempty open subset $D^0(t_0) \subseteq D$, with $x_i^0 \in D^0(t_0)$ $(i=1,2,\ldots,N)$, such that $x_i \in D$ for all $t \geqslant t_0$, $i=1,2,\ldots,N$, and

$$\lim_{t\to\infty} \|x_i(t;t_0,x_1^0,\ldots,x_N^0) - x_j(t;t_0,x_1^0,\ldots,x_N^0)\|_2 = 0 \quad \text{for } 1 \leqslant i,\ j \leqslant N,$$

then the dynamical network (5) is said to realize *synchronization* and $D^0(t_0) \times \cdots \times D^0(t_0)$ is called the *region of synchrony* for the dynamical network (5).

It is noticed that the diffusive coupling condition (2) on dynamical network (3) ensures that the synchronous solution $x_1(t;t_0,x_1^0,\ldots,x_N^0) = x_2(t;t_0,x_1^0,\ldots,x_N^0) = \cdots = x_N(t;t_0,x_1^0,\ldots,x_N^0)(x_1^0 = \cdots = x_N^0 \in D)$ be a solution of an individual node $\dot{x} = f(x)$, denoted $s(t)$, namely,

$$\dot{s}(t) = f(s(t)). \tag{6}$$

Obviously, synchronization in dynamical network (3) corresponds to the motion in the invariant manifold: $x_1(t) = x_2(t) = \cdots = x_N(t)$. Here, $s(t)$ can correspond to an equilibrium point, a periodic orbit, or a chaotic orbit.

Hereafter, we only consider the case of *chaos synchronization* of the dynamical network (3). We first present a rigorous definition for the transverse errors of the synchronous manifold.

Definition 2. From Definition 1, if system (6) is chaotic and

$$\lim_{t\to\infty} \|x_i(t;t_0,x_1^0,\ldots,x_N^0) - x_1(t;t_0,x_1^0,\ldots,x_N^0)\|_2 = 0 \quad \text{for} \quad 2 \leqslant i \leqslant N,$$

then the network (3) achieves synchronization, where the errors

$$\eta_i(t) = x_i(t;t_0,x_1^0,\ldots,x_N^0) - x_1(t;t_0,x_1^0,\ldots,x_N^0), \quad 2 \leqslant i \leqslant N$$

are called the *transverse errors* of the synchronous manifold $x_1(t)=x_2(t)=\cdots=x_N(t)$, where $x_1(t)$ is called the *reference direction* of the synchronous manifold. When all the transverse errors $\eta_i(t)$ $(2 \leqslant i \leqslant N)$ exponentially tend to zero, the chaotic synchronous state $x_1(t) = x_2(t) = \cdots = x_N(t)$ is called *exponentially stable*. Moreover, when $\eta_i(t)$ $(2 \leqslant i \leqslant N)$ uniformly exponentially tend to zero, the chaotic synchronous state $x_1(t) = x_2(t) = \cdots = x_N(t)$ is called *uniformly exponentially stable*.

Obviously, if all the transverse errors $\eta_i(t)$ $(2 \leqslant i \leqslant N)$ uniformly exponentially tend to zero, then the chaotic network (3) realizes synchronization.

It is very important to point out that since a chaotic attractor is an attracting invariant set, the uniformly exponential stability of a chaotic synchronous state $x_1(t) = x_2(t) = \ldots = x_N(t)$ is equivalent to the uniformly exponential stability of the zero *transverse errors* of the synchronous manifold for the dynamical network (3). However, it is quite different from the general solution case, since we do not know the stability of $s(t)$. The uniformly exponential stability of synchronous solution $(x_1(t), x_2(t), \ldots, x_N(t))$ of network (3) is equivalent to the uniformly exponential stability of the *error vector* $(\xi_1(t), \xi_2(t), \ldots, \xi_N(t))$ about its zero solution, where $\xi_i(t) = x_i(t) - s(t)$ $(i = 1, 2, \ldots, N)$. Therefore, from the diffusive coupling condition (2), if the general solution $s(t)$ itself is not uniformly exponentially stable, then it is impossible for the synchronous solution $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ to be uniformly exponentially stable. Hence, the main results of this paper hold particularly for chaotic networks.

Without loss of generality, let $x_1(t) = s(t)$ be the reference direction of the synchronous manifold $x_1(t) = x_2(t) = \cdots = x_N(t)$. Then we have

$$\eta_1(t) = x_1(t) - s(t) \equiv 0 \tag{7}$$

and

$$x_i(t) = s(t) + \eta_i(t), \quad i = 1, 2, \ldots, N . \tag{8}$$

Substituting (8) into network (3) yields

$$\dot{\eta}_i(t) = f(s(t) + \eta_i(t)) - f(s(t)) + \sum_{j=2}^{N} c_{ij}(t)A(t)\eta_i(t), \quad 2 \leqslant i \leqslant N . \tag{9}$$

Denote

$$\bar{\eta}(t) = \begin{pmatrix} \eta_2(t) \\ \vdots \\ \eta_N(t) \end{pmatrix}, \quad \eta_i(t) = \begin{pmatrix} \eta_{i1}(t) \\ \vdots \\ \eta_{in}(t) \end{pmatrix}, \quad \bar{S}(t) = \begin{pmatrix} s(t) \\ \vdots \\ s(t) \end{pmatrix} \in \mathbf{R}^{n(N-1)} . \tag{10}$$

Then Eq. (9) can be written as

$$\dot{\bar{\eta}}(t) = F(t, \bar{\eta}(t)) \tag{11}$$

and the Jacobian matrix of $F(t, \bar{\eta})$ at $\bar{\eta} = 0$ is

$$DF(t,0) = \begin{pmatrix} Df(s(t)) + c_{22}(t)A(t) & c_{23}(t)A(t) & \cdots & c_{2N}(t)A(t) \\ c_{32}(t)A(t) & Df(s(t)) + c_{33}(t)A(t) & \cdots & c_{3N}(t)A(t) \\ \vdots & \vdots & \ddots & \vdots \\ c_{N2}(t)A(t) & c_{N3}(t)A(t) & \cdots & Df(s(t)) + c_{NN}(t)A(t) \end{pmatrix} . \tag{12}$$

In the following, we introduce some general concepts of diffusively coupled matrix and nonnegative diffusively coupled matrix.

Definition 3. For a given real matrix $C=(c_{ij})_{N\times N}$, if the sum of all elements in each row is equal to zero, that is,

$$\sum_{j=1}^{N} c_{ij}=0, \quad i=1,2,\ldots,N\ , \tag{13}$$

then the matrix C is called a *diffusively coupled matrix*. In addition, if all the off-diagonal elements of C are nonnegative, that is,

$$c_{ij}\geqslant 0, \quad \text{for } i\neq j\ (1\leqslant i,\ j\leqslant N)\ , \tag{14}$$

then the matrix C is called a *nonnegative diffusively coupled matrix*. The set of all nonnegative diffusively coupled matrices is denoted T.

Here, we review the Gershgorin's circle theorem [20] for convenience.

Definition 4. Let C be a square complex matrix. Around every element c_{ii} of the diagonal of the matrix on the complex plane, draw a circle with radius equal to $\sum_{j\neq i}|c_{ij}|$. Such circles are called *Gershgorin discs*.

Lemma 1 (Gershgorin's circle theorem [20]). *Every eigenvalue of C lies in one of the Gershgorin discs.*

Based on the Gershgorin's circle theorem, we deduce several important properties of the eigenvalues of diffusively coupled matrices and summarized them in the following lemma:

Lemma 2. *If C is a diffusively coupled matrix, then the following results hold*:

(i) 0 *is an eigenvalue of matrix C, associated with eigenvector* $(1,1,\ldots,1)^{\mathrm{T}}$.
(ii) *If $C\in T$, then the real parts of all eigenvalues of matrix C are less than or equal to* 0 *and all eigenvalues with zero real part are the real eigenvalue* 0.
(iii) *If $C\in T$ is irreducible, then* 0 *is its eigenvalue of multiplicity* 1.

Proof. Let $\bar{c}_j=(c_{1j},c_{2j},\ldots,c_{Nj}t)^{\mathrm{T}}$ $(j=1,2,\ldots,N)$ be the column vectors of matrix $C=(c_{ij})_{N\times N}$. According to (13), we have

$$\sum_{j=1}^{N}\bar{c}_j=0\ ,$$

that is, the column vectors of matrix C are linearly relative. Therefore, $|C|=0$ and 0 is an eigenvalue of matrix C. It is easy to verify that $(1,1,\ldots,1)^{\mathrm{T}}$ is the corresponding eigenvector of eigenvalue 0. So (i) is clearly true.

If $C \in T$, then from Gerschgorin's circle theorem (Lemma 1 above), for any eigenvalue λ of C, there exists a diagonal element c_{ii}, such that

$$\lambda = c_{ii} + r\cos(\theta) + \mathrm{i}r\sin(\theta) ,$$

where $0 \leqslant r \leqslant \sum_{j\neq i} c_{ij} = -c_{ii}$, $\theta \in [0, 2\pi)$. Therefore, the real part of λ satisfies

$$\mathrm{Re}(\lambda) = c_{ii} + r\cos(\theta) \leqslant c_{ii} + (-c_{ii}) = 0 \tag{15}$$

and $\mathrm{Re}(\lambda) = 0$ iff $r = -c_{ii}$ and $\theta = 0$. When $\theta = 0$, $\lambda = 0$. Thus, (ii) holds.

If $C \in T$ is irreducible, let α be such that $\alpha I + C$ is a nonnegative matrix. According to the Perron–Frobenius theorem [20], $\alpha I + C$ has a real nonnegative eigenvalue $\lambda_0 \geqslant 0$, such that $\lambda_0 \geqslant |\lambda'|$, where λ' is any eigenvalue of $\alpha I + C$, and λ_0 is a simple eigenvalue associated to a positive eigenvector. Since 0 is an eigenvalue of C, α is an eigenvalue of $\alpha I + C$ and $\alpha \leqslant \lambda_0$. From Gerschgorin's circle theorem, similar to (15), we have

$$\mathrm{Re}(\lambda') = \alpha + c_{ii} + r\cos(\theta) \leqslant \alpha ,$$

where λ' is any eigenvalue of $\alpha I + C$. That is, $\mathrm{Re}(\lambda_0) = \lambda_0 \leqslant \alpha$. Therefore, $\lambda_0 = \alpha$. Since α is a simple eigenvalue of $\alpha I + C$, 0 is a simple eigenvalue of C. That is, 0 is an eigenvalue of C with multiplicity 1. So (iii) follows.

The proof is thus completed. □

For the proof of network synchronization, we need the following lemma:

Lemma 3. *If $C = (c_{ij})_{N\times N}$ is a diffusively coupled matrix and can be diagonalized, then there exists a nonsingular matrix $\Phi = (\phi_1, \ldots, \phi_N)$, such that*

$$C^{\mathrm{T}}\phi_k = \lambda_k\phi_k, \quad k = 1, 2, \ldots, N , \tag{16}$$

where the eigenvalue $\lambda_1 = 0$, and $\Phi^{-1} = (\phi_{1'}^{\mathrm{T}}, \ldots, \phi_{N'}^{\mathrm{T}})^{\mathrm{T}}$ with $\phi_1' = (1, 1, \ldots, 1)$.

Lemma 3 follows from Lemma 2 and the proof is omitted here.

Definition 5. For a given matrix $A = (a_{ij})_{n\times n}$, the characteristic quantity $\mu(A)$ for matrix A is defined by

$$\mu(A) = \lim_{h\to 0^+} \frac{|I + hA| - 1}{h} .$$

When $|A|$ is chosen as the square root of the maximum eigenvalue of matrix AA^{T}, $\mu(A)$ is the maximum eigenvalue of matrix $\frac{1}{2}(A^{\mathrm{T}} + A)$ [21].

Obviously, if the coupling configuration matrix $C(t) = (c_{ij}(t))_{N\times N}$ is a nonnegative diffusively coupled matrix for all time t, then all eigenvalues of $C(t)$ satisfy $\lambda_1(t) = 0$ and $\mathrm{Re}(\lambda_k(t)) < 0$ for $i = 2, 3, \ldots, N$, for any t.

3.2. Chaos synchronization theorems of complex dynamical networks

In this section, we introduce two chaotic network synchronization theorems. We use the Lyapunov's direct method to prove a network synchronization theorem of the chaotic network (3). On the other hand, we use the mode decomposition method [16,18] to translate the stability problem of a network synchronous solution into the stability problem of $N-1$ independent n-dimensional linear time-varying systems, whose stabilities can be completely determined by the inner coupled link matrix $A(t)$, the eigenvalues and the corresponding eigenvectors of the coupled configuration matrix $C(t)$. Especially, in Theorem 2, we derive a sufficient and necessary condition of chaos synchronization for the time-varying complex dynamical network (3), where $\dot{x}=f(x)$ is a chaotic system defined on a convex set Θ and $s(t)$ corresponds to an orbit of a chaotic attractor of the system.

Definition 6. If $\|x(t)-s(t)\|_2 \to 0$ uniformly and exponentially as $t\to\infty$, then $x(t)-s(t)$ is said to be *uniformly exponentially stable*.

Theorem 1. *Assume that* $F:\Omega\to\mathbf{R}^{n(N-1)}$ *is continuously differentiable on the positively invariant set* $\Omega=\{x\in\mathbf{R}^{n(N-1)}|\ \|x\|_2<r\}$. *The chaotic synchronous state* $x_1(t)=x_2(t)=\cdots=x_N(t)=s(t)$ *is uniformly exponentially stable for dynamical network* (3) *if there exist two symmetric positive definite matrices* $P,Q\in\mathbf{R}^{n(N-1)\times n(N-1)}$, *such that*

$$P(DF(t,0))+(DF(t,0))^{\mathrm{T}}P \leqslant -Q \leqslant -c_1 I\ ,$$

where $c_1>0$, *and*

$$(\Gamma(t,y)-\Gamma(t,\bar{S}(t)))^{\mathrm{T}}P+P(\Gamma(t,y)-\Gamma(t,\bar{S}(t)))\leqslant c_2 I < c_1 I\ ,$$

where $\Gamma(t,y(t)) = \operatorname{diag}\{Df(y_1(t)),\ldots,Df(y_{N-1}(t))\}$, $y(t) = (y_1^{\mathrm{T}}(t),y_2^{\mathrm{T}}(t),\ldots,y_{N-1}^{\mathrm{T}}(t))^{\mathrm{T}}$, $y_i(t)=s(t)+\theta_i(t)\eta_{i+1}(t)$ *with* $0\leqslant\theta_i(t)\leqslant 1$ *for all* $1\leqslant i\leqslant N-1$, $\bar{S}(t)$ *is defined in* (10), *and* $y-\bar{S}(t)\in\Omega$.

Proof. Since f is continuously differentiable on the convex set Θ, it follows from the mean-value theorem that

$$f(\eta_i(t)+s(t))-f(s(t))=Df(y_{i-1}(t,\eta_i(t)))\eta_i(t),\qquad i=2,\ldots,N\ .$$

According to Eqs. (9)–(12), we have

$$\begin{aligned}\dot{\bar{\eta}}(t)&=\begin{pmatrix} f(\eta_2(t)+s(t))-f(s(t))\\ \vdots\\ f(\eta_N(t)+s(t))-f(s(t))\end{pmatrix}+\begin{pmatrix} c_{22}(t)A(t) & \cdots & c_{2N}(t)A(t)\\ \vdots & \ddots & \vdots\\ c_{N2}(t)A(t) & \cdots & c_{NN}(t)A(t)\end{pmatrix}\bar{\eta}(t)\\ &=\begin{pmatrix} Df(y_1(t,\eta_2(t)))\eta_2(t)\\ \vdots\\ Df(y_{N-1}(t,\eta_N(t)))\eta_N(t)\end{pmatrix}+\begin{pmatrix} c_{22}(t)A(t) & \cdots & c_{2N}(t)A(t)\\ \vdots & \ddots & \vdots\\ c_{N2}(t)A(t) & \cdots & c_{NN}(t)A(t)\end{pmatrix}\bar{\eta}(t)\end{aligned}$$

$$= \begin{pmatrix} Df(y_1(t,\eta_2(t))) - Df(s(t)) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Df(y_{N-1}(t,\eta_N(t))) - Df(s(t)) \end{pmatrix} \bar{\eta}(t)$$
$$+ DF(t,0)\bar{\eta}(t)$$
$$= [\Gamma(t,y(t)) - \Gamma(t,\bar{S}(t)) + DF(t,0)]\bar{\eta}(t) \, .$$

Defining a Lyapunov candidate, using $\bar{\eta}(t)$ given in (10), by

$$V(t) = \bar{\eta}(t)^{\mathrm{T}} P \bar{\eta}(t)$$

and then differentiating it with respect to t, we have

$$\begin{aligned} \dot{V}(t) &= \dot{\bar{\eta}}(t)^{\mathrm{T}} P \bar{\eta}(t) + \bar{\eta}(t)^{\mathrm{T}} P \dot{\bar{\eta}}(t) \\ &= \bar{\eta}(t)^{\mathrm{T}} \left[(DF(t,0))^{\mathrm{T}} P + P\,(DF(t,0)) \right] \bar{\eta}(t) \\ &\quad + \bar{\eta}(t)^{\mathrm{T}} \left[\left(\Gamma(t,y) - \Gamma(t,\bar{S}(t))\right)^{\mathrm{T}} P + P\left(\Gamma(t,y) - \Gamma(t,\bar{S}(t))\right) \right] \bar{\eta}(t) \\ &\leqslant -\bar{\eta}(t)^{\mathrm{T}} Q \bar{\eta}(t) + c_2 \bar{\eta}(t)^{\mathrm{T}} I \bar{\eta}(t) \\ &\leqslant (c_2 - c_1)\bar{\eta}(t)^{\mathrm{T}} \bar{\eta}(t) < 0 \, . \end{aligned}$$

From the Lyapunov stability theory, the synchronization errors $\bar{\eta}(t)$ will uniformly exponentially converge to zero. That is, the synchronous state $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ is uniformly exponentially stable for the chaotic network (3). The proof is thus completed. □

Theorem 2. *Assume that* $F : \Omega \to \mathbf{R}^{n(N-1)}$ *is continuously differentiable on* $\Omega = \{x \in \mathbf{R}^{n(N-1)} \mid \|x\|_2 < r\}$, *with* $F(t,0) = 0$ *for all* t, *and the Jacobian* $DF(t,x)$ *is bounded and Lipschitz on* Ω, *uniformly in* t. *Assume that there exists a bounded nonsingular real matrix* $\Phi(t)$, *such that* $\Phi^{-1}(t)(C(t))^{\mathrm{T}}\Phi(t) = \mathrm{diag}\,\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\}$, $\exists t_0 \geqslant 0$, *for any* $\lambda_i(t)(1 \leqslant i \leqslant N)$, *either* $\lambda_i(t) \neq 0$ *for all* $t > t_0$, *or* $\lambda_i(t) \equiv 0$ *for all* $t > t_0$, *and* $\dot{\Phi}^{-1}(t)\Phi(t) = \mathrm{diag}\,\{\beta_1(t), \beta_2(t), \ldots, \beta_N(t)\}$. *Then, the chaotic synchronous state* $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ *is exponentially stable for dynamical network* (3) *if and only if the linear time-varying systems*

$$\dot{w} = [Df(s(t)) + \lambda_k(t)A(t) - \beta_k(t)I_n]w, \quad k = 2, \ldots, N \tag{17}$$

are exponentially stable about the zero solution.

Proof. For the dynamical network (3), to investigate the exponential stability of its synchronous state, we only need to analyze the exponential stability of the zero transverse errors of the synchronous manifold $x_1(t) = x_2(t) = \cdots = x_N(t)$.

From (11), we have

$$\dot{\bar{\eta}}(t) = F(t, \bar{\eta}(t)) \tag{18}$$

and its corresponding linear system at $\bar{\eta} = 0$ is

$$\dot{\bar{\eta}}(t) = DF(t, 0)\bar{\eta}(t) . \tag{19}$$

Since the Jacobian $DF(t,x)$ is bounded and Lipschitz on Ω, uniformly in t, according to the Lyapunov converse theorem [22], the origin is an exponentially stable equilibrium point for the nonlinear system (18) if and only if it is an exponentially stable equilibrium point for the linear time-varying system (19). Moreover, the origin is an exponentially stable equilibrium point for the linear system (19) if and only if $\bar{\eta}(t) \to 0$ exponentially as $t \to +\infty$.

According to Eqs. (7), (8) and (19), we get

$$\begin{aligned} \dot{\eta}_i(t) &= Df(s(t))\eta_i(t) + \sum_{j=1}^{N} c_{ij}(t)A(t)\eta_j(t) \\ &= Df(s(t))\eta_i(t) + A(t)(\eta_1(t), \ldots, \eta_N(t))(c_{i1}(t), \ldots, c_{iN}(t))^{\mathrm{T}}, \quad i = 1, 2, \ldots, N , \end{aligned}$$

where $\eta(t) = (\eta_1(t), \ldots, \eta_N(t)) \in \mathbf{R}^{n \times N}$, $Df(s(t)) \in \mathbf{R}^{n \times n}$ is the Jacobian of $f(x)$ at $x = s(t)$. That is,

$$\dot{\eta}(t) = Df(s(t))\eta(t) + A(t)\eta(t)(C(t))^{\mathrm{T}} , \tag{20}$$

since $\eta_1 \equiv 0$, $\bar{\eta}(t) \to 0$ exponentially as $t \to \infty$ is equivalent to $\eta(t) \to 0$ exponentially as $t \to \infty$.

From the hypothesis of Theorem 2, we have

$$(C(t))^{\mathrm{T}}\Phi(t) = \Phi(t)\Lambda(t),$$

where $\Lambda(t) = \mathrm{diag}\,(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t))$. According to Lemmas 2 and 3, $\lambda_1(t) = 0$ for all $t > t_0$, and $\Phi^{-1}(t) = \left(\phi'_1(t), \ldots, \phi'_N(t)\right)^{\mathrm{T}}$ with $\phi'_1(t) = (1, 1, \ldots, 1)$ for all $t > t_0$. Moreover, since $\dot{\Phi}^{-1}(t)\Phi(t) = \mathrm{diag}\,\{\beta_1(t), \beta_2(t), \ldots, \beta_N(t)\}$, $\beta_1(t) = 0$ for all $t > t_0$.

Consider the following nonsingular linear transformation:

$$\eta(t) = v(t)\Phi^{-1}(t) . \tag{21}$$

According to (20), the matrix vector $v(t) = (v_1(t), \ldots, v_N(t)) \in \mathbf{R}^{n \times N}$ satisfies the equation

$$\dot{v}(t) = Df(s(t))v(t) + A(t)v(t)\Lambda(t) - v(t)\dot{\Phi}^{-1}(t)\Phi(t) , \tag{22}$$

namely,

$$\dot{v}_k(t) = [Df(s(t)) + \lambda_k(t)A(t) - \beta_k(t)I_n]v_k(t), \quad k = 1, 2, \ldots, N . \tag{23}$$

Thus, we have translated the exponential stability problem of the chaotic synchronous state into the exponential stability problem of N independent n-dimensional linear time-varying systems (23), which has the same form as (17).

From (21), we have

$$(\eta_1(t), \eta_2(t), \ldots, \eta_N(t)) = (v(t)\phi_1''(t),\ v(t)\phi_2''(t), \ldots, v(t)\phi_N''(t)) \ ,$$

where $\phi_j''(t) = (\phi_{1j}'(t), \phi_{2j}'(t), \ldots, \phi_{Nj}'(t))^{\mathrm{T}}$ for $j = 1, 2, \ldots, N$. That is,

$$0 \equiv \eta_1(t) = v(t)\phi_1''(t) = \sum_{k=1}^{N} \phi_{k1}'(t) v_k(t) \ .$$

According to Lemma 3, $\phi_{11}'(t) = 1 \neq 0$. Hence,

$$v_1(t) = -\sum_{k=2}^{N} \phi_{k1}'(t) v_k(t) \ .$$

If $\eta(t) \to 0$, then $v(t) = \eta(t)\Phi(t) \to 0$. On the other hand, if $v_k(t) \to 0$ for $k = 2, \ldots, N$, then $v_1(t) = -\sum_{k=2}^{N} \phi_{k1}'(t) v_k(t) \to 0$. Therefore, $\eta(t) = v(t)\Phi^{-1}(t) \to 0$. That is, $\eta(t) \to 0$ if and only if $v_k(t) \to 0$ for $k = 2, \ldots, N$. Here, we have transferred the exponential stability problem of the synchronous states to the exponential stability problem of the $N - 1$ independent n-dimensional linear time-varying systems (17).

Therefore, the synchronous state of the chaotic network (3) is exponentially stable if and only if the linear time-varying systems (17) are exponentially stable about the zero solution. The proof is thus completed. □

Remarks. (1) Theorem 1 presents a sufficient condition for synchronization of the chaotic network (3). There are no restrictive conditions for the coupled configuration matrix $C(t)$. And it is a rather general condition for network synchronization.

(2) Theorem 2 proposes a necessary and sufficient condition for synchronization of the chaotic network (3). Especially, Eq. (17) shows that the synchronization of a time-varying chaotic network (3) is determined by means of its inner coupled link matrix $A(t)$, the eigenvalues $\lambda_k(t)$ ($k = 2, \ldots, N$) and the corresponding eigenvectors (the functions $\phi_k(t)$, $k = 2, \ldots, N$) of the coupled configuration matrix $C(t)$, rather than the conventional eigenvalues λ_k ($k = 2, \ldots, N$) of the coupled configuration matrix C for the uniform network (4). It is noticed that the coupled configuration matrix $C(t)$ is a diffusively coupled matrix, and it does not have to be a nonnegative diffusively coupled matrix. Moreover, we do not assume the symmetry of $C(t)$ in the network (3).

4. Some specific cases

It is noticed that Theorem 2 does not give precise conditions on the coupled matrix $A(t)$ and the coupled configuration matrix $C(t)$ under which network (3) can synchronize. In some special cases, however, we are able to do so, namely, to give some detailed conditions on $A(t)$ and $C(t)$, as demonstrated in this section.

4.1. Several corollaries

In this section, we present several useful corollaries based on Theorem 2 and show that the results of Wang and Chen [11,12] are special cases of our new results here.

For simplicity of notation and discussion, we make the following two hypotheses:

Hypothesis 1 (H1). Assume that $F:\Omega \rightarrow \mathbf{R}^{n(N-1)}$ is continuously differentiable on $\Omega = \{x \in \mathbf{R}^{n(N-1)} | \, \|x\|_2 < r\}$, with $F(t,0)=0$ for all t, the Jacobian $DF(t,x)$ is bounded and Lipschitz on Ω, uniformly in t, and the coupling configuration matrix $C(t)$ can be diagonalized by using a constant nonsingular matrix, and $\exists t_0 \geqslant 0$, for any $\lambda_i(t)(1 \leqslant i \leqslant N)$, either $\lambda_i(t) \neq 0$ for all $t > t_0$, or $\lambda_i(t) \equiv 0$ for all $t > t_0$.

Hypothesis 2 (H2). Assume that $C(t) \in T$, where T was defined after (14), and its eigenvalues are real numbers satisfying $\lambda_1(t) \geqslant \lambda_2(t) \geqslant \cdots \geqslant \lambda_N(t)$ for all $t > t_0 \geqslant 0$.

Note that the matrix $\Phi(t)$ of Theorem 2 is a function of time t. When it is a constant matrix, we can derive the following corollary:

Corollary 1. *Suppose* (H1) *holds. Then the chaotic synchronous state* $x_1(t)=x_2(t)=\cdots=x_N(t)=s(t)$ *of the chaotic network* (3) *is exponentially stable if and only if the linear time-varying systems*

$$\dot{w} = [Df(s(t)) + \lambda_k(t)A(t)]w, \quad k = 2,\ldots,N \tag{24}$$

are exponentially stable about the zero solution.

Proof. Since there exists a constant matrix Φ, such that $\Phi^{-1}C(t)^{\mathrm{T}}\Phi = \mathrm{diag}\{\lambda_1(t),\ldots,\lambda_N(t)\}$, $\dot{\Phi}^{-1}\Phi = 0$, Corollary 1 follows from Theorem 2. This completes the proof. □

When $\Phi(t)$ is a constant nonsingular matrix, Eq. (17) can be simplified as Eq. (24). In this case, the synchronization of a time-varying chaotic network (3) is completely determined by the inner coupled matrix $A(t)$ and the eigenvalues $\lambda_k(t)$ $k=2,\ldots,N$, of the coupled configuration matrix $C(t)$. And the corresponding eigenvectors of the coupled configuration matrix $C(t)$ have no effect on the stability of the network synchronous solution.

For some real-world complex networks, the matrix $(A(t)+(A(t))^{\mathrm{T}})$ is a semi-positive-definite matrix, e.g., when $A(t)$ is a diagonal matrix with nonnegative diagonal elements. In this case, we have the following simple corollary:

Corollary 2. *Suppose* (H1) *and* (H2) *hold. If* $(A(t)+(A(t))^{\mathrm{T}})$ *is a semi-positive-definite matrix, and the maximum eigenvalue* $\mu[Df(s(t)) + \lambda_2(t)A(t)] \leqslant a < 0$ *for all* $t \geqslant t_0$, *then the chaotic synchronous state* $x_1(t)=x_2(t)=\cdots=x_N(t)=s(t)$ *of network* (3) *is exponentially stable.*

Proof. Since $\mu[Df(s(t)) + \lambda_2(t)A(t)] \leqslant a < 0$ for all $t \geqslant t_0$, the linear time-varying system

$$\dot{x} = [Df(s(t)) + \lambda_2(t)A(t)]x \tag{25}$$

is uniformly asymptotically stable about its zero solution. Because the exponential stability of a linear time-varying system is equivalent to the uniform asymptotic stability, the linear time-varying system (25) is also exponentially stable.

Consider the linear time-varying systems

$$\dot{x} = [Df(s(t)) + \lambda_i(t)A(t)]x, \quad i = 3, 4, \ldots, N \; . \tag{26}$$

Since $(A(t) + (A(t))^{\mathrm{T}})$ is a semi-positive-definite matrix, from (H2), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}\|x\|^2}{\mathrm{d}t} &= \frac{1}{2}\frac{\mathrm{d}(x^{\mathrm{T}}x)}{\mathrm{d}t} \\ &= x^{\mathrm{T}}\left(\frac{(Df(s(t)) + \lambda_2(t)A(t))^{\mathrm{T}} + (Df(s(t)) + \lambda_2(t)A(t))}{2}\right)x \\ &\quad +(\lambda_i(t) - \lambda_2(t))x^{\mathrm{T}}\left(\frac{A^{\mathrm{T}}(t) + A(t)}{2}\right)x \\ &\leqslant x^{\mathrm{T}}\left(\frac{(Df(s(t)) + \lambda_2(t)A(t))^{\mathrm{T}} + (Df(s(t)) + \lambda_2(t)A(t))}{2}\right)x \; ,\end{aligned}$$

where $i = 3, 4, \ldots, N$. When $t \geqslant t_0$, $\mu[Df(s(t)) + \lambda_2(t)A(t)] \leqslant a < 0$, so that

$$\|x(t)\| \leqslant \|x(t_0)\|\mathrm{e}^{at} \; .$$

That is, linear time-varying systems (26) are exponentially stable. According to Corollary 1, the chaotic synchronous state $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ of network (3) is exponentially stable. The proof is thus completed. □

Sometimes, we can deduce the exponential stability of the network synchronous solution from the coupled matrix $A(t)$ and the least eigenvalue $\lambda_N(t)$ of the coupled configuration matrix $C(t)$. The result is summarized in the following corollary, which is very useful for judging the synchronization of the chaotic network (3):

Corollary 3. *Suppose* (H1) *and* (H2) *hold. If*

$$\int_{t_0}^{\infty} |\lambda_N(t)A(t)| \,\mathrm{d}t < \infty$$

and the linear time-varying system

$$\dot{w} = [Df(s(t)) + \lambda_N(t)A(t)]w \tag{27}$$

is exponentially stable about the zero solution, then the chaotic synchronous state $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ *of network* (3) *is exponentially stable.*

Proof. Since the linear time-varying system (27) is exponentially stable about the zero solution, it is also uniformly asymptotically stable. From (H2), we get

$$\int_{t_0}^{\infty} |\lambda_k(t)A(t)| \,\mathrm{d}t \leqslant \int_{t_0}^{\infty} |\lambda_N(t)A(t)| \,\mathrm{d}t < \infty \quad \text{for } k = 2, \ldots, N-1 \; .$$

Therefore, the linear time-varying systems

$$\dot{w} = [Df(s(t)) + \lambda_k(t)A(t)]\,w, \quad k = 2, \ldots, N-1$$

are all uniformly asymptotically stable, and therefore they are also exponentially stable. According to Corollary 1, the chaotic synchronous state $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ of network (3) is exponentially stable. This thus completes the proof. □

Suppose that $C(t) \in T$ can be diagonalized by using a constant nonsingular matrix and its eigenvalues are real numbers satisfying $\lambda_1(t) \geqslant \lambda_2(t) \geqslant \cdots \geqslant \lambda_N(t)$. According to Lemma 2, $\lambda_1(t) = 0$ and $0 > \lambda_2(t) \geqslant \cdots \geqslant \lambda_N(t)$ for all $t \geqslant t_0$. Hence, there exists a non-positive real number λ_2^0 such that for all $t \geqslant t_0$,

$$0 \geqslant \sup_{t \geqslant t_0} \lambda_2(t) = \lambda_2^0 \geqslant \lambda_2(t) . \tag{28}$$

From Theorem 2 or Corollary 1, we can derive the same results of Refs. [11,12]. It is therefore clear that our results for time-varying network (3) is rather more general than the results for the uniform network (4) studied in Refs. [11,12]. Corollaries 4 and 5 show the corresponding results.

Corollary 4. *Suppose* (H1) *and* (H2) *hold. If there exists an* $n \times n$ *positive-definite matrix* B, *such that*

$$[Df(s(t)) + \mathrm{d}A(t)]^{\mathrm{T}} B + B[Df(s(t)) + \mathrm{d}A(t)] \leqslant -I_n \tag{29}$$

for all $\mathrm{d} \leqslant \lambda_2^0$ *and* $t \geqslant t_0$, *then the chaotic synchronous state* $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ *of network* (3) *is exponentially stable.*

Proof. Construct Lyapunov functions

$$V_k = w^{\mathrm{T}} B w, \quad k = 2, \ldots, N .$$

According to (H2) and Eq. (29), we have

$$[Df(s(t)) + \lambda_k(t)A(t)]^{\mathrm{T}} B + B[Df(s(t)) + \lambda_k(t)A(t)] \leqslant -I_n, \quad k = 2, \ldots, N . \tag{30}$$

Obviously, Corollary 4 follows from Eq. (30) and Corollary 1. The proof is thus completed. □

Corollary 5. *Suppose* (H1) *and* (H2) *hold. If there exists an* $n \times n$ *positive-definite matrix* B, *such that for all* $t \geqslant t_0$, $(A(t))^{\mathrm{T}} B + BA(t)$ *is a semi-positive-definite matrix, and*

$$[Df(s(t)) + \lambda_2^0 A(t)]^{\mathrm{T}} B + B[Df(s(t)) + \lambda_2^0 A(t)] \leqslant -I_n ,$$

then the chaotic synchronous state $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ *of network* (3) *is exponentially stable.*

Proof. Since $(A(t))^{\mathrm{T}} B + BA(t)$ is a semi-positive-definite matrix for all $t \geqslant t_0$, from (28), we have

$$\begin{aligned}
&[Df(s(t)) + \lambda_k(t)A(t)]^{\mathrm{T}} B + B[Df(s(t)) + \lambda_k(t)A(t)] \\
&\quad \leqslant [Df(s(t)) + \lambda_2^0 A(t)]^{\mathrm{T}} B + B[Df(s(t)) + \lambda_2^0 A(t)]
\end{aligned}$$

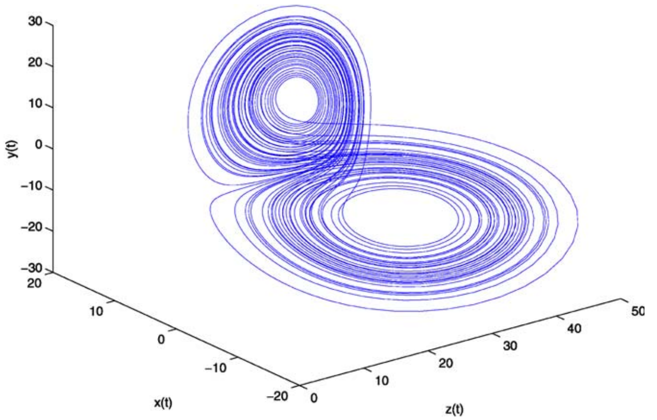

Fig. 5. Lorenz chaotic attractor, with parameters $a = 10, b = \frac{8}{3}, c = 28$.

$$
\begin{aligned}
&+(\lambda_k(t) - \lambda_2^0)[(A(t))^{\mathrm{T}}B + BA(t)] \\
&\leqslant [Df(s(t)) + \lambda_2^0 A(t)]^{\mathrm{T}}B + B[Df(s(t)) + \lambda_2^0 A(t)] \leqslant -I_n .
\end{aligned}
$$

According to Corollary 1, the result of Corollary 5 clearly holds. This completes the proof. □

For the uniform chaotic network (4), we can easily derive Lemma 1 of Ref. [12] from Corollary 1. Similarly, we can easily derive Lemma 1 of Ref. [11] and Lemma 2 of Ref. [12] from Corollary 4 above. Therefore, the results of Refs. [11,12] are special cases of our results obtained here.

4.2. An example

In this section, we illustrate Theorem 2 by using the Lorenz system as a node in network (3). For simplicity, we consider a three-node network. A single Lorenz system of node i is described by [18]:

$$
\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} a(x_{i2} - x_{i1}) \\ cx_{i1} - x_{i1}x_{i3} - x_{i2} \\ x_{i1}x_{i2} - bx_{i3} \end{pmatrix}, \quad i = 1, 2, 3 , \tag{31}
$$

which is chaotic when $a = 10,\ b = \frac{8}{3},\ c = 28$ (Fig. 5), and its Jacobian is

$$
Df(x_i) = \begin{pmatrix} -a & a & 0 \\ c - x_{i3} & -1 & -x_{i1} \\ x_{i2} & x_{i1} & -b \end{pmatrix} .
$$

Choose the inner coupled matrix be

$$A(t)=\begin{pmatrix}(a+c)^2(1+e^{-t}) & 0 & 0\\ 0 & b^2(1+e^{-3t}) & 0\\ 0 & 0 & b^2(1+e^{-2t})\end{pmatrix}$$

and the coupled configuration matrix be

$$C(t)=\frac{1}{2e^2-e-1}\begin{pmatrix}c_{11}(t) & c_{12}(t) & c_{13}(t)\\ c_{21}(t) & c_{22}(t) & c_{23}(t)\\ c_{31}(t) & c_{32}(t) & c_{33}(t)\end{pmatrix},$$

where

$$\begin{aligned}&c_{11}(t)=(e^2-1)\,\mathrm{th}(t)+e\arctan(t), \quad c_{12}(t)=(1-e)\,\mathrm{th}(t)-2e\arctan(t),\\ &c_{13}(t)=(e-e^2)\,\mathrm{th}(t)+e\arctan(t), \quad c_{21}(t)=2(e^2-1)\,\mathrm{th}(t)+e^2\arctan(t),\\ &c_{22}(t)=2(1-e)\,\mathrm{th}(t)-2e^2\arctan(t), \quad c_{23}(t)=2(e-e^2)\,\mathrm{th}(t)+e^2\arctan(t),\\ &c_{31}(t)=3(e^2-1)\,\mathrm{th}(t)+\arctan(t), \quad c_{32}(t)=3(1-e)\,\mathrm{th}(t)-2\arctan(t),\\ &c_{33}(t)=3(e-e^2)\,\mathrm{th}(t)+\arctan(t),\end{aligned}$$

with $\mathrm{th}(t)=(e^t-e^{-t})/(e^t+e^{-t})$.

It is easy to verify that there exists a nonsingular real matrix,

$$\Phi(t)=\frac{1}{2e^2-e-1}\begin{pmatrix}3e^2-2 & (1-e^2)e^t & -e^{1+\sin(t)}\\ 1-3e & (e-1)e^t & 2e^{1+\sin(t)}\\ 2e-e^2 & (e^2-e)e^t & -e^{1+\sin(t)}\end{pmatrix}$$

such that

$$\Phi^{-1}(t)\,(C(t))^{\mathrm{T}}\,\Phi(t)=\mathrm{diag}\,\{0,-\mathrm{th}(t),-\arctan(t)\}$$

and

$$\dot{\Phi}^{-1}(t)\Phi(t)=\mathrm{diag}\,\{0,-1,-\cos(t)\}.$$

Obviously, the conditions of Theorem 2 hold. Therefore, the chaotic synchronous state $x_1(t)=x_2(t)=x_3(t)=s(t)$ of network (3) is exponentially stable if and only if the linear time-varying systems

$$\dot{w}=[Df(s(t))+\lambda_k(t)A(t)-\beta_k(t)I_3]w, \quad k=2,3 \tag{32}$$

are exponentially stable about the zero solution.

When $k=2$, $\lambda_2(t)=-\mathrm{th}(t)$, $\beta_2(t)=-1$, we have

$$\dot{w}=\begin{pmatrix}1-a-a_{11}(t)\,\mathrm{th}(t) & a & 0\\ c-x_3(t) & -a_{22}(t)\,\mathrm{th}(t) & -x_1(t)\\ x_2(t) & x_1(t) & 1-b-a_{33}(t)\,\mathrm{th}(t)\end{pmatrix}w\,, \tag{33}$$

where $a_{11}(t) = (a + c)^2(1 + e^{-t})$, $a_{22}(t) = b^2(1 + e^{-3t})$, $a_{33}(t) = b^2(1 + e^{-2t})$, and $(x_1(t),\ x_2(t),\ x_3(t)) = s(t)$.

Denoting (33) as $\dot{w} = W(t)w$, we get

$$\frac{W(t)+W^{\mathrm{T}}(t)}{2} = \begin{pmatrix} 1-a-a_{11}(t)\,\mathrm{th}(t) & \frac{a+c-x_3(t)}{2} & \frac{x_2(t)}{2} \\ \frac{a+c-x_3(t)}{2} & -a_{22}(t)\,\mathrm{th}(t) & 0 \\ \frac{x_2(t)}{2} & 0 & 1-b-a_{33}(t)\,\mathrm{th}(t) \end{pmatrix} \tag{34}$$

and its corresponding characteristic equation is

$$\lambda^3 + p(t)\lambda^2 + q(t)\lambda + r(t) = 0\ , \tag{35}$$

where

$$p(t) = a + b - 2 + \mathrm{th}(t)(a_{11}(t) + a_{22}(t) + a_{33}(t))\ ,$$

$$q(t) = a_{22}(t)\,\mathrm{th}(t)(a - 1 + a_{11}(t)\,\mathrm{th}(t)) - \frac{x_2^2(t)}{4} - \frac{(a + c - x_3(t))^2}{4}$$
$$+(b - 1 + a_{33}(t)\,\mathrm{th}(t))(a - 1 + a_{11}(t)\,\mathrm{th}(t) + a_{22}(t)\,\mathrm{th}(t))\ ,$$

$$r(t) = a_{22}(t)\,\mathrm{th}(t)(a - 1 + a_{11}(t)\,\mathrm{th}(t))(b - 1 + a_{33}(t)\,\mathrm{th}(t))$$
$$-\frac{x_2^2(t)}{4}a_{22}(t)\,\mathrm{th}(t) - \frac{(a + c - x_3(t))^2}{4}(b - 1 + a_{33}(t)\,\mathrm{th}(t))\ . \tag{36}$$

It has been shown that there exists a bounded region $\Gamma \subset R^3$ containing the whole Lorenz attractor such that each orbit of (31) never leaves it [16,23,24]:

$$\Gamma = \{(x, y, z) \in R^3\,|\,x^2 + y^2 + (z - a - c)^2 = C\}\ ,$$

where $C = b^2(a + c)^2/4(b - 1)$.

Since $s(t) = (x_1(t), x_2(t), x_3(t))$ is a solution of an individual node (31), we can get the boundary as

$$x_2^2(t) + (x_3^2(t) - a - c)^2 \leqslant \frac{b^2(a + c)^2}{4(b - 1)}\ .$$

Obviously, $a_{11}(t) = (a + c)^2(1 + e^{-t}) \geqslant (a + c)^2$, $a_{22}(t) = b^2(1 + e^{-3t}) \geqslant b^2$, $a_{33}(t) = b^2(1 + e^{-2t}) \geqslant a_{22}(t) \geqslant b^2$ for all $t > 0$. Since $\lim_{t\to+\infty} e^{-t} = 0$ and $\lim_{t\to+\infty} \mathrm{th}(t) = 1$, from (36), we have $p(t) > 0$ and

$$q(t) > a_{22}(t)\,\mathrm{th}(t)(a - 1 + a_{11}(t)\,\mathrm{th}(t)) - \frac{x_2^2(t)}{4} - \frac{(a + c - x_3(t))^2}{4}$$
$$> a_{11}(t)a_{22}(t)\,\mathrm{th}^2(t) - \frac{1}{4}(x_2^2(t) + (a + c - x_3(t))^2)$$
$$> \frac{1}{4}b^2(a + c)^2 - \frac{b^2(a + c)^2}{4(b - 1)} > 0\ ,$$

$$r(t) > a_{22}(t)\,\mathrm{th}(t)(a-1+a_{11}(t)\,\mathrm{th}(t))(b-1+a_{33}(t)\,\mathrm{th}(t))$$
$$-\frac{1}{4}(x_2^2(t)+(a+c-x_3(t))^2)(b-1+a_{33}(t)\,\mathrm{th}(t))$$
$$> (b-1+a_{33}(t)\,\mathrm{th}(t))\left[\frac{1}{4}b^2(a+c)^2-\frac{b^2(a+c)^2}{4(b-1)}\right] > 0\,,$$

for large enough $t > 0$.

Since (34) is a real symmetric matrix, all its eigenvalues are real numbers. Furthermore, since the coefficients of Eq. (35) are positive, all these eigenvalues are negative numbers. Therefore, $\lambda_{\max}(t) < 0$. It is noticed that the coefficients $p(t)$, $q(t)$, $r(t)$ are continuous functions of time t, so $\lambda_{\max}(t)$ is also a continuous function of time t. Moreover, $\lim_{t\to+\infty}\lambda_{\max}(t)=\lambda_{\max}(+\infty) < 0$. Thus, there exists a $t_0 > 0$ and a $\varepsilon > 0$, such that $\lambda_{\max}(t) < -\varepsilon < 0$ for all $t > t_0$. According to the stability theory of linear time-varying systems [22], system (33) is exponentially stable about the zero solution.

When $k=3$, $\lambda_3(t)=-\arctan(t)$, $\beta_3(t)=-\cos(t)$, we have

$$\dot{w}=\begin{pmatrix} w_{11}(t) & a & 0 \\ c-x_3(t) & w_{22}(t) & -x_1(t) \\ x_2(t) & x_1(t) & w_{33}(t) \end{pmatrix} w\,, \tag{37}$$

where $w_{11}(t)=\cos(t)-a-a_{11}(t)\arctan(t), w_{22}(t)=\cos(t)-1-a_{22}(t)\arctan(t), w_{33}(t)=\cos(t)-b-a_{33}(t)\arctan(t)$. Similarly, we can show that the linear time-varying system (37) is exponentially stable about its zero solution.

Therefore, according to Theorem 2, the chaotic synchronous state $x_1(t)=x_2(t)=x_3(t)=s(t)$ of network (3) is exponentially stable.

Notice that, in this example, $C(t)$ is a nonsymmetric matrix, and it is not a nonnegative diffusively coupled matrix [25,26].

5. Conclusions

We have introduced a general complex dynamical network model and presented two chaotic network synchronization theorems. We have shown that the synchronization of a time-varying network is determined by means of the inner coupled link matrix $A(t)$, and the eigenvalues and the corresponding eigenvectors of the coupled configuration matrix $C(t)$, rather than the conventional eigenvalues of the coupled configuration matrix C for a uniform network. Especially, we do not assume that $C(t)$ is symmetric and all the off-diagonal elements of $C(t)$ are nonnegative in the chaotic network (3).

Our results summarize that the synchronization of a time-varying revolving dynamical network is more complex than that of a uniform dynamical network which is completely determined by the second largest eigenvalue. Moreover, we have found the key quantities—the link matrix $A(t)$, and the eigenvalues and eigenvectors of the coupled matrix $C(t)$, which determine the synchronizability of the general time-varying complex dynamical network model. Future work includes applying the obtained network synchronous theorems to construct some robust time-varying complex chaotic networks for possible engineering applications.

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