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Chaotic Behavior in the Real Dynamics of a One Parameter Family of Functions

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Abstract: The chaotic behavior in the real dynamics of a one parameter family of nonlinear functions is studied in the present paper. For this purpose, the function $f_\lambda(x) = \lambda \cdot xe^x/(x-1)$ $\lambda > 0,\ x \in R \setminus \{1\}$ is considered. The fixed points, periodic points and their nature are investigated for the function $f_\lambda(x)$. Bifurcation is shown to occur in the dynamics of $f_\lambda(x)$. Period doubling, which is a route of chaos in the real dynamics, is also shown to take place in the real dynamics of $f_\lambda(x)$. The orbits of the dynamics of $f_\lambda(x)$ are graphically represented by time series graphs. Moreover, the chaotic behavior in the dynamics of $f_\lambda(x)$ is found by computing positive Lyapunov exponents.

Keywords: Bifurcation; chaos; dynamics; fixed point; Lyapunov exponents.

1. Introduction

The study of real dynamics has attracted much interest of researchers in the last thirty years due to the fast growing availability of computer software for both simulation and graphics. These computer implementations demonstrate and disclose unexpected and aesthetic patterns, generating a number of predictions that lead to new developments in nonlinear phenomena in dynamical systems such as the phenomenon of chaos. In recent years, chaos theory molds the highest quality research in many fields such as different disciplines of science, engineering and technology [1, 2, 3]. Chaos theory is a combination of theoretical, computational, numerical simulation and computer graphics.

Historically, the word *chaos* was first used in a technical sense to describe an irregular behavior seen in mathematical systems by Li and Yorke [4]. Chaos is a phenomenon that takes place when a deterministic system behaves unpredictably, i.e., a system which is governed by fixed and precise rules starts behaving in a complex and an unpredictable manner in the long run [3, 5, 6]. Chaos is recognized as a primary research area by many scientists, engineers and mathematicians since the last decade of 20th Century.

The phenomenon of chaos is generally associated with the field known as dynamical systems and includes researches from many different fields [7, 8]. Chaos can be defined in many different

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forms according to conditions, observations or applications of the object. It is commonly characterized by the sensitive dependence on the initial conditions of the dynamics. There are many methods to identify and quantify chaos in dynamics. It can be identified by looking for period doubling or observing time series behavior and it can be quantified by computing Lyapunov exponents [5].

Many engineering and scientific systems in the real world can be thought of as being governed by equations that are applied repeatedly. Scientists were able to find the chaos using iterations or dynamics [6].

Chaos in the real dynamics of logistic map $f(x) = \alpha x(1-x), x \in [0,1], a > 0$ was vastly explored numerically, computationally and theoretically by many mathematicians and scientists [9, 10, 11, 12]. Recently, chaos in logistic-like maps $f(x) = rx^{\lambda}(1-x)^{\mu}$, $x \in [0,1]$ for real positive parameters r, λ and μ was studied by Stavroulaki and Sotiropoulos [13]. Several special cases of this map were also included in [13] which had been investigated by other researchers. For real dynamics of sine families $f(x) = \lambda \sin x$ and $f(x) = \lambda \sin^2 x$, chaotic behaviors were discussed and found to exist [14]. Many researchers have widely described the dynamics of the exponential family λe^{z} [15, 16, 17, 18, 19]. An image encryption via smooth exponential map was proposed for fast and highly secure image transmission [20]. The chaotic behavior in the Newton iterative function associated with Kepler's equation was considered in [21]. Some applications of transition from Newton's laws to deterministic chaos in mechanics were well represented by Florian Scheck [2]. Deshpande et al. [22] have shown the chaotic behavior in dislocation dynamics by calculating positive Lyapunov exponent.

To give some introductory definitions, let $x_n = f(x_{n-1}) = f^n(x_0)$, n = 0, 1, 2, …where $f^n = f.f.f.\cdots n$ times The point x_n is called the nth iterate of x_0 under the function $f(x)$. A point χ is said to be a periodic point of period ρ for a function $f(x)$ if $f^p(x) = x$. If $p=1$ ($f(w)=w$), then w is called a fixed point of $f(x)$. The periodic point x_0 of period p is classified according to the magnitude of $\left(f^p\right)'(x_0)$, where $(\ )' = d(\ )/dx$, as follows:

(i) If $\left|\left(f^p\right)'(x_0)\right| < 1$, then the periodic point x_0 is called attracting.

(ii) If $\left|\left(f^p\right)'(x_0)\right| > 1$, then the periodic point x_0 is called repelling.

(iii) If $\left|\left(f^p\right)'(x_0)\right| = 1$, then the periodic point x_0 is called neutral or indifferent.

The dynamics of the function changes when the parameter value crosses through a certain point. Each of such a change is called a bifurcation. Lyapunov Exponents, for kth iterate x_k of the function $f_{\lambda}(x)$, is mathematically defined as:

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln\left|f_{\lambda}'(x_k)\right| \tag{1}$$

It is known that a positive value of Lyapunov exponents for a particular dynamical system is a quantitative measure of chaos [1].

In this research work, the dynamics of a one parameter family of nonlinear functions of the form: $f_\lambda(x) = \lambda \frac{\chi}{\chi - 1} e^x$, $\lambda > 0$, $x \in R \setminus \{1\}$ is considered. This paper is organized as follows. In Section 2, the fixed points, periodic points and their nature for the function $f_\lambda(x)$ are found. Bifurcation diagram and time series graphs are given in Section 3. The chaotic behavior in the dynamics of $f_\lambda(x)$ is found by computing positive Lyapunov exponents in Section 4. Finally, some conclusions are presented in Section 5.

2. Real Fixed Points, Periodic Points and their Nature

The existence of real fixed points and periodic points of the functions $f_\lambda(x) = \lambda \frac{x}{x-1} e^x$ and their nature is investigated in the present section.
Let

$$\phi(x) = (x-1)e^{-x} \tag{2}$$

The function $\phi(x)$ is continuously differentiable in $(-\infty, \infty)$, $\phi(x) > 0$ for $(1, \infty)$ and $\phi(x) < 0$ for $(-\infty, 1)$. Since $\phi'(x) = -(x-2)e^{-x}$, then $\phi'(x) > 0$ for $x < 2$ and $\phi'(x) < 0$ for $x > 2$. It follows that $\phi(x)$ is strictly increasing in $(-\infty, 2)$ and strictly decreasing in $(2, \infty)$ (see Figure 1).

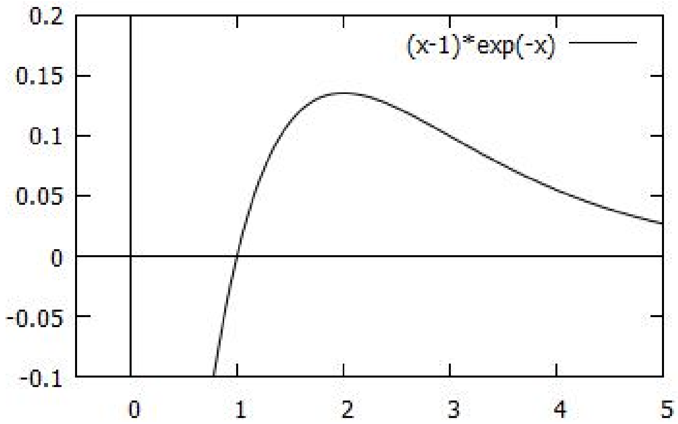


Figure 1. Graph of $\phi(x) = (x-1)e^{-x}$

Let us denote

$$\lambda^* = \phi\left(\sqrt{2}\right) \tag{3}$$

The nature of real fixed points of the function $f_\lambda(x)$ f is described here for different value of the parameter λ. Since $f_\lambda(0) = 0$ for all $\lambda > 0$, the point is a fixed point of all functions $f_\lambda(x)$ for all $\lambda > 0$. The non-zero real fixed points of the function $f_\lambda(x)$ are the solutions of the equation $\phi(x) = \lambda$, where $\phi(x)$ is given by Equation (3). Since $\phi(x)$ is strictly increasing in $(-\infty, 2)$ and strictly decreasing in $(2, \infty)$, it follows that the horizontal line $u = \lambda$ intersects the graph of $\phi(x)$ at exactly two points for $0 < \lambda < e^{-2}$, exactly one point for

$\lambda = e^{-2}$ (which is $x = 2$) and no intersection for $\lambda > e^{-2}$. Therefore, $f_\lambda(x)$ has only two non-zero real fixed points lying in (1, 2) and (2, ∞) respectively for $0 < \lambda < e^{-2}$, only one non-zero fixed point at $x = 2$ for $\lambda = e^{-2}$ and no non-zero fixed point for $\lambda > e^{-2}$.

Since the non-zero fixed points of $f_\lambda(x)$ are solutions of the equation $\phi(x) = \lambda$ and $x - 1 > 0$ for $\lambda > 0$, then the multiplier $f'_\lambda(x_f)$ of the fixed point x_f is given by

$$\left|f'_\lambda(x_f)\right| = \frac{\left|x_f^2 - x_f - 1\right|}{x_f - 1} \tag{4}$$

Let $p(x) = \left|x^2 - x - 1\right| - (x - 1)$. The function $p(x)$ can be written as

$$p(x) = \begin{cases} x^2 - 2x & \text{for} \quad x \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, \infty\right) \\ \frac{1-\sqrt{5}}{2} & \text{for} \quad x = \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & \text{for} \quad x = \frac{1-\sqrt{5}}{2} \\ -x^2 + 2 & \text{for} \quad x \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right) \end{cases}$$

It is seen that $p(x)$ is continuous and has two zeros at $x = \sqrt{2}$ and $x = 2$. It follows that $p(x) < 0$ for $x \in (\sqrt{2}, 2)$ and $p(x) > 0$ for $x \in (-\infty, \sqrt{2}) \cup (2, \infty)$. Therefore, from Equation (4), we get

$$\left|f'_\lambda(x_f)\right| < 1 \quad \text{for} \quad x_f \in \left(\sqrt{2}, 2\right) \tag{5}$$

$$\left|f'_\lambda(x_f)\right| = 1 \quad \text{for} \quad x_f = \sqrt{2}, 2 \tag{6}$$

$$\left|f'_\lambda(x_f)\right| > 1 \quad \text{for} \quad x_f \in \left(-\infty, \sqrt{2}\right) \cup (2, \infty) \tag{7}$$

The nature of the fixed points are categorised in the following cases:

Case(i). If $0 < \lambda < \lambda^*$, then the fixed points $r_{1,\lambda} \in (1, \sqrt{2})$ and $r_{2,\lambda} \in (x^*, \infty)$ are repelling and the fixed point *0* is attracting, where x^* is a solution of $\phi(x) = \lambda^*$.

To prove this, by Inequality (7), $\left|f'_\lambda(r_{1,\lambda})\right| > 1$ and $\left|f'_\lambda(r_{2,\lambda})\right| > 1$ since $r_{1,\lambda} \in (1, \sqrt{2})$ and $r_{2,\lambda} \in (x^*, \infty)$. Therefore, fixed points $r_{1,\lambda}$ and $r_{2,\lambda}$ are repelling. Further, $\left|f'_\lambda(0)\right| = |\lambda| < 1$ for $0 < \lambda < \lambda^*$, it shows that the point *0* is an attracting fixed point of $f_\lambda(x)$.

Case(ii). If $\lambda = \lambda^*$, then the fixed point $\sqrt{2}$ is rationally indifferent, the fixed point x^* is repelling and the fixed point *0* is attracting.

To show this, using $\left|f'_\lambda(\sqrt{2})\right| = 1$, it gives that the point $\sqrt{2}$ is rationally indifferent fixed point. From Inequality (7), $\left|f'_\lambda\left(x^*\right)\right|>1$, it follows that the point $x=x^*$ is repelling fixed point. Further, $\left|f'_{\lambda^*}\left(0\right)\right|<1$, the point *0* is an attracting fixed point of $f_\lambda\left(x\right)$.

Case(iii). If $\lambda^*<\lambda<e^{-2}$, then the fixed point $\alpha_\lambda\in(\sqrt{2}$, 2) is attracting, the fixed points $r_\lambda\in(2,\ x^*)$ are repelling and the fixed point *0* is attracting.

For fixed point $\alpha_\lambda\in(\sqrt{2}$, 2), by Inequality (5), $\left|f'_\lambda\left(a_\lambda\right)\right|<1$ so that a_λ is an attracting fixed point. Since for the fixed points $r_\lambda\in(2, x^*)$, by Inequality (7), $\left|f'_\lambda\left(r_\lambda\right)\right|>1$, then r_λ are repelling fixed point. Further, $\left|f'_\lambda\left(0\right)\right|=\left|\lambda\right|<1$ for $\lambda<e^{-2}$, it gives that the point *0* is an attracting fixed point of $f_\lambda\left(x\right)$.

Case(iv). If $\lambda=e^{-2}$, then the fixed point *2* is rationally indifferent and the fixed point *0* is attracting.

From Equation (6), $\left|f'_\lambda\left(2\right)\right|=1$. Therefore, the point *2* is a rationally indifferent fixed point. Further, $\left|f'_\lambda\left(0\right)\right|=\left|\lambda\right|=e^{-2}<1$, the point *0* is an attracting fixed point of $f_\lambda\left(x\right)$.

Case(v). If $e^{-2}<\lambda<1$, then the fixed point *0* is attracting.
Since $\left|f'_\lambda\left(0\right)\right|=\left|\lambda\right|<1$, it follows that the point *0* is an attracting fixed point of $f_\lambda\left(x\right)$.

Case(vi). If $\lambda=1$, then the fixed point *0* is rationally indifferent.
Since $\left|f'_\lambda\left(0\right)\right|=1$, then the point *0* is rationally indifferent fixed point of $f_\lambda\left(x\right)$.

The next case is more interesting because it shows the existence of chaotic behavior in the dynamics of $f_\lambda\left(x\right)$.

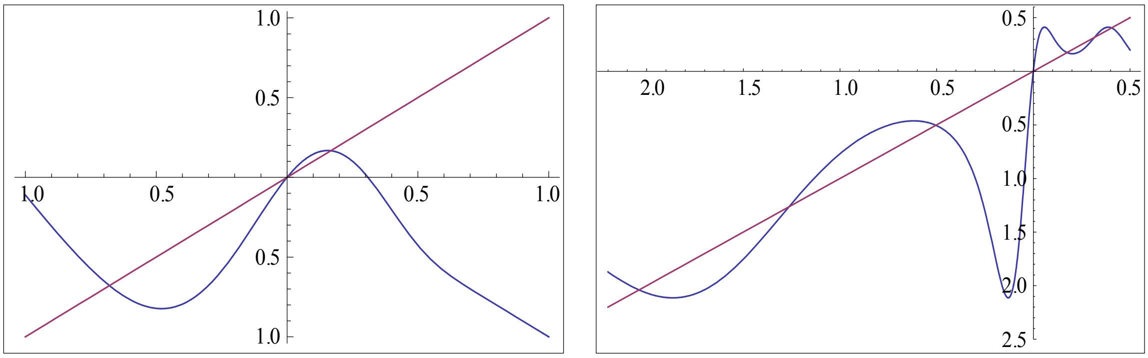
Case(vii). If $\lambda>1$, then the fixed point *0* is repelling and periodic points of period 2 or more exist.
Proof of this case is as follows:
Since $\left|f'_\lambda\left(0\right)\right|=\left|\lambda\right|>1$, it gives that the point *0* is repelling fixed point of $f_\lambda\left(x\right)$.

Since the function $f_\lambda\left(x\right)$ is highly nonlinear, the analytical calculations are very complicated (or almost impossible). So we use the numerical techniques to find the periodic cycles of the function $f_\lambda\left(x\right)$. The periodic points are roots of $f_\lambda^n\left(x\right)=x$, i. e.,

$$\lambda\frac{f_\lambda^{n-1}\left(x\right)e^{f_\lambda^{n-1}(x)}}{f_\lambda^{n-1}\left(x\right)-1}=x$$

It is found that the function $f_\lambda\left(x\right)$ has periodic cycle of period greater than or equal to *2*. This cycle may be attracting, indifferent or repelling for $\lambda>1$.

To show this, let's consider the case $n=2$. (i) For $\lambda=1.1$, the periodic points of period two of $f_\lambda\left(x\right)$ are $p_{11}=0.194059$ and $p_{12}=-0.321589$, so that $f_\lambda(p_{11})=-2.37778$ and $f_\lambda(p_{12})=$ -0.262541. It follows that $\left|\ f'_\lambda(p_{11})f'_\lambda(p_{12})\ \right|=-0.624267<1$. Therefore, the periodic 2-cycle of $f_\lambda(x)$ is attracting for $\lambda = 1.1$. (ii) For $\lambda = 1.74$, $f_\lambda(x)$ has 2-cycle periodic points $q_{11} = 0.309609$ and then $f_\lambda(q_{11}) = -6.03881$ and $f_\lambda(q_{12}) = 0.168524$. Hence, $|f'_\lambda(q_{11}) f'_\lambda(q_{12})| = 1.01768 > 1$. It follows that the periodic 2-cycle of $f_\lambda(x)$ is repelling for $\lambda = 1.74$. While, for λ =1.730634, $f_\lambda(x)$ has 2-cycle periodic points $s_{11} = 0.309255$ and $s_{12} = -1.05563$ $s_{12} = -1.05563$, so that $f_\lambda(s_{11}) = -5.99735$, $f_\lambda(s_{12}) = 0.166741$ and consequently, $|f'_\lambda(s_{11}) f'_\lambda(s_{12})| = 1$. Therefore, it follows that the periodic 2-cycle of $f_\lambda(x)$ is indifferent for λ=1.730634. Thus, for $\lambda > 1$, the periodic cycle of period greater than or equal to *2* of $f_\lambda(x)$ may be attracting, indifferent or repelling (Figure 2(a) and (b)). This proves Case (vii).

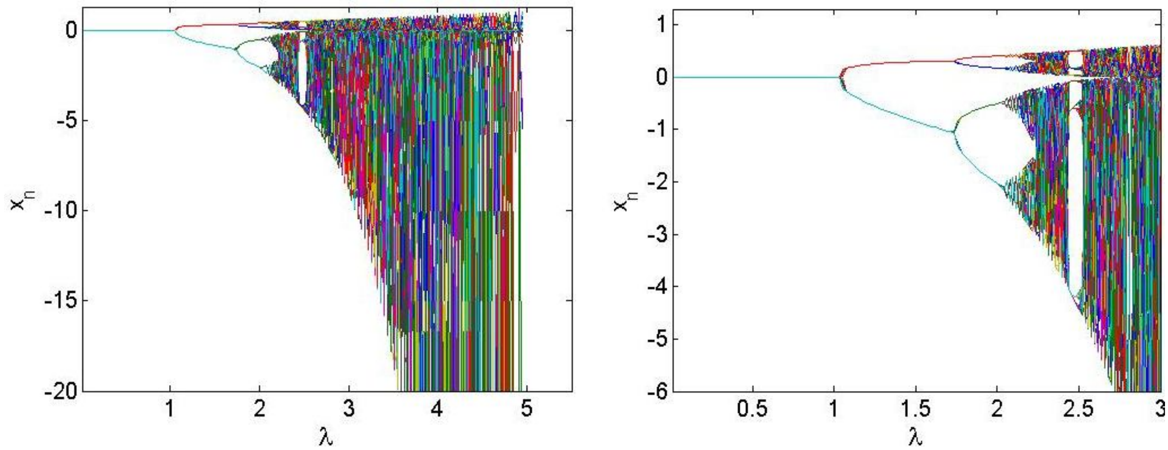


(a) Periodic 2-cycles for $\lambda = 1.1$ (b) Periodic 4-cycles for $\lambda = 2$

Figure 2. Periodic cycles of $f_\lambda(x)$

3. Bifurcation and time series graphs

The investigation of the periodic points of $f_\lambda(x)$ becomes significantly more complicated when the parameter value λ increases beyond the values obtained in Section 2. Many new periodic cycles of period *4, 8, 16,…* come into the picture (exist) when λ increases. Figure 3(a) shows the limiting behavior of orbits for values of λ in the range $0 < \lambda < 5.5$. The values of the parameter λ and values of x are plotted on horizontal axis and vertical axis respectively. Bifurcation in the dynamics of $f_\lambda(x)$ occurs at several parameter values $\lambda = (\sqrt{2}-1)e^{-\sqrt{2}}$, $\lambda = e^{-2}$, $\lambda = 1$ and so on. It is observed that the periodic doubling occurs which represents the presence of chaos.



(a) (b)

Figure 3. Bifurcation diagrams of dynamics of $f_\lambda(x)$

It can be clearly seen from Figure 3(b), which is a zoom of Figure 3(a), that a gap window is visible near λ=2.45. This shows that the chaotic region break up into non-chaotic region temporarily, and then go back to being chaotic. Bifurcation diagram is seen to be self-similar because if a small portion of the diagram is chosen, it will look similar to the whole diagram and this shows the fractal nature in the dynamics.

Remark:

For $\lambda > 1$, it is found that

(i)the function $f_\lambda(x)$ does not have any periodic point for $x > 1$.

(ii) $f_\lambda^n(x) \to \infty$ for $x > 1$.

To prove this remark:

Let $g_\lambda(x) = f_\lambda(x) - x$. It is easily seen that $g_\lambda(x) = x\left(\frac{\lambda e^x}{x-1} - 1\right) > \frac{2}{x-1} > 0$ for $x > 1$ and $\lambda > 1$. Therefore, $f_\lambda(x) > x$ for $x > 1$ and $\lambda > 1$. Again, let $h_\lambda(x) = f_\lambda{}^2(x) - x$. Then, for $x > 1$ and $\lambda > 1$

$$h_\lambda(x) = x\left[\frac{\lambda^2 e^x e^{\frac{\lambda x e^x}{x-1}}}{(x-1)\left(\frac{\lambda x e^x}{x-1} - 1\right)} - 1\right] > \frac{1}{\left(\frac{\lambda x e^x}{x-1} - 1\right)}\left[\lambda \frac{e^x}{x-1}\left(\lambda e^{\frac{\lambda x e^x}{x-1}} - x\right) + 1\right] > 0$$

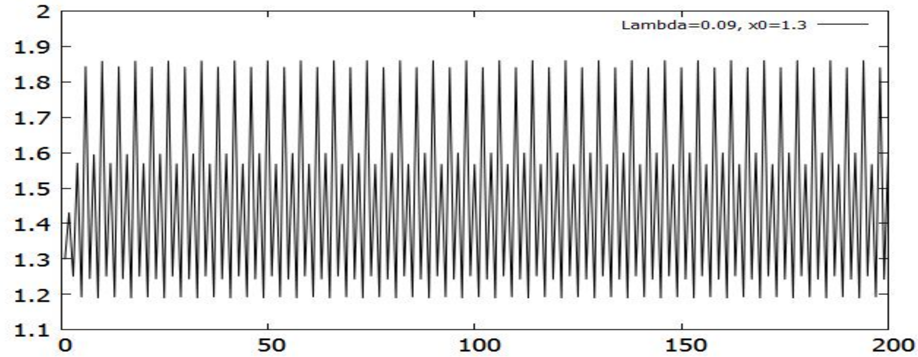
It follows that $f_\lambda^2(x) > x$ for $x > 1$ and $\lambda > 1$. To prove result for *n,* we use induction process. Let us consider the result true for $n-1$ that is $f_\lambda^{n-1}(x) > x$ for $x > 1$ and $\lambda > 1$.

$$f_\lambda^n(x) - x = f_\lambda^n(x) - f_\lambda^{n-1}(x) + f_\lambda^{n-1}(x) - x > f_\lambda^n(x) - f_\lambda^{n-1}(x) = \frac{f_\lambda^{n-1}(x)}{f_\lambda^{n-1}(x) - 1}\left[\lambda e^{f_\lambda^{n-1}} - f_\lambda^{n-1}(x) + 1\right] > 0$$

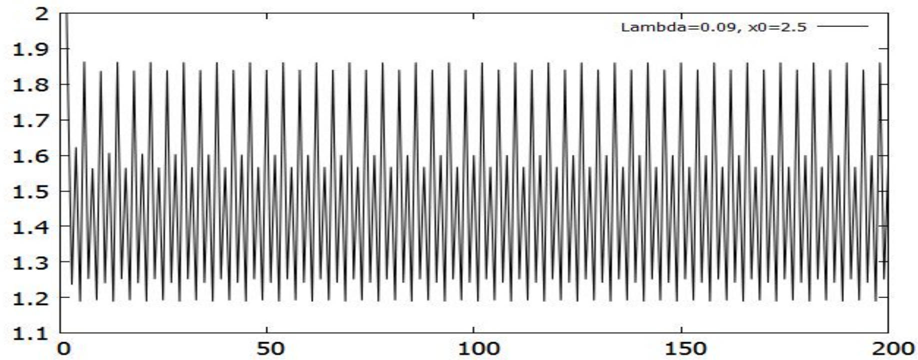
Thus, the function $f_{\lambda}(x)$ does not have periodic point for $x>1$ and this proves (i).

Since $f_{\lambda}^{n}(x)-x>0$ for $x>1$ and $\lambda>1$, it follows that $f_{\lambda}^{n}(x)>x$. Consequently, $f_{\lambda}^{n}(x)\to\infty$ since $x>1$ and this proves (ii).

The orbits of the dynamics of $f_{\lambda}(x)$ are graphically represented by time series graphs for different values of parameter λ. For λ=0.009, time series plots are given in Figure 4(a) and (b) using initial values x_0=1.3 and x_0=2.5 respectively.



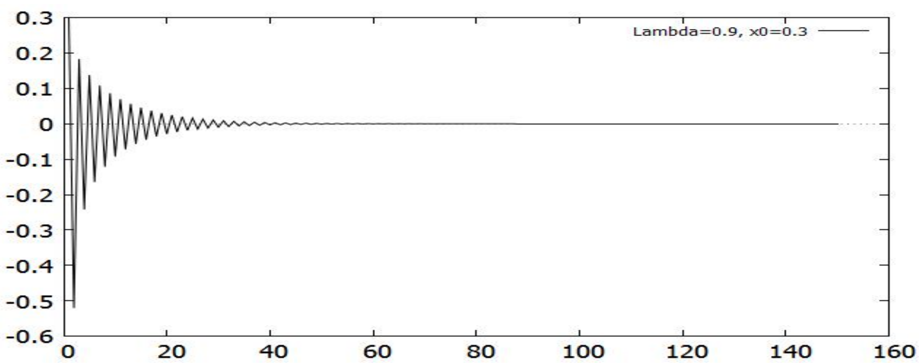
(a)



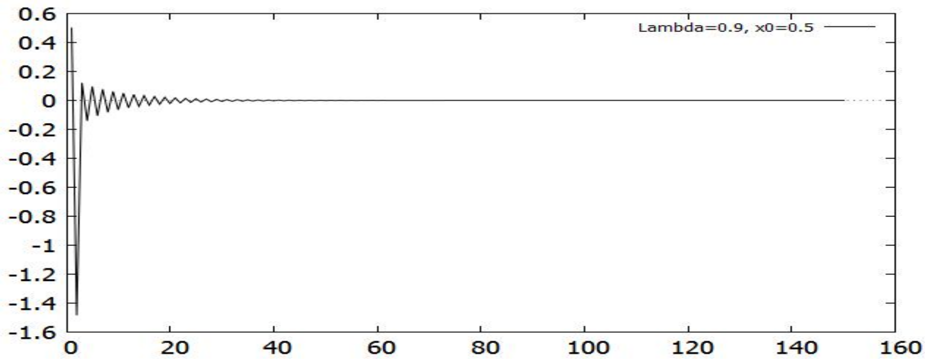
(b)

Figure 4. For λ=0.09 (a) x_0=1.3 (b) x_0=2.5

For λ=0.9 and λ=1.74, time series plots are given in Figure 5(a), (b) and Figure 6(a), (b) using initial values x_0=0.3 and x_0=0.5 respectively.

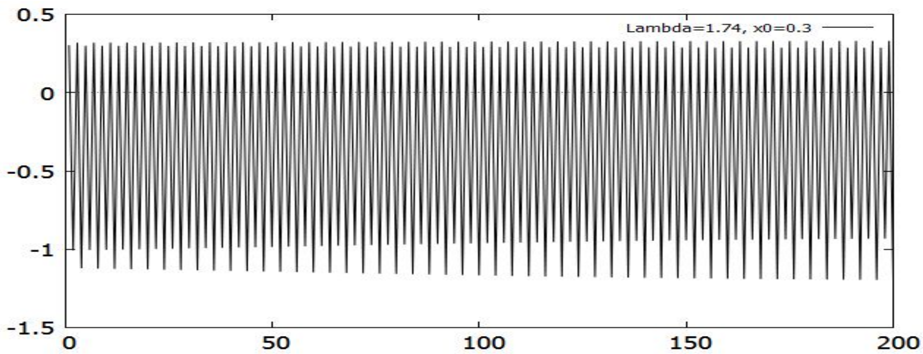


(a)

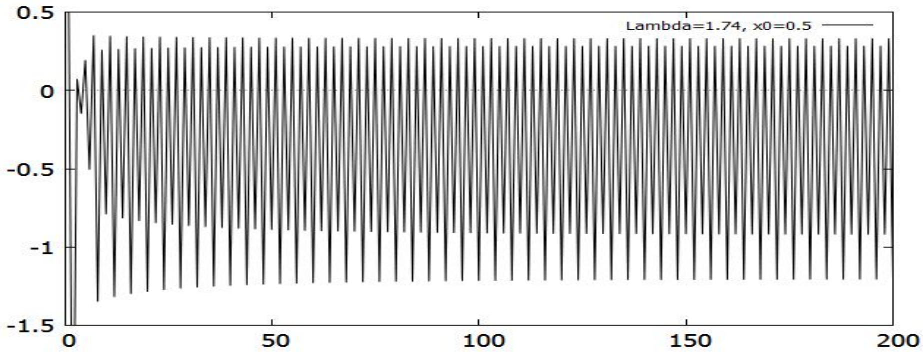


(b)

Figure 5. For λ=0.09 (a) x_0=0.3 (b) x_0=0.5



(a)

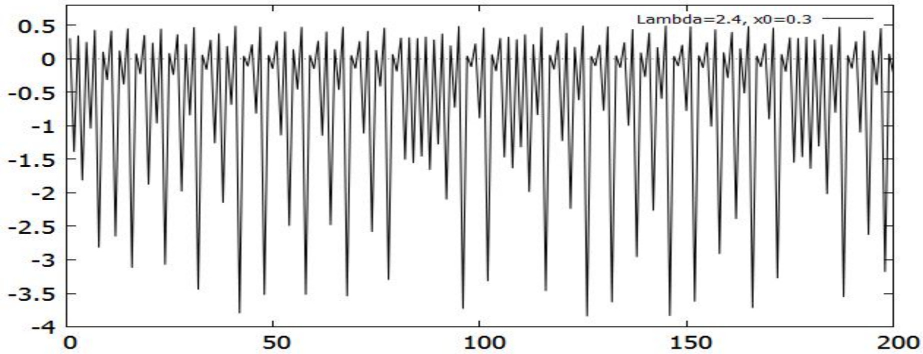


(b)

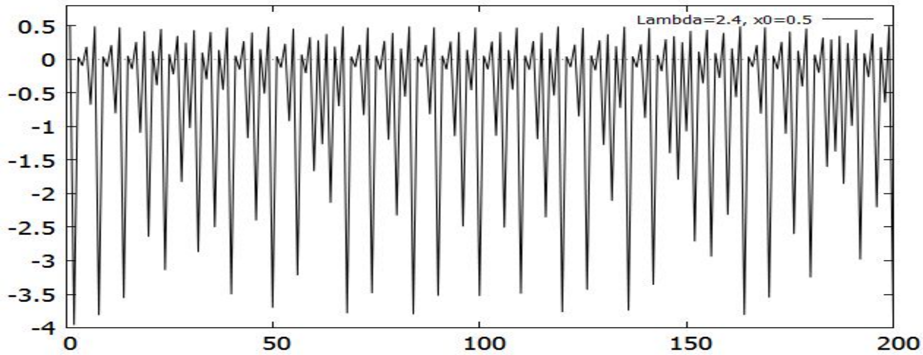
Figure 6. For λ=1.74 (a) x_0=0.3 (b) x_0=0.5

For $\lambda = 2.4$ and $\lambda = 3$, time series plots are given in Figure 7(a), (b) using initial values $x_0 = 0.3$ and $x_0 = 0.5$ respectively.

Time series graphs of the dynamics of $f_\lambda(x)$ show the different behavior of trajectories for different values of λ. For some values of λ, time series are stable (Figure 4 and Figure 5). But as λ increases, trajectories become unstable or periodic (Figure 6). Further increase of λ, initiates period doubling as can be clearly seen in Figure 7. This means that chaos exists in the dynamics of $f_\lambda(x)$.



(a)



(b)

Figure 7. For $\lambda = 2.4$ (a) $x_0 = 0.3$ (b) $x_0 = 0.5$

4. Lyapunov exponents

The quantification of chaos in the dynamics of functions is found by computing Lyapunov exponents. The Lyapunov exponent at a point *x* measures the average loss of information during successive iterations of points near *x*. The Lyapunov exponents of the dynamics of $f_\lambda(x)$ can be calculated by using Equation (1). Doing that yields,

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \ln\left(\lambda \frac{\left|x_k^2 - x_k - 1\right| e^{x_k}}{\left(x_k - 1\right)^2} \right) \tag{8}$$

For some values of λ, the computed Lyapunov exponents are presented in Table 1 using initial value $x_0 = 0.3$ and number of iterations $n = 2000$. It can be seen from Table 1 that Lyapunov exponents are positive for λ values of 2.3, 2.5, 3.0 and 4.0. It follows that the dynamics of $f_\lambda(x)$ is chaotic for these values of λ. For λ values of 0.5, 1.5, 2.45, and 3.04, Lyapunov exponents are negative, which means that non-chaotic regions are present in the dynamics of $f_\lambda(x)$. Lyapunov exponents of the dynamics of $f_\lambda(x)$ for $0 < \lambda < 5$ are plotted in Figure 8.

It is clear from Figure 8 that for certain ranges of parameter values λ, chaos exists in the dynamics of $f_\lambda(x)$ since Lyapunov exponents are positive for these parameter values.

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Table 1. Computed Lyapunov exponents for some values of parameter λ

Parameter (λ)	Lyapunov Exponents (L)
0.5	-0.30126
1.5	-0.14317
2.3	0.078802
2.45	-0.12816
2.5	0.010228
3.0	0.124923
3.04	-0.06647
4.0	0.090559

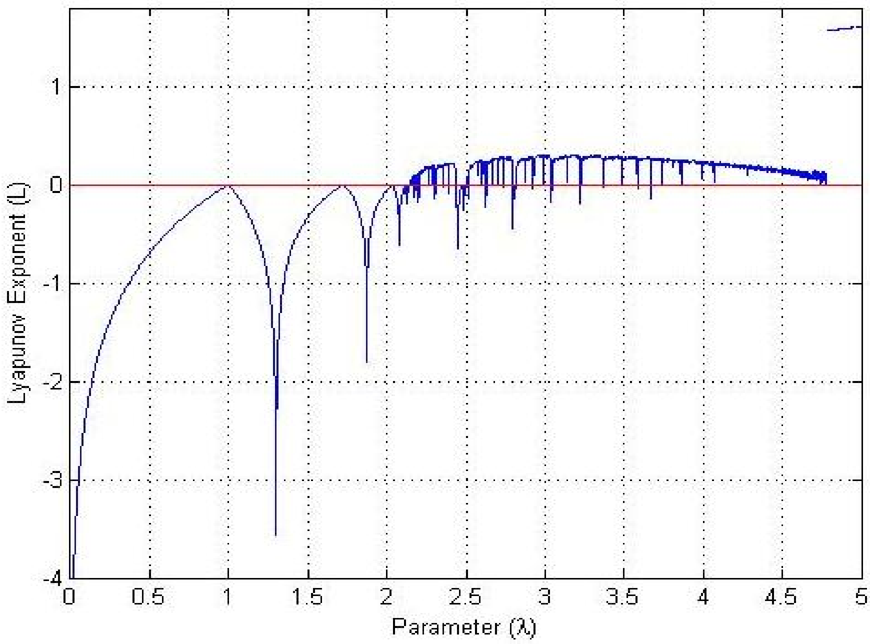

Figure 8. Lyapunov exponents of dynamics of $f_\lambda(x)=\lambda\frac{x}{x-1}e^x$

5. Conclusions

In this research work, the chaotic behavior in the real dynamics of a one parameter family of nonlinear functions that are of the form $f_\lambda(x) = \lambda \frac{\chi}{\chi - 1} e^x,\ \ \lambda > 0,\ \ x \in R \setminus \{1\}$ is studied. It is found that chaos exists in the dynamics of the considered family of functions. The existence of chaotic behavior is observed due to periodic doubling which was found to take place in the dynamics of the functions considered. Bifurcation diagrams of the considered functions show the fractal nature in the dynamics. Chaos is quantified by computing positive Lyapunov exponents. These findings are expected to partially fulfil some gap in the dynamics of the family of nonlinear functions considered.

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