Chaotic difference equations
are dense

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A continuous function mapping a compact interval of the real line
into itself is called chaotic if the difference equation defined
in terms of it behaves chaotically in the sense of Li and Yorke.
The set of all such chaotic functions is shown to be a dense
subset of the space of continuous mappings of that interval into
itself with the max norm. This result indicates the structural
instability of nonchaotic difference equations with respect to
chaotic behaviour.

1. Introduction

Numerical studies of the solutions of one-dimensional deterministic
difference equations

(1.1) $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$,

describing meteorological ([7, 8]) and biological ([10, 11]) phenomena have
indicated behaviour so apparently irregular that it might well be termed
chaotic. In a recent paper Li and Yorke [6] clearly specified what might
heuristically be thought of as chaotic behaviour in a difference equation
(1.1) and, indeed, it was they who suggested the term *chaos* to describe
such behaviour.

In this paper functions f in difference equation (1.1) will be
considered which are continuous and map a compact interval X of the real
line into itself. The space of all such functions with the max norm will
be denoted by $C(X)$ and any such function for which difference equation

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(1.1) behaves chaotically in the sense of Li and Yorke will be called
chaotic. It will be proved that the subset of all such chaotic functions
is a dense subset of $C(X)$. This will be done by actually constructing
for each function f in $C(X)$ and each desired degree of accuracy
$\varepsilon > 0$, a chaotic function in $C(X)$ which lies within ε of f with
respect to the max norm on $C(X)$.

Although this result suggests the structural instability with respect
to chaotic behaviour of nonchaotic difference equations, it is not quite so
serious as it may at first seem on account of the localised nature of the
chaotic behaviour in the constructed function. It does, however, indicate
the need for some kind of measure of the extent of chaotic behaviour and
the need to restrict a discussion of such structural instability to
parametric perturbations rather than arbitrary perturbations.

2.   Chaotic difference equations

Let $X = [a, b]$ be a compact interval and let $C(X)$ denote the space
of all continuous functions $f : X \to X$ with the max norm
$$\|f\| = \max_{x \in X} |f(x)| .$$

Associated with each f in $C(X)$ there is a difference equation

(2.1) $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$,

for which a set of $k = 1, 2, 3, \ldots$ distinct points $p_0, p_1, \ldots, p_{k-1}$ is
called a *cycle of order* k if $p_i = f(p_{i-1})$ for $i = 1, 2, \ldots, k-1$ and
$p_0 = f(p_{k-1})$. Note that cyclic points of any order k are fixed points
of the kth-iterate $f^k = f \circ f \circ \ldots \circ f$ (k times) of f and that each
f in $C(X)$ has at least one fixed point (of order 1).

For a given difference equation (2.1) an initial point x_0 in X is
said to be *asymptotically periodic* if there exists a cyclic point p_0 of
some order such that

$$\lim_{n \to \infty} \left| f^n(x_0) - f^n(p_0) \right| = 0 .$$

This represents a highly desirable form of behaviour for all points

x_0 in X and their associated iterative sequences. Coppel [5] has shown
this to be the case for difference equations having cycles of only finitely
many different orders $\left(\text{necessarily all of the form }2^n\right)$, though the
following theorem of Li and Yorke [6] shows that it is not obtainable in
all difference equations.

THEOREM 1. *Suppose for* f *in* $C(X)$ *that the difference equation*
(2.1) *has a cycle of order three. Then it has cycles of all orders and*
there exists an uncountable subset S *of* X *containing no cyclic points*
such that for all x_0, y_0 *in* S *and all cyclic points* p_0 *of any order*
the following hold:

(i) $\lim\limits_{n\to\infty} \sup \left| f^n(x_0) - f^n(y_0) \right| > 0$,

(ii) $\lim\limits_{n\to\infty} \inf \left| f^n(x_0) - f^n(y_0) \right| = 0$, *and*

(iii) $\lim\limits_{n\to\infty} \sup \left| f^n(x_0) - f^n(p_0) \right| > 0$.

This behaviour was described quite aptly by Li and Yorke as being
chaotic.

Actually related and more powerful results appeared earlier in the
Eastern European mathematical literature (for example, [1, 2, 3, 4, 12,
13]). In particular, the work of the Ukrainian mathematician A.N.
Šarkovskiĭ (for example, [12, 13]) lead Kloeden, Deakin, Tirkel [9] to
weaken the conclusions of Theorem 1 above in defining chaotic behaviour to
allow the absence of some (in fact, possibly infinitely many) cyclic
orders, provided that there is always at least one cycle present with order
of the form $(2k+1).2^n$ for some nonzero k . In this paper, however, the
more restrictive definition of Li and Yorke will be used and any f in
$C(X)$ with difference equation (2.1) behaving in such a manner will be
called *chaotic*.

Examples of chaotic functions occurring in meteorological and
biological modelling are given in [7, 8, 10, 11]. Other examples can be
found in the papers cited above. The following example is a chaotic
function and will be used in proving Theorem 2.

EXAMPLE 1. Let $\eta > 0$ and let $\Lambda \in C([0, 2\eta])$ be defined as

$$\Lambda(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq \eta \text{ ,} \\[2em] -2x + 4\eta & \text{for } \eta \leq x \leq 2\eta \text{ .} \end{cases}$$

This has a cycle of order three for any value of $\eta > 0$, namely $p_0 = \frac{4}{7}\eta$, $p_1 = \frac{8}{7}\eta$, $p_2 = \frac{12}{7}\eta$, and by Theorem 1, is thus a chaotic function. Its maximum value is 2η and occurs at $x = \eta$. It is illustrated together with its nth-iterates Λ^n in Figure 4 of [1] for $\eta = \frac{1}{2}$.

3. Main result

The following theorem is the main result of this paper. Recall that a subset of a topological space is *dense* if its closure is equal to the whole space.

THEOREM 2. *The chaotic functions are dense in* $C(X)$.

Alternatively, for any $f \in C(X)$ *and any* $\varepsilon > 0$ *there is a chaotic function* $g \in C(X)$ *such that*

(3.1) $\|f-g\| < \varepsilon$.

Proof. Let $f \in C(X)$ and $\varepsilon > 0$ be given, but otherwise arbitrary. The theorem will be proved by constructing a chaotic function $g \in C(X)$ for which inequality (3.1) is satisfied.

Let x^* be a fixed point of f and assume for the present that it is not equal to b. Then as f is continuous at x^* there is a $\delta = \delta(x^*, \varepsilon) > 0$, which can be chosen to satisfy $\delta < \varepsilon/2$, such that

(3.2) $|f(x)-x^*| < \varepsilon/2$

holds for all $x \in [x^*, x^*+\delta]$.

Let $\eta = \delta/4$ and define $\Lambda^* \in C([x^*, x^*+2\eta])$ as

$$\Lambda^*(x) = \begin{cases} 2x - x^* & \text{for } x^* \leq x \leq x^* + \eta \text{ ,} \\[2em] -2x + 3x^* + 4\eta & \text{for } x^* + \eta \leq x \leq x^* + 2\eta \text{ ,} \end{cases}$$

which has the same graph as function Λ of Example 1, but with the origin

translated to the point $(-x^*, -x^*)$.

Also, let $\gamma : [x^*+2\eta, x^*+3\eta] \to X$ be defined as

$$\gamma(x) = x^* + (x-x^*-2\eta)\left[\frac{f(x^*+3\eta)-x^*}{\eta}\right] ,$$

the graph of which is the straight line segment joining the points $(x^*+2\eta, x^*)$ and $\left(x^*+3\eta, f(x^*+3\eta)\right)$ in the plane.

Finally, define a function $g : X \to X$ as

$$g(x) = \begin{cases} f(x) & \text{for } x \in X\backslash(I \cup J) \ , \\ \Lambda^*(x) & \text{for } x \in I \ , \\ \gamma(x) & \text{for } x \in J \ , \end{cases}$$

where $I = [x^*, x^*+2\eta]$ and $J = [x^*+2\eta, x^*+3\eta]$.

The graph of such a g is illustrated in Figure 1 (see page 376). It is obvious that g is a continuous function; that is, $g \in C(X)$. Note that

$$\left|f(x)-g(x)\right| = \begin{cases} 0 & \text{for } x \in X\backslash(I \cup J) \ , \\ \left|f(x)-\Lambda^*(x)\right| & \text{for } x \in I \ , \\ \left|f(x)-\gamma(x)\right| & \text{for } x \in J \ . \end{cases}$$

Hence

$$\begin{aligned} \|f-g\| &= \max\Big\{\max_{x \in I} \left|f(x)-\Lambda^*(x)\right|, \max_{x \in J} \left|f(x)-\gamma(x)\right| \\ &\leq \max\Big\{\max_{x \in I} \left|f(x)-x^*\right| + \max_{x \in I} \left|x^*-\Lambda^*(x)\right|, \max_{x \in J} \left|f(x)-x^*\right| + \max_{x \in J} \left|x^*-\gamma(x)\right|\Big\} \\ &< \max\{\varepsilon/2+\varepsilon/4, \ \varepsilon/2+\varepsilon/2\} \\ &= \varepsilon \end{aligned}$$

on account of inequality (3.2), the definitions of intervals I and J , and the two inequalities

$$\max_{x \in I} \left|x^*-\Lambda^*(x)\right| = 2\eta = \delta/2 < \varepsilon/4 \ ,$$

$$\max_{x \in J} \left|x^*-\gamma(x)\right| = \left|x^*-f(x^*+3\eta)\right| < \varepsilon/2 \ .$$

That this g is a chaotic function follows immediately from Theorem 1 and the fact that Λ^* , and hence g itself, has a cycle of order three, namely $p_0 = x^* + \frac{4}{7}\eta \ , \quad p_1 = x^* + \frac{8}{7}\eta \ , \quad p_2 = x^* + \frac{12}{7}\eta \ .$
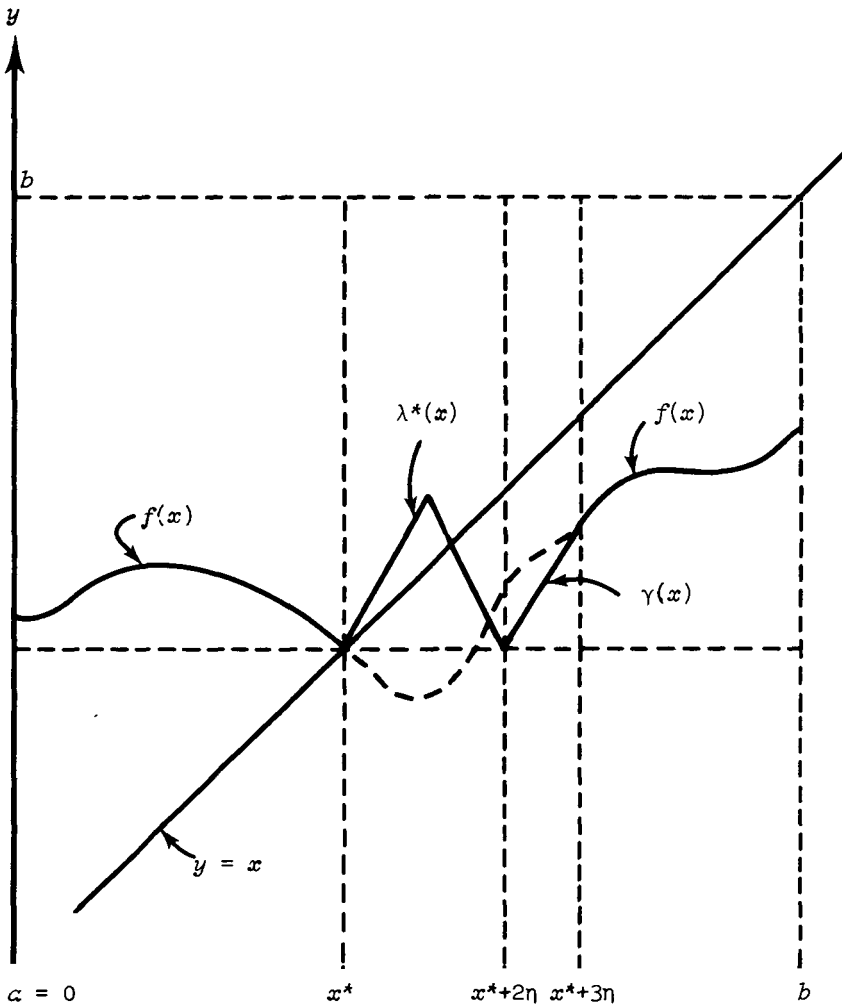
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FIGURE 1. Function g when $x^* \neq b$.

        It now remains to consider the case where the only fixed point of f
is larger endpoint b of the interval X . Here a chaotic function g
satisfying inequality (3.1) can be constructed in the same way as above,
except with the Λ^* perturbation now the reflection in the point (x^*, x^*)
of that above. This is illustrated in Figure 2.

        This completes the proof of the theorem.

        REMARK. Since a function which is chaotic in the sense of Li and
Yorke is also chaotic in the weaker sense of Kloeden, Deakin, Tirkel the

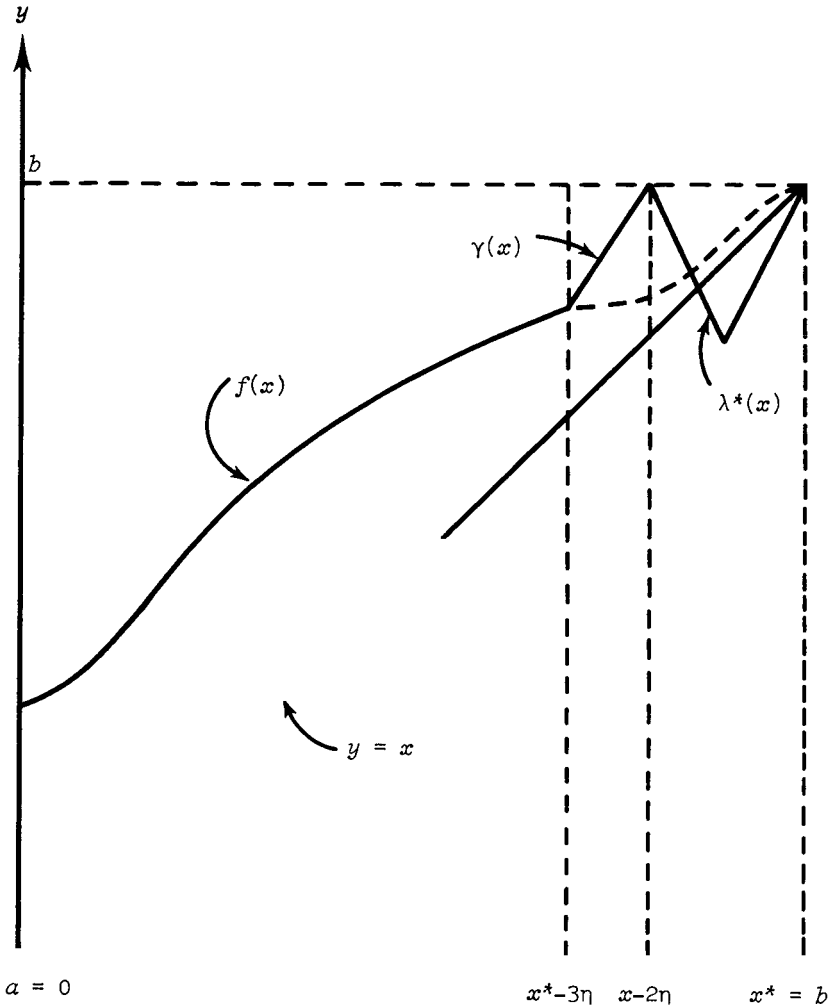above theorem is also true for this weaker definition of chaotic functions.

FIGURE 2. Function g when $x^* = b$.

4. Concluding remarks

Theorem 2 can be interpreted as saying that nonchaotic difference equations are structurally unstable with respect to chaotic behaviour. The proof of the theorem shows, however, that this is not quite so devastating as it may at first seem.

Observe that, for sufficiently small bound ε on the perturbations,

it is unlikely, with the perturbations used, that the behaviour of the
perturbed difference equation would differ much from that of the original
one outside of the interval $[x^*, x^*+3\eta]$ in which the perturbations are
made.   Indeed the chaotic behaviour of the perturbed difference equation
will most likely be restricted entirely to the interval $[x^*, x^*+2\eta]$ in
which the perturbation Λ^* is defined.   Iterative sequences entering this
interval from outside it will be trapped inside it and may then begin to
behave chaotically.   In constructing a mathematical model from physical
measurements of such behaviour, given the limited accuracy of any physical
measurements, this would probably be mistaken for convergence to the fixed
point x^* .   This suggests that chaotic functions should be further
classified in terms of the size of the region in which their iterative
sequences behave chaotically.

A second point is that Λ^* is not what one would consider as a
naturally occurring perturbation.   Indeed, in [9] Kloeden, Deakin, Tirkel
suggest that a discussion on structural instability of difference equations
with respect to chaotic behaviour should be related to parametric
perturbations in the functions being considered, rather than a somewhat
arbitrary and unnatural perturbation like Λ^* .

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