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CHAOTIC DIFFERENCE EQUATIONS IN $\mathbf{R}^n$

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Abstract

Sufficient conditions are given for the chaotic behaviour of difference equations defined in terms of continuous mappings in $\mathbf{R}^n$. These conditions are applicable to both difference equations with snap-back repellors and with saddle points. They are applied here to the twisted-horseshoe difference equation of Guckenheimer, Oster and Ipaktchi.

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1. Introduction

Recently Marotto [5] proved that a difference equation in $\mathbf{R}^n$ with a snap-back repellor behaves chaotically. His proof is a generalization of the period 3 implies chaos result of Li and Yorke [4] for difference equations in $\mathbf{R}^1$. It differs in that the mappings defining the difference equations are required to be continuously differentiable rather than continuous. This is so the inverse mapping theorem and the Brouwer fixed point theorem can be used to obtain the existence of continuous inverse functions and periodic points. Marotto's result is however applicable only to difference equations with repellors and not to those with saddle points. It thus cannot be used for difference equations involving the horseshoe mappings of Smale [7] or the twisted-horseshoe mappings of Guckenheimer, Oster and Ipaktchi [3].

Sufficient conditions for the chaotic behaviour of difference equations in $\mathbf{R}^n$ are given in this paper, which are applicable to difference equations with saddle points as well as to those with repellors. These conditions are valid for difference

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equations defined in terms of continuous mappings and in the special case of a difference equation with a snap-back repellor are more readily tested than those listed by Marotto in [5; Remark 2.2]. The proof that they imply chaotic behaviour is a modification of the proof used by Marotto in [5]. There are however two important differences. Firstly the mappings in the difference equations are assumed to be continuous rather than continuously differentiable. The existence of continuous inverse mapping then follows from the fact that continuous one-to-one mappings have continuous inverses on compact sets. Using this result rather than the inverse mapping theorem considerably simplifies the proof. Secondly the Brouwer fixed point theorem is used on a homeomorph of an l-ball for some $1 \leqslant l \leqslant n$ rather than on a homeomorph of an n-ball as in Marotto's proof. This allows saddle points to be considered as well as repellors.

In the next section the sufficient conditions are stated and are proved to imply chaotic behaviour. Then in Section 3 these conditions are used in two examples, one of which is the twisted-horseshoe example of Guckenheimer, Oster and Ipaktchi [3].

2. Sufficient conditions

Some definitions and preliminary results will be given first.

In the sequel $f\colon \mathbf{R}^n \to \mathbf{R}^n$ will be a continuous mapping, and which is associated the first order difference equations

$$x_{k+1} = f(x_k). \tag{2.1}$$

A point $y_0 \in \mathbf{R}^n$ is a *periodic point of period p* if the points y_0, $f(y_0)$, $f^2(y_0), \dots, f^{p-1}(y_0)$ are pairwise different and if $f^p(y_0) = y_0$. Difference equation (2.1) is *chaotic* if there exists

(i) a positive integer N such that (2.1) has a periodic point of period p for each $p \geqslant N$;

(ii) a scrambled set of (2.1) that is an uncountable set S containing no periodic points of (2.1) such that:

(a) $f(S) \subset S$,

(b) for every $x_0, y_0 \in S$ with $x_0 \neq y_0$

$$\limsup_{x \to \infty} \|f^k(x_0) - f^k(y_0)\| > 0,$$

(c) for every $x_0 \in S$ and any periodic point y_0 of (2.1)

$$\limsup_{k \to \infty} \|f^k(x_0) - f^k(y_0)\| > 0;$$

(iii) an uncountable subset S_0 of S such that for every $x_0, y_0 \in S_0$:

$$\liminf_{k\to\infty} \|f^k(x_0) - f^k(y_0)\| = 0.$$

An l-ball is a closed ball of finite radius in $\mathbf{R}^l$ in terms of the euclidean distance on $\mathbf{R}^l$. Such a ball of radius r centred on a point $z_0 \in \mathbf{R}^l$ will be denoted by $B^l(z_0; r)$. A mapping f: $\mathbf{R}^n \to \mathbf{R}^n$ is *expanding* on a set $A \subset \mathbf{R}^n$ if there exists a constant $\lambda > 1$ such that

$$\lambda\|x - y\| \leqslant \|f(x) - f(y)\| \tag{2.2}$$

for all $x, y \in A$. Note that such a mapping is one-to-one on A.

The following two lemmas will be needed in the proof of the theorem below. The proof of the first lemma is straightforward and will be omitted. A proof of the second lemma can be found in Diamond [2].

LEMMA 1. *Let f: $\mathbf{R}^n \to \mathbf{R}^n$ be a continuous mapping which is one-to-one on a compact subset $K \subset \mathbf{R}^n$. Then there exists a continuous mapping g: $f(K) \to K$ such that $g(f(x)) = x$ for all $x \in K$.*

The mapping g in Lemma 1 is a continuous inverse of mapping f on the compact set K. It will be denoted by f_K^{-1} in the sequel.

LEMMA 2. *Let f: $\mathbf{R}^n \to \mathbf{R}^n$ be a continuous mapping and let $\{K_i\}_{i=0}^{\infty}$ be a sequence of compact sets in $\mathbf{R}^n$ such that $K_{i+1} \subseteq f(K_i)$ for $i = 0, 1, 2, \ldots$. Then there exists a nonempty compact set $K \subseteq K_0$ such that $f^i(x_0) \in K_i$ for all $x_0 \in K$ and all $i \geqslant 0$.*

The following theorem is the principal result of this paper.

THEOREM. *Let f: $\mathbf{R}^n \to \mathbf{R}^n$ be a continuous mapping and suppose that there exist nonempty compact sets A and B, and integers $1 \leqslant l \leqslant n$ and $n_1, n_2 \geqslant 1$ such that:*

(1) *A is homeomorphic to an l-cell;*

(2) $A \subseteq f(A)$;

(3) *f is expanding on A;*

(4) $B \subseteq A$;

(5) $f^{n_1}(B) \cap A = \varnothing$;

(6) $A \subseteq f^{n_1+n_2}(B)$;

(7) *$f^{n_1+n_2}$ is one-to-one on B.*

Then difference equation (2.1) *defined in terms of mapping f is chaotic.*

PROOF. The proof here is similar to that used by Marotto in [5], except Lemma 1 is used instead of the inverse mapping theorem and the Brouwer fixed point theorem is used on homeomorphisms of l-balls rather than n-balls.

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From the continuity of f and (6) there exists a nonempty, compact subset $C \subseteq B$ such that $A = f^{n_1+n_2}(C)$. By (7) $f^{n_1+n_2}$ is one-to-one on C, so by Lemma 1 there exists a continuous function $g: A \to C$ such that $g(f^{n_1+n_2}(x)) = x$ for all $x \in C$. Note that by (5) $f^{n_1}(C) \cap A = \varnothing$.

From (3) f is one-to-one on A, so by Lemma 1 f has a continuous inverse f_A^{-1}: $f(A) \to A$. By (2) $C \subset A \subseteq f(A)$, from which it follows that $f_A^{-k}(C) \subset A$ for all $k \geqslant 0$.

For each $k \geqslant 0$ the mapping $f_A^{-k} \circ g: A \to A$ is a continuous mapping from a homeomorph of an l-ball into itself, so by the Brouwer fixed point theorem there exists a point $y_k \in A$ such that $f_A^{-k}(g(y_k)) = y_k$. In fact $y_k \in f^{-k}(C)$ and so $f^{n_1+k}(y_k) = f^{n_1+k}(f_A^{-k}(g(y_k))) = f^{n_1}(g(y_k)) \in f^{n_1}(C)$ as $g(y_k) \in C$. Hence $f^{n_1+n_k}(y_k) \notin A$ as $f^{n_1}(C) \cap A = \varnothing$. Also $f^{n_1+n_2+k}(y_k) = f^{n_1+n_2}(g(y_k)) = y_k$.

Now for $k \geqslant n_1 + n_2$ the point y_k is a periodic point of period $p = n_1 + n_2 + k$. To see this note that p cannot be less than or equal to k because $f^j(y_k) \in f_A^{-k+j}(C) \subset A$ for $1 \leqslant j \leqslant k$ and then the whole cycle would belong to A in contradiction to the fact that $f^{n_1+k}(y_k) \notin A$. Also p cannot lie between k and $n_1 + n_2 + k$ when $k \geqslant n_1 + n_2$ because $f^{n_1+n_2+k}(y_k) = y_k$ and so p would have to divide $n_1 + n_2 + k$ exactly, which is impossible when $k > n_1 + n_2$.

Hence difference equation (2.1) has a periodic point of period p for each $p \geqslant N = 2(n_1 + n_2)$.

Let $D = f^{n_1}(C)$ and $h = f^N$. Then $A \cap D = \varnothing$

$$h(D) = f^N(D) = f^{2n_1+n_2}(f^{n_2}(D)) = f^{2n_1+n_2}(A) \supseteq A \tag{2.3}$$

in view of (2) and the definition of C. Also

$$h(A) = f^N(A) \supseteq A \tag{2.4}$$

by (2) and

$$h(A) = f^N(A) \supseteq f^{2(n_1+n_2)}\left(f_A^{-n_1-2n_2}(C)\right) = f^{n_1}(C) = D \tag{2.5}$$

as $f_A^{-n_1-2n_2}(C) \subset A$,. Moreover as A and D are nonempty, disjoint compact sets it follows that

$$\inf\{\|x - y\|; x \in A, y \in D\} > 0. \tag{2.6}$$

The existence of a scrambled set S then follows exactly as in Marotto's proof [5; page 208] or in Li and Yorke [4]. It will be briefly outlined here for completeness.

Let $\mathbf{E}$ be the set of sequences $\mathcal{E} = \{E_k\}_{k=1}^{\infty}$ where E_k is either A or D, and $E_{k+1} = E_{k+2} = A$ if $E_k = D$. Let $r(\mathcal{E}, k)$ be the number of sets E_j equal to D for $1 \leqslant j \leqslant k$ and for each $\eta \in (0, 1)$ choose $\mathcal{E}^{\eta} = \{E_k^{\eta}\}_{k=1}^{\infty}$ to be a sequence in $\mathbf{E}$ satisfying

$$\lim_{k\to\infty} \frac{r(\mathcal{E}^{\eta}, k^2)}{k} = \eta.$$

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Let $\mathbf{F} = \{\mathcal{E}^\eta;\ \eta \in (0, 1)\} \subset \mathbf{E}$. Then $\mathbf{F}$ is uncountable. Also from (2.3)–(2.5) $h(E_k^\eta) \supseteq E_{k+1}^\eta$ and so by Lemma 2 for each $\mathcal{E}^\eta \in \mathbf{F}$ there is a point $x_\eta \in A \cup D$ with $h^k(x_\eta) \in E_k^\eta$ for all $k \geqslant 1$. Let $S_h = \{h^k(x_\eta);\ k \geqslant 0$ and $\mathcal{E}^\eta \in \mathbf{F}\}$. Then $h(S_h) \subset S_h$, S_h contains no periodic points of h, and there exists an infinite number of k's such that $h^k(x) \in A$ and $h^k(y) \in D$ for any $x, y \in S_h$ with $x \neq y$. Hence from (2.6) for any $x, y \in S_h$ with $x \neq y$

$$L_1 = \limsup_{k \to \infty} \|h^k(x) - h^k(y)\| > 0.$$

Thus letting $S = \{f^k(x);\ x \in S_h$ and $k \geqslant 0\}$ it follows that $f(S) \subset S$, S contains no periodic points of f and for any $x, y \in S$ with $x \neq y$

$$\limsup_{k \to \infty} \|f^k(x) - f^k(y)\| \geqslant L_1 > 0.$$

This proves that the set S has properties (ii)a and (ii)b of a scrambled set. The remaining property (ii)c can be proved similarly. See Li and Yorke [4] for further details.

It remains now to establish the existence of an uncountable subset S_0 of the scrambled set S with the properties listed in part (iii) of the definition of chaotic behaviour. In contrast with Marotto's proof this is the first place where assumption (3) that f is expanding on A is required. Until now all that has been required is that f is one-to-one on A. From this, (2) and Lemma 1 follows the existence of a continuous inverse f_A^{-1}: $A \to A$. Hence by the Brouwer fixed point theorem there exists a point $a \in A$ such that $f_A^{-1}(a) = a$, or equivalently $f(a) = a$.

Now because f is expanding on A it follows that f_A^{-1} is contracting A, that is

$$\|f_A^{-1}(x) - f_A^{-1}(y)\| \leqslant \lambda^{-1}\|x - y\|$$

for all $x, y \in A$ where $\lambda > 1$ is the coefficient of expansion of f on A. Hence for any $k \geqslant 1$ and all $x, y \in A$

$$\|f_A^{-k}(x) - f_A^{-k}(y)\| \leqslant \lambda^{-k}\|x - y\|,$$

and in particular for any $x \in C \subset A$ and for $y = a$

$$\|f_A^{-k}(x) - a\| \leqslant \lambda^{-k}\|x - a\|, \tag{2.7}$$

so $f_A^{-k}(x) \to a$ as $k \to \infty$ for all $x \in C$. Consequently for any $\varepsilon > 0$ there exists an integer $j = j(x, \varepsilon)$ such that $f_A^{-j}(x) \in A \cap B^n(a; \varepsilon)$. Then by continuity there exists a $\delta = \delta(x, \varepsilon) > 0$ such that $f_A^{-1}(A \cap \operatorname{int} B^n(x; \delta)) \subset A \cap B^n(a; \varepsilon)$. Now the collection $\mathcal{C} = \{\operatorname{int} B^n(x; \delta);\ x \in C\}$ constitutes an open cover of the compact set C, so there exists a finite sub-collection $\mathcal{C}_0 = \{\operatorname{int} B^n(x_i; \delta_i);\ i = 1, 2, \ldots, L\}$ which also covers C. Let $T = T(\varepsilon) = \max\{j(x_i; \varepsilon);\ i = 1, 2, \ldots, L\}$. Then $f_A^{-T}(x) \in B^n(a; \varepsilon) \cap A$ for all $x \in C$ and so by (2.7) $f_A^{-k}(C) \subset B^n(a; \varepsilon) \cap A$ for all $k \geqslant T(\varepsilon)$.

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Let $H_k = h_A^{-k}(C)$ for all $k \geqslant 0$ where h_A^{-1} is a continuous inverse of $h = f^N$ on A. Then for any $\varepsilon > 0$ there exists a $J = J(\varepsilon)$ such that $\|x - a\| < \varepsilon/2$ for all $x \in H_k$ and all $k > J$.

The remainder of the proof parallels that in Marotto [5; page 208] and in Li and Yorke [4]. The sequences $\mathcal{E}^\eta = \{E_k^\eta\}_{k=1}^\infty \in \mathbf{E}$ will be further restricted as follows: if $E_k^\eta = D$ then $k = m^2$ for some integer m and if $E_k^\eta = D$ for both $k = m^2$ and $k = (m+1)^2$ then $E_k^\eta = H_{2m-j}$ for $k = m^2 + j$ for $j = 1, 2, \ldots, 2m$. Finally for the remaining k's, $E_k^\eta = A$. Now these sequences still satisfy $h(E_k^\eta) \supset E_{k+1}^\eta$, so by Lemma 2 there exists a point x_η with $h^k(x_\eta) \in E_k^\eta$ for all $k \geqslant 0$. Let $S_0 = \{x_\eta \colon \eta \in (\frac{4}{5}, 1)\}$. Then S_0 is uncountable, $S_0 \subset S_h \subset S$ and for any $s, t \in (\frac{4}{5}, 1)$ there exist infinitely many m's such that $h^k(x_s) \in E_k^s = H_{2m-1}$ and $h^k(x_t) \in E_k^t = H_{2ml-1}$ where $k = m^2 + 1$. But from above, given any $\varepsilon > 0$, $\|x - a\| < \varepsilon/2$ for all $x \in H_{2m-1}$ provided m is sufficiently large. Hence for any $\varepsilon > 0$ there exists an integer m such that $\|h^k(x_s) - h^k(x_t)\| < \varepsilon$ where $k = m^2 + 1$. As $\varepsilon > 0$ is arbitrary it follows that

$$L_2 = \liminf_{k \to \infty} \|h^k(x_s) - h^k(x_t)\| = 0.$$

Thus for any $x, y \in S_0$

$$\liminf_{k \to \infty} \|h^k(x_s) - h^k(x_t)\| \leqslant L_2 = 0.$$

This completes the proof of the theorem.

For one-dimensional difference equations conditions (3) and (7) of the theorem are superfluous as the intermediate value theorem can be used to show the existence of periodic points. Without these two conditions, the theorem then contains the sufficient conditions of Barna [1] and Sharkovsky [6] as special cases. Indeed as can be seen from Sharkovsky [6; Lemma 1] the remaining conditions are also necessary conditions for chaotic behaviour.

3. Examples

Two examples are given in this section to illustrate the application of the preceding theorem. The first example is a one-dimensional difference equation with a snap-back repellor. It forms one of the components of the second example, which is a two-dimensional twisted-horseshoe difference equation considered by Guckenheimer, Oster and Ipaktchi [3]. This second example has a saddle point.

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EXAMPLE 1. Consider the difference equation on the unit interval I which is defined in terms of the continuous mapping f where

$$f(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \\ 2 - 2x & \text{for } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

This mapping f maps I into itself and has two fixed points 0 and $\frac{2}{3}$, both of which are easily seen to be snap-back repellors.

The conditions of the theorem are satisfied by $A = [\frac{9}{16}, \frac{7}{8}]$, $B = [\frac{3}{4}, \frac{7}{8}]$, $n = l = 1$ and $n_1 = n_2 = 1$. To see this note that

$$f(A) = \left[\frac{1}{4}, \frac{7}{8}\right], \qquad f(b) = \left[\frac{1}{4}, \frac{1}{2}\right], \qquad f^2(B) = \left[\frac{1}{2}, 1\right]$$

so

$$f(A) \supset A, \qquad f(B) \cap A = \varnothing \quad \text{and} \quad f^2(B) \supset A.$$

Also f is expanding on A because for $x, y \in A$

$$|f(x) - f(y)| = |(2 - 2x) - (2 - 2y)| = 2|x - y|$$

and f^2 is one-to-one on B because for all $x \in B$

$$f^2(x) = 2(2 - 2x) = 4 - 4x.$$

Hence this difference equation exhibits chaotic behaviour.

EXAMPLE 2. Consider the difference equation on the unit square I^2 in $\mathbf{R}^2$ which is defined in terms of the continuous mapping $f = (f_1, f_2)$ where

$$f_1(x, y) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \\ 2 - 2x & \text{for } \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

and

$$f_2(x, y) = \frac{x}{2} + \frac{y}{10} + \frac{1}{4}.$$

This mapping describes a twisted-horseshoe on I^2 and has been investigated in detail by Guckenheimer, Oster and Ipaktchi [3]. It has a fixed point $(\bar{x}, \bar{y}) = (\frac{2}{3}, \frac{35}{54})$, which is a saddle point with eigenvalues -2 and $\frac{1}{10}$. Consequently Marotto's snap-back repellor theorem cannot be used here, but the preceding theorem can.

Let L_1 be the line $90x + 378y = 305$ and L_2 the line $90x - 378y = -125$. Then $(\bar{x}, \bar{y}) \in L_1$. Also let

$$A = \left\{(x, y) \in L_1;\ \frac{9}{16} \leqslant x \leqslant \frac{7}{8}\right\} \quad \text{and} \quad B = \left\{(x, y) \in L_1;\ \frac{3}{4} \leqslant x \leqslant \frac{7}{8}\right\}.$$

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Then

$$f(A) = \left\{(x,y) \in L_1;\ \frac{1}{4} \leqslant x \leqslant \frac{7}{8}\right\},\quad f(B) = \left\{(x,y) \in L_1;\ \frac{1}{4} < x < \frac{1}{2}\right\},$$

$$f^2(B) = \left\{(x,y) \in L_2;\ \frac{1}{2} < x < 1\right\},\quad f^3(B) \supset L_1 \cap I^2.$$

Hence $f(A) \supset A$, $f(B) \cap A = \varnothing$ and $f^3(B) \supset A$, so conditions (2), (4), (5) and (6) of the theorem are satisfied with $n_1 = 1$ and $n_2 = 2$. Also A is homeomorphic to a 1-ball and f is expanding on A because for $(x, y) \in A$

$$f_1(x,y) = 2 - 2x, \qquad f_2(x,y) = \frac{35}{18} - 2y$$

so for any two points $(x', y'), (x'', y'') \in A$

$$\|f(x',y') - f(x'',y'')\| = 2\|(x',y') - (x'',y'')\|.$$

Finally for all $(x, y) \in B$

$$f_1^3(x,y) = 2 - 2(2(2 - 2x)) = 8 - 8x$$

and

$$f_2^3(x,y) = \frac{381}{200}x + \frac{1}{1000}y - \frac{249}{400}$$

which gives the nonsingular Jacobian matrix

$$\begin{bmatrix} -8 & 0 \\ \dfrac{381}{200} & \dfrac{1}{1000} \end{bmatrix}.$$

Hence f^3 is one-to-one on B.

All of the conditions of the theorem are thus satisfied, so this twisted-horseshoe difference equation behaves chaotically.

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