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Chaotic Motion of Wheels

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Summary

Trolleys have wheels which can choose the direction of their rolling. Studying the motion of a wheel like this, we can often find periodic motions ("shimmy") or even chaotic ones. It has also been experienced that the chaotic motions sometimes disappear quite unexpectedly. A strongly simplified model of these systems is analysed in the paper by means of the methods of bifurcation theory. Analytical and numerical results are shown to characterize the system, including simulation results. Similar behaviour can be found in more complicated systems as well, like the trailers or the nose-gears of aeroplanes. The development of the so-called transient chaotic motion is explained in these systems.

1. Introduction

The special literature has investigated the problem of shimmying wheels for several years and a great number of mechanical models have been developed. The central problem of these mechanical models is the mathematical description of the constraining force at the contact point of the wheel and the ground. The application of the so-called creep-force provides the most realistic models and it is widely used in the literature (see [1,2] or also [3] for more details).

However, the present research concentrates on the topological description of possible stable and unstable chaotic behaviour in the motion of a wheel with towed axis. The understanding of this behaviour needs the simplest model (see e.g. p. 22 in [3]) which enables us to describe the system in low-dimensional phase spaces by the geometry of the trajectories. This has



Figure 1: Mechanical model

resulted in the application of the classical Coulomb friction force in the model which, of course, have a number of well-known disadvantages and makes the model quantitatively unrealistic with respect to some of the parameters (like the speed of towing) [3].

2. Model

The model in question, possibly the simplest one, is presented in Fig. 1. The system has n = 3 degrees of freedom if the geometrical constraints are considered only. The corresponding three general coordinates are q (pin position), ϑ (bar angle) and φ (wheel rotation angle).

The parameters are as follows. Body 1 is assumed to be massless, i.e. $M_1 \approx 0$ while M_2 and M_3 denote the masses of the homogeneous bar (body 2) and disc (body 3, i.e. the wheel). There are two geometrical parameters l and R which are the length of the bar and the radius of the wheel, respectively. Body 1 is supported by linear springs having an overall stiffness k. The velocity v is constant which means that the geometrical constraints are non-stationary in the system.

The cases of rolling and slipping wheel are considered separately by means of the Coulomb friction as explained in the *Introduction*. When the friction force is great enough to provide rolling then a kinematical constraint is considered in the form

$$|\mathbf{v}_P| = \mathbf{0}$$

where P denotes the ideal contact point of the wheel and the ground. This means that the velocity of this contact point is zero and it results two first order scalar differential equations with respect to the general coordinates. If the Appell-Gibbs equations are applied for this anholonomic rheonom system in the same way as it is shown in [4], the following 4-dimensional system of ordinary differential equations yields:

$$\begin{split} \dot{\vartheta} &= \nu \,,\\ \dot{\vartheta} &= \nu \,,\\ \dot{\nu} &= -\frac{\frac{v}{l} \left(\frac{1}{\cos^2 \vartheta} - \frac{1}{2} + \frac{3M_3}{2M_2} \tan^2 \vartheta\right) \nu + \frac{k}{lM_2} q + \left(1 + \frac{3M_3}{2M_2}\right) \frac{\tan \vartheta}{\cos \vartheta} \nu^2}{\left(\frac{1}{3} + \tan^2 \vartheta\right) \cos \vartheta + \frac{M_3}{4M_2} \left(\frac{R^2}{l^2} \cos \vartheta + 6 \tan^2 \vartheta \cos \vartheta\right)} \,,\\ \dot{q} &= v \tan \vartheta + \frac{\nu l}{\cos \vartheta} \,,\\ \dot{\varphi} &= \frac{v + \nu l \sin \vartheta}{R \cos \vartheta} \,, \end{split}$$
(1)

where ν stands for the angular velocity of the bar.

Since the angle φ is a cyclic coordinate, it does not appear on the righthand-side of the first three equations (1a-c). Thus, the dynamics of rolling can be described uniquely in the three dimensional phase-space of the coordinates ϑ , ν and q.

The dynamics of the slipping wheel can be given by Lagrange's equations of the second kind since there are no kinematical constraints in this case. The simplest presentation of these equations is the following system of three second order ordinary differential equations:

$$\begin{pmatrix} \left(\frac{1}{3}M_{2} + (1 + \frac{R^{2}}{4t^{2}})M_{3}\right)l^{2} & -(\frac{1}{2}M_{2} + M_{3})l\cos\vartheta & 0\\ -(\frac{1}{2}M_{2} + M_{3})l\cos\vartheta & M_{2} + M_{3} & 0\\ 0 & 0 & \frac{1}{2}M_{3}R^{2} \end{pmatrix} \begin{pmatrix} \ddot{\vartheta}\\ \ddot{q}\\ \ddot{\varphi} \end{pmatrix} = \\ \begin{pmatrix} \frac{v_{Px}}{\sqrt{v_{Px}^{2} + v_{Py}^{2}}} (\frac{1}{2}M_{2} + M_{3})l\mu_{s}g\\ -(\frac{1}{2}M_{2} + M_{3})l\dot{\vartheta}^{2}\sin\vartheta - kq - \frac{v_{Px}\sin\vartheta + v_{Py}\cos\vartheta}{\sqrt{v_{Px}^{2} + v_{Py}^{2}}} (\frac{1}{2}M_{2} + M_{3})\mu_{s}g\\ \frac{v_{Py}}{\sqrt{v_{Px}^{2} + v_{Py}^{2}}} (\frac{1}{2}M_{2} + M_{3})R\mu_{s}g \end{pmatrix} .$$
(2)

In these equations μ_s denotes the constant coefficient of Coulomb friction when the wheel is slipping and g stands for the gravitational acceleration. The velocity coordinates of the point P of the wheel which is in contact with the ground should be substituted into the right-hand-side of Equ.(2) in the following form:

$$\mathbf{v}_P = \begin{pmatrix} v_{Px} \\ v_{Py} \\ v_{Pz} \end{pmatrix} = \begin{pmatrix} v\cos\vartheta + \dot{q}\sin\vartheta - R\dot{\varphi} \\ -v\sin\vartheta + \dot{q}\cos\vartheta - l\dot{\vartheta} \\ 0 \end{pmatrix},$$

and this completes the differential equations related to the dynamics of the slipping wheel. This system can be represented in the 6-dimensional phase-space of the coordinates ϑ , $\dot{\vartheta}$, q, \dot{q} , φ and $\dot{\varphi}$.

There is still one question left: when will the dynamics of rolling change to the dynamics of slipping, and when will this happen in the other way round? If $\mathbf{Q}_P = \operatorname{col}(Q_{Px} \ Q_{Py} \ Q_{Pz})$ denotes the constraining force at the point P of the wheel then the condition of rolling is expressed by the following inequality:

$$\sqrt{Q_{Px}^2 + Q_{Py}^2} \le \mu_r Q_{Pz} = \mu_r (\frac{1}{2}M_2 + M_3)g, \qquad (3)$$

where $\mu_r (\geq \mu_s)$ is the coefficient of Coulomb friction in case of rolling. The friction force $(Q_{Px} \ Q_{Py} \ 0)$ can, of course, be expressed by the coordinates $\vartheta, \nu = \dot{\vartheta}$ and q if equations (1a-c) are also used. Without going into the complicated details of the transformation of condition (3) into the 3-dimensional phase-space of the dynamics of rolling, we present the following inequality which is, in practice, a good approximation of (3):

$$\frac{1}{4}M_3^2l^2\nu^4 + v^2(M_3 + \frac{3M_3R^2 + M_2l^2}{3M_3R^2 + 4M_2l^2}M_2)^2\nu^2 - \mu_r^2(M_3 + \frac{1}{2}M_2)^2g^2 \le 0.$$
(4)

This is a condition for $\nu = \dot{\vartheta}$ only, which can be given analytically by means of the positive zero ν^2 of the polynomial on the left hand side in (4). Note that the condition of rolling does depend on ϑ and q as well, but their effect is negligible as compared to that of the angular velocity of the bar.

The condition of switching to the dynamics of rolling from the dynamics of slipping is much simpler. Because of the strong dissipation due to Coulomb friction, the speed of the contact point P of the wheel decreases during slipping and becomes zero. If the condition of rolling is also satisfied at this instant, the wheel rolls again. This can easily be followed when Equ.(2) is solved numerically.

3. Dynamics of Rolling

The equations (1a-c) can clearly be analysed with the help of the Hopf Bifurcation Theorem [5]. It is the method presented in [4] which has been followed during the analysis of Equ.(1). The main steps are as follows.

The stability of the trivial solution of (1a-c) can be investigated by means of the variational system

$$\begin{pmatrix} \dot{\vartheta} \\ \dot{\nu} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{M_2 v}{(\frac{2}{3}M_2 + \frac{1}{2}M_3\frac{R^2}{l^2})l} & -\frac{k}{(\frac{1}{3}M_2 + \frac{1}{4}M_3\frac{R^2}{l^2})l} \\ v & l & 0 \end{pmatrix} \begin{pmatrix} \vartheta \\ \nu \\ q \end{pmatrix} \,.$$

The Routh-Hurwitz criterion applied to the coefficient matrix results in the following condition of the exponential asymptotical stability of zero solution:

$$M_3 < M_{3cr} = \frac{2}{3} M_2 \frac{l^2}{R^2} \,. \tag{5}$$

At the critical value of the parameter M_3 the characteristic roots $\lambda_{1,2,3}$ are given by the formulae

$$M_3 = M_{3cr} \quad \Rightarrow \quad \lambda_{1,2} = \pm i\omega \,, \ \omega = \sqrt{\frac{2k}{M_2}} \,; \quad \lambda_3 = -\frac{v}{l} \,.$$
 (6)

As a consequence, Hopf bifurcation can be expected. The behaviour of the system can analytically be approximated for those parameters close to the critical ones in (5). This approximation is based on the truncated power series of the non-linear terms in (1) at the critical parameters which results the system (i) = (i)

$$\begin{pmatrix} \vartheta \\ \dot{\nu} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{v}{l} & -\frac{2k}{M_2 l} \\ v & l & 0 \end{pmatrix} \begin{pmatrix} \vartheta \\ \nu \\ q \end{pmatrix} + \\ \begin{pmatrix} -\frac{v}{2l} \vartheta^2 \nu + \frac{3k}{M_2 l} (1 + \frac{4l^2}{3R^2}) \vartheta^2 q - 2(1 + \frac{l^2}{R^2}) \vartheta \nu^2 \\ \frac{v}{3} \vartheta^3 + \frac{l}{2} \vartheta^2 \nu \end{pmatrix} .$$

The plane spanned by the eigenvectors

$$\mathbf{s}_1 = \begin{pmatrix} \frac{\omega^2 l}{v^2 + \omega^2 l^2} \\ -\frac{\omega^2 v}{v^2 + \omega^2 l^2} \\ 1 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} \frac{\omega v}{v^2 + \omega^2 l^2} \\ -\frac{\omega^3 l}{v^2 + \omega^2 l^2} \\ 0 \end{pmatrix}$$

related to $\lambda_{1,2} = \pm i\omega$ is tangent to the attractive centre manifold at the origin. Using also the third eigenvector $\mathbf{s}_3 = \operatorname{col}(\frac{l}{\omega} - \frac{v}{\omega} \ 0)$ and the new variable $\mathbf{x} \in \mathbf{R}^3$ defined by

$$\begin{pmatrix} \vartheta \\ \nu \\ q \end{pmatrix} = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

we get the so-called Poincaré normal form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & -\frac{v}{l} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \sum_{\substack{j+k=3 \\ j,k\geq 0}} a_{jk} x_1^j x_2^k + \dots \\ \sum_{\substack{j+k=3 \\ j,k\geq 0}} b_{jk} x_1^j x_2^k + \dots \end{pmatrix} .$$

Only the coefficients

$$a_{30} = -\frac{v\omega^6 l^3}{6(v^2 + \omega^2 l^2)^3}, \quad a_{12} = \frac{v^3\omega^4 l + 2v\omega^6 l^3}{2(v^2 + \omega^2 l^2)^3},$$

$$b_{21} = \frac{v\omega^4 l}{(v^2 + \omega^2 l^2)^3} (2v^2 + \frac{13}{2}\omega^2 l^2 + 2\frac{l^2}{R^2}(v^2 + 4\omega^2 l^2)),$$

$$b_{03} = -\frac{v\omega^4 l}{(v^2 + \omega^2 l^2)^3} (\frac{1}{2}v^2 + 2\omega^2 l^2 + 2\frac{l^2}{R^2}\omega^2 l^2)$$

are needed for the calculation of the parameter

$$\delta = \frac{1}{8}(3a_{30} + a_{12} + b_{21} + 3b_{03}) = \frac{v\omega^4 l}{8(v^2 + \omega^2 l^2)^2}(1 + 2\frac{l^2}{R^2}) > 0.$$

As shown in [5], the positiveness of δ proves the existence of a subcritical Hopf bifurcation, that is an unstable periodic motion exists around the stable stationary motion when $M_3 < M_{3cr}$. If we take the value of ω from (6) then

$$\delta = \frac{k^2 v l}{2(M_2 v^2 + 2kl^2)^2} (1 + 2\frac{l^2}{R^2}) \,.$$

It is easy to calculate the other important parameter

$$\operatorname{Re} \frac{d\lambda_1}{dM_3}\Big|_{M_3=M_{3cr}} = \frac{R^2 kv}{2lM_2(2kl^2 + M_2v^2)},$$

which is used to estimate the amplitude r of the unstable periodic motion in the plane (x_1, x_2) (see details in [5]):

$$r = \sqrt{-\frac{\operatorname{Re}\lambda'_{M_3}|_{cr}}{\delta}(M_3 - M_{3cr})} \,.$$



Figure 2: Phase-space structure in case of rolling

By means of the transformation matrix $(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_3)$, the unstable periodic solution can be transformed back to the phase-space (ϑ, ν, q) . Since the last row of the transformation matrix is just $(1 \ 0 \ 0)$, the simplest result is obtained for the amplitude A_q with respect to the coordinate q which has the actual form

$$A_q = r = \sqrt{\frac{M_2 v^2 + 2kl^2}{k(1+2\frac{l^2}{R^2})}} (\frac{2}{3} - \frac{M_3 R^2}{M_2 l^2}) \ .$$

The formulae are similar but more complicated for A_{ϑ} and A_{ν} , and the unstable limit cycle can be placed into the phase-space onto the centre manifold.

These analytical results and two trajectories are presented in Fig. 2. The trajectories are obtained from the numerical solutions of Equ.(1) by means of fourth order Runge-Kutta method on an IBM PC. The parameters are fixed as follows:

$$M_2 = 1.5 [kg], M_3 = 3.75 [kg],$$

$$l = 0.2 [m], R = 0.1 [m],$$

$$k = 75 [N/m], v = 1 [m/s]$$
(7)

while

$$M_{3cr} = 4 \, [kg], \ \omega = 10 \, [1/s], \ A_{\vartheta} = 0.10 \, [rad] \approx 6^{\circ}, \ A_q \approx 20 \, [mm]$$

Fig. 2 shows the 3-dimensional phase-space of the dynamics of rolling. The plane presented in this space is a good approximation of the centre manifold, and the cylinder, determined by means of the third eigenvector \mathbf{s}_3 , estimates the domain of attraction of the stable zero solution. The unstable limit cycle is located at the cross section of the plane and the cylinder.

All the solutions approach the attractive plane quickly. The trajectory which starts inside the cylinder (with initial conditions $\vartheta_0 = -0.24 [rad]$, $\dot{\vartheta}_0 = \nu_0 = 0.4 [rad/s]$, $q_0 = 0$) tends to the zero solution then, but the one starting outside the cylinder ($\vartheta_0 = -0.24 [rad]$, $\nu_0 = 0$, $q_0 = 0$) seems to tend to infinity (or at least to some singularities at $\vartheta = \pm \pi/2$).

These simulation results show a good agreement with the analytical approximations. However, it is obvious that the model of rolling cannot describe a physically realistic situation since the vibrations should not increase till infinity. Equ.(1), of course, is valid only where condition (4) is satisfied. In other cases, Equ.(2) describes the system. The next section presents results involving both dynamics.

4. Chaos

Condition (4) of rolling appears in the 3-dimensional phase-space of rolling as two planes parallel to the plane (ϑ, q) . These planes are designated by their shaded corners in *Fig. 3*.

Thus, the wheel rolls till the trajectory runs between these "walls". With the parameters (7) and $\mu_r = 0.12$, the unstable limit cycle is situated within the walls. As a consequence, those trajectories, starting outside the cylinder, will cross the walls as they spiral outwards. When they reach the wall, the wheel starts slipping and the trajectories have to be calculated as the solutions of Equ.(2) with initial conditions given by continuity. The coefficient $\mu_s = 0.06$ is used in (2).

Note that the trajectories related to the dynamics of slipping run in the 6-dimensional phase-space $(\vartheta, \dot{\vartheta}, q, \dot{q}, \varphi, \dot{\varphi})$. In Fig. 3, they are represented with dotted lines by projecting them onto the 3-dimensional space (ϑ, ν, q)



Figure 3: Chaotic behaviour

used in case of rolling. The trajectories do not stay for too long in the 6-dimensional phase-space of the dynamics of slipping since the Coulomb friction causes a strong dissipation. As explained at the end of Section 2, the wheel may roll again when the trajectory crosses the 3-dimensional space (ϑ, ν, q) right between the walls. If the trajectory arrives back in this phase-space of the dynamics of rolling outside the cylinder, it will, sooner or later, leave this space again. There are several switches between the dynamics of rolling and slipping and, as it often happens in systems with similar structures, this may refer to the presence of a chaotic attractor. The numerical simulation does present this chaotic attractor as shown in Fig. 3.

Figure 4(a) shows a one-dimensional discrete mapping. This presents the qualitative structure of an approximative Poincaré mapping when the trajectories in the 3-dimensional phase-space are cut by the half-plane (ϑ_+, q) . Since the centre manifold is very attractive, the trajectories run quite close to it, and two subsequent intersection points (of subscript j and j + 1) can fairly well be represented by their ϑ coordinates only. At the unstable limit cycle, there is an unstable fix point $\vartheta_U (\approx 6^o)$ in the map, while the trivial



Figure 4: Discrete maps with chaos (a) and transient chaos (b)

solution $\vartheta_S = 0$ is stable, of course. Outside the limit cycle, there is a sharp discontinuity at ϑ^* in the map which refers to the change from rolling to slipping. The narrow part of the map on the right side of this discontinuity describes the switch back to the dynamics of rolling. The structure of the map in the upper right corner of Fig. 4(a) shows a typical case when chaotic iteration occurs (see the Lorenz map, p. 98 in [6]). This approximate discrete mapping based on numerical results does not provide a mathematical proof of the existence of chaos here, but makes it very likely and geometrically well interpreted.

This result is quite reasonable since the chaotic dance of the wheels of trolleys can often be experienced. But this chaos is not always attractive, i.e. the strange solution may be unstable for some parameters. This is explained in the last section.

5. Transient Chaos

If $\mu_r = 0.115$ is used when solving the equations (1) and (2) with condition (4), and the trajectory starts outside the cylinder (i.e. outside the domain of attractivity of the trivial solution), the trajectory behaves chaotically only for a limited time. After some switches between the two dynamics to an fro, the trajectory once arrives back to the phase-space of rolling within the cylinder, and after that it gradually approaches the zero solution. Since the chaotic behaviour is temporary, this effect is often referred as transient chaos or preturbulence (see p. 315 in [6]).



Figure 5: Transient chaos

This situation is shown in Fig. 5. Instead of the 3-dimensional phasespace of rolling, the plane of ϑ and $\nu = \dot{\vartheta}$ is used, and instead of the additional 3 dimensions required for the dynamics of slipping, the absolute value of the velocity of the wheel contact point P is given. When the wheel rolls, the trajectory runs in the horizontal plane within the planes of dashed corners. If the trajectory leaves this plane at the wall, the wheel is slipping. After some time, the trajectory suddenly gets "inside" the direct domain of attractivity of the origin and it tends to the zero solution right in the middle of the plane (ϑ, ν) , as represented by the dark disc in Fig. 5 covered by trajectories spiralling inwards.

The same process can be followed in Fig. 4(b) which shows the corresponding one-dimensional discrete map as explained in Section 4. Now, the narrow part of the map on the right side of ϑ^* enables the chaotic-like iteration in the upper right corner to escape and to get inside ϑ_U as indicated by the arrow in the figure.

It is not easy to explain this situation either topologically in the phasespace or physically in the real space. The analysis of the structure of this transient chaos shows that this is an unstable strange solution. Thus, there is only one attractor in this case, and it is the zero solution. Moreover, the domain of attractivity of the trivial solution is not only the cylinder, but "almost" the whole phase space. But the presence of this unstable chaos produces a fractal structure in this space which makes it difficult for the solutions to reach the stable origin. They do reach it sooner or later, but after a "stochastically" varying time of chaotic-like behaviour only.

This transient chaos is also experienced in practice when the chaotic dance of the trolley-wheel suddenly disappears and the motion becomes regular and stationary.

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