

Chapter 2

Jump-Diffusion Models for Asset Pricing in Financial Engineering

S.G. Kou

Department of Industrial Engineering and Operations Research, Columbia University
E-mail: sk75@columbia.edu

Abstract

In this survey we shall focus on the following issues related to jump-diffusion models for asset pricing in financial engineering. (1) The controversy over tailweight of distributions. (2) Identifying a risk-neutral pricing measure by using the rational expectations equilibrium. (3) Using Laplace transforms to pricing options, including European call/put options, path-dependent options, such as barrier and lookback options. (4) Difficulties associated with the partial integro-differential equations related to barrier-crossing problems. (5) Analytical approximations for finite-horizon American options with jump risk. (6) Multivariate jump-diffusion models.

1 Introduction

There is a large literature on jump-diffusion models in finance, including several excellent books, e.g. the books by [Cont and Tankov \(2004\)](#), [Kijima \(2002\)](#). So a natural question is why another survey article is needed. What we attempt to achieve in this survey chapter is to emphasize some points that have not been well addressed in previous surveys. More precisely we shall focus on the following issues.

- (1) The controversy over tailweight of distributions. An empirical motivation for using jump-diffusion models comes from the fact that asset return distributions tend to have heavier tails than those of normal distribution. However, it is not clear how heavy the tail distributions are, as some people favor power-type distributions, others exponential-type distributions. We will stress that, quite surprisingly, it is very difficult to distinguish power-type tails from exponential-type tails from empirical data unless one has extremely large sample size perhaps in the order of tens of thousands or even hundreds of thousands. Therefore,

whether one prefers to use power-type distributions or exponential-type distributions is a subjective issue, which cannot be easily justified empirically. Furthermore, this has significant implications in terms of defining proper risk measures, as it indicates that robust risk measures, such as VaR, are *desirable* for external risk management; see Heyde et al. (2006).

- (2) Identifying a risk-neutral pricing measure by using the rational expectations equilibrium. Since jump-diffusion models lead to incomplete markets, there are many ways to choose the pricing measure; popular methods include mean–variance hedging, local mean–variance hedging, entropy methods, indifference pricing, etc. Here we will use the rational expectations equilibrium, which leads to a simple transform from the original physical probability to a risk-neutral probability so that we can price many assets, including zero-coupon bonds, stocks, and derivatives on stocks, simultaneously all in one framework.
- (3) Using Laplace transforms to price options, including European call and put options, path-dependent options, such as barrier options and lookback options. We shall point out that even in the case of European call and put options, Laplace transforms lead to simpler expressions and even faster computations, as direct computations may involve some complicated special functions which may take some time to compute while Laplace transforms do not.
- (4) Difficulties associated with the partial integro-differential equations related to barrier-crossing problems. For example: (i) Due to nonsmoothness, it is difficult to apply Itô formula and Feynman–Kac formula directly. (ii) It is generally difficult to solve the partial integro-differential equations unless the jump sizes have an exponential-type distribution. (iii) Even renewal-type arguments may not lead to a unique solution. However martingale arguments may be helpful in solving the problems.
- (5) Two analytical approximations for finite-horizon American options, which can be computed efficiently and with reasonable accuracy.
- (6) Multivariate jump-diffusion models.

In a survey article, inevitably I will skip some important topics which are beyond the expertise of the author. For example, I will omit numerical solutions for jump-diffusion models; see Cont and Tankov (2004), Cont and Voltchkova (2005) and d’Halluin et al. (2003) on numerical methods for solving partial integro-differential equations, and Feng and Linetsky (2005) and Feng et al. (2004) on how to price path-dependent options numerically via variational methods and extrapolation. Two additional topics omitted are hedging (for a survey, see the book by Cont and Tankov, 2004) and statistical inference and econometric analysis for jump-diffusion models (for a survey, see the book by Singleton, 2006). Due to the page limit, I will also skip various applications of the jump-diffusion models; see the references in Glasserman and Kou (2003) for applications of jump-diffusion models in fixed income derivatives and term

structure models, and [Chen and Kou \(2005\)](#) for applications in credit risk and credit derivatives.

2 Empirical stylized facts

2.1 Are returns normally distributed

Consider the daily closing prices of S&P 500 index (SPX) from Jan 2, 1980 to Dec 31, 2005. We can compute the daily returns of SPX, either using the simple returns or continuously compounded returns. The (one-period) simple return is defined to be $R_t = \{S(t) - S(t - 1)\}/S(t - 1)$ at time t , where $S(t)$ is the asset price. For mathematical convenience, the continuously compounded return (also called log return) at time t , $r_t = \ln \frac{S(t)}{S(t-1)}$, is very often also used, especially in theoretical modeling. The difference between simple and log returns for daily data is quite small, although it could be substantial for monthly and yearly data. The normalized daily simple returns are plotted in [Fig. 1](#), so that the daily simple returns will have mean zero and standard deviation one.

We see big spikes in 1987. In fact the max and min (which all occurred during 1987) are about 7.9967 and -21.1550 standard deviation. The continuously compounded returns show similar features. Note that for a standard normal

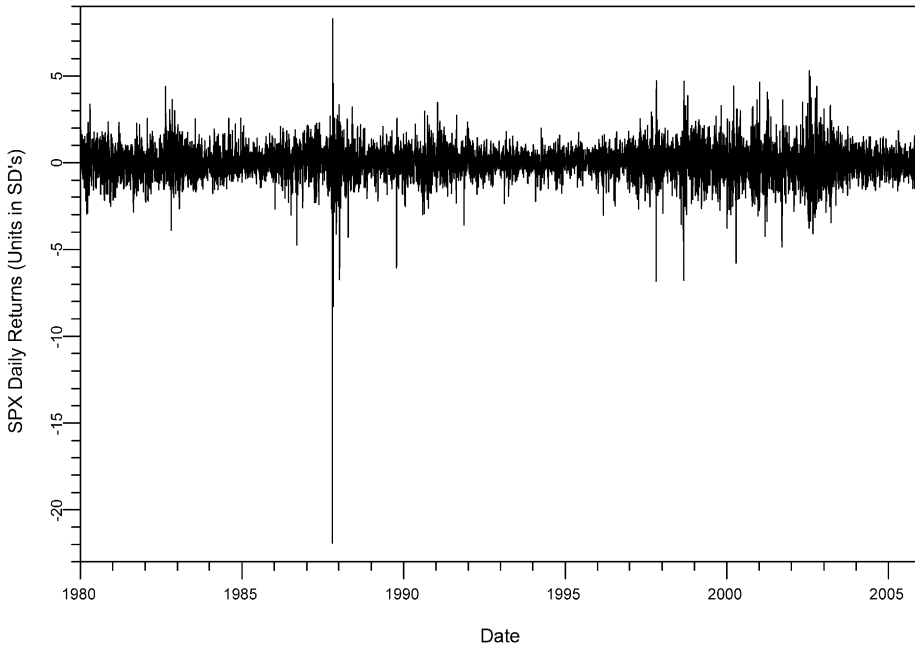


Fig. 1. The normalized daily simple returns of S&P 500 index from Jan 2, 1980 to Dec 31, 2005. The returns have been normalized to have mean zero and standard deviation one.

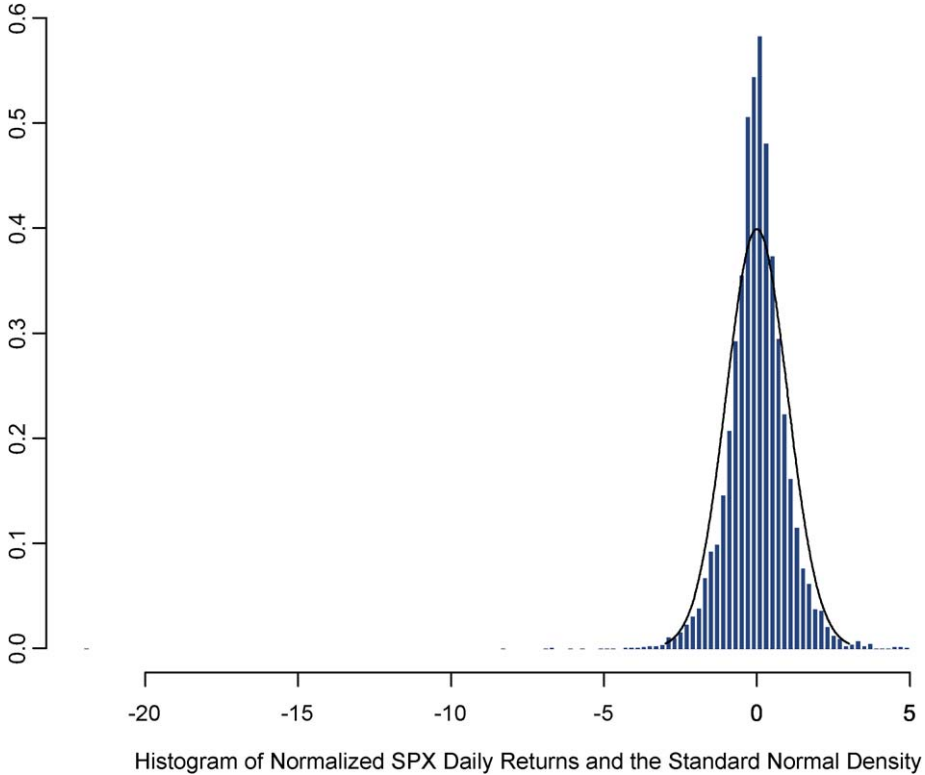


Fig. 2. Comparison of the histogram of the normalized daily returns of S&P 500 index (from Jan 2, 1980 to Dec 31, 2005) and the density of $N(0, 1)$. The feature of a high peak and two heavy tails (i.e. the leptokurtic feature) is quite evident.

random variable Z , $P(Z < -21.1550) \approx 1.4 \times 10^{-107}$; as a comparison, note that the whole universe is believed to have existed for 15 billion years or 5×10^{17} seconds.

Next we plot the histogram of the daily returns of SPX. Figure 2 displays the histogram along with the standard normal density function, which is essentially confined within $(-3, 3)$.

2.1.1 Leptokurtic distributions

Clearly the histogram of SPX displays a high peak and asymmetric heavy tails. This is not only true for SPX, but also for almost all financial asset prices, e.g. US and world wide stock indices, individual stocks, foreign exchange rates, interest rates. In fact it is so evident that a name “leptokurtic distribution” is given, which means the kurtosis of the distribution is large. More precisely, the kurtosis and skewness are defined as $K = E\left(\frac{(X-\mu)^4}{\sigma^4}\right)$, $S = E\left(\frac{(X-\mu)^3}{\sigma^3}\right)$; for the standard normal density $K = 3$. If $K > 3$ then the distribution will be called

leptokurtic and the distribution will have a higher peak and two heavier tails than those of the normal distribution. Examples of leptokurtic distributions include: (1) double exponential distribution with the density given by

$$f(x) = p \cdot \eta_1 e^{-x\eta_1} 1_{\{x>0\}} + (1-p) \cdot \eta_2 e^{x\eta_2} 1_{\{x<0\}}.$$

(2) t -distribution, etc.

To estimate skewness and kurtosis, we shall use

$$\hat{S} = \frac{1}{(n-1)\hat{\sigma}^3} \sum_{i=1}^n (X_i - \bar{X})^3, \quad \hat{K} = \frac{1}{(n-1)\hat{\sigma}^4} \sum_{i=1}^n (X_i - \bar{X})^4$$

as sample skewness and sample kurtosis, where $\hat{\sigma}$ is the sample standard deviation. For the daily returns of the SPX data, the sample kurtosis is about 42.23. The skewness is about -1.73 ; the negative skewness means the return has a heavier left tail than the right tail.

The leptokurtic feature has been observed since 1950's. However classical finance models simply ignore this feature. For example, in the Black-Scholes Brownian motion model, the stock price is modeling as a geometric Brownian motion, $S(t) = S(0)e^{\mu t + \sigma W(t)}$, where the Brownian motion $W(t)$ has a normal distribution with mean 0 and variance t . Here μ is called the drift, which measures the average return, and σ is called the volatility which measures the standard deviation of the return distribution. In this model, the continuous compounded return, $\ln(S(t)/S(0))$, has a normal distribution, which it is not consistent with leptokurtic feature. Many alternative models, e.g. models with jumps and/or stochastic volatility, have been proposed to incorporate the feature, as we will discuss some of them shortly.

2.1.2 Power tails and exponential tails

It is clear that the returns of stocks have two tail distributions heavier than those of normal distribution. However, how heavy the stock tail distributions are is a debatable question. Two main classes proposed in the literature are power-type tails and exponential-type tails. For example, we say that the right tail of a random variable X has a power-type tail if $P(X > x) \approx \frac{c}{x^\alpha}$, $x > 0$, as $x \rightarrow \infty$, and the left tail of X has a power-type tail if $P(X < -x) \approx \frac{c}{x^\alpha}$, $x > 0$, as $x \rightarrow \infty$. Similarly, we say that X has a right exponential-type tail if $P(X > x) \approx ce^{-\alpha x}$, $x > 0$, and a left exponential-type tail if $P(X < -x) \approx ce^{-\alpha x}$, $x < 0$, as $x \rightarrow \infty$.

As pointed out by Kou (2002, p. 1090), one problem with using power-type right tails in modeling return distributions is that the power-type right tails cannot be used in models with continuous compounding. More precisely, suppose that, at time 0, the daily return distribution X has a power-type right tail. Then in models with continuous compounding, the asset price tomorrow $A(\Delta t)$ is given by $A(\Delta t) = A(0)e^X$. Since X has a power-type right tail, it is clear that $E(e^X) = \infty$. Consequently,

$$E(A(\Delta t)) = E(A(0)e^X) = A(0)E(e^X) = \infty.$$

In other words, the asset price tomorrow has an infinite expectation! The price of call option may also be infinite, if under the risk-neutral probability the return has a power-type right distribution. This is because

$$E^*[(S(T) - K)^+] \geq E^*[S(T)] - K = \infty.$$

In particular, these paradoxes hold for any t -distribution with any degrees of freedom which has power tails, as long as one considers financial models with continuous compounding. Therefore, the only relevant models with t -distributed returns outside these paradoxes are models with discretely compounded simple returns. However, in models with discrete compounding analytical solutions are in general impossible.

2.1.3 Difficulties in statistically distinguish power-type tails from exponential-type tails

Another interesting fact is that, for a sample size of 5000 (corresponding to about 20 years of daily data), it may be very difficult to distinguish empirically the exponential-type tails from power-type tails, although it is quite easy to detect the differences between them and the tails of normal density; see [Heyde and Kou \(2004\)](#). A good intuition may be obtained by simply looking at the quantile tables for both standardized Laplace and standardized t -distributions with mean zero and variance one. Recall that a Laplace distribution has a symmetric density $f(x) = \frac{1}{2}e^{-x}I_{[x>0]} + \frac{1}{2}e^xI_{[x<0]}$. The right quantiles for the Laplace and normalized t densities with degrees of freedom from 3 to 7 are given in [Table 1](#).

[Table 1](#) shows that the Laplace distribution may have higher tail probabilities than those of t -distributions with low degrees of freedom, even if asymptotically the Laplace distribution should have lighter tails than those of t -distributions. For example, the 99.9% percentile of the Laplace distribution is actually bigger than that of t -distribution with d.f. 6 and 7! Thus, regardless of the sample size, the Laplace distribution may appear to be heavier tailed than a t -distribution with d.f. 6 or 7, up to the 99.9% percentile. In order to distinguish the distributions it is necessary to use quantiles with very low p values and correspondingly large samples.

If the true quantiles have to be estimated from data, then the problem is even more serious, as confidence intervals need to be considered, resulting

Table 1.
Percentiles of Laplace and t -distributions

Prob.	Laplace	t7	t6	t5	t4	t3
1%	2.77	2.53	2.57	2.61	2.65	2.62
0.1%	4.39	4.04	4.25	4.57	5.07	5.90
0.01%	6.02	5.97	6.55	7.50	9.22	12.82
0.001%	7.65	8.54	9.82	12.04	16.50	27.67

in sample sizes typically in the tens of thousands or even hundreds of thousands necessary to distinguish power-type tails from exponential-type tails; see Heyde and Kou (2004).

2.1.4 Practical implications for risk measures

The difficulties in distinguishing tail distributions also have implications in risk management. For example, a controversy in axiomatic approaches to risk measures is whether one use should Value-at-Risk (or VaR), which is a measure based on quantiles, or the tail conditional expectation. Unlike the tail conditional expectation, VaR does not in general satisfy an axiom of subadditivity (Artzner et al., 1999). However, VaR is more robust against model assumptions and misspecifications, thus making the VaR more suitable to be used for external risk regulations, because VaR can produce more consistent results which are essential for external law enforcement (see Heyde et al., 2006). Furthermore, VaR at a higher quantile (e.g. 97.5%) can also be represented as tail conditional median (e.g. at 95%), thus taking into consideration of the loss beyond the threshold just as the tail conditional mean does. Indeed, VaR is widely used in practice, e.g. in the recent Basel (II) governmental regulation.

It can be shown that VaR also satisfies a different set of axioms based on commonotonic subadditivity (Heyde et al., 2006), which is consistent with both prospect theory in behavior finance and robustness requirement for external law enforcement. Furthermore, the intuition behind subadditivity (which is the theoretical basis for “coherent risk measures” such as tail conditional expectations) that merger reduces risk is not true in general, in particular in presence of the limited liability law. For details, see Heyde et al. (2006).

In short, although one may use various risk measures for internal risk management, robust risk measures, such as VaR, are needed for external risk regulations. In addition, VaR, though simple, is not irrational because it also satisfies a different set of axioms.

2.2 Are stock returns predictable: introduction to the dependent structure of stock returns

To study the question on whether future stock returns can be predicted from the current returns, we can formulate this question mathematically by asking whether returns are correlated in some ways, so that the current returns will provide some information about future returns. For a weakly stationary discrete time series $\{r_t\}$, where the index $t \in (-\infty, \infty)$ can only take integer values (i.e. we have $\dots, r_{-2}, r_{-1}, r_0, r_1, r_2, \dots$), we can define the lag- k autocovariance $\gamma_k = \text{Cov}(r_t, r_{t-k}) = \text{Cov}(r_t, r_{t+k})$; the two covariances are equal due to the definition of the weak stationarity. Similarly, we can define lag- k autocorrelation ρ_k :

$$\begin{aligned}\rho_k &= \text{Cor } r(r_t, r_{t-k}) = \text{Cor } r(r_t, r_{t+k}) = \frac{\text{Cov}(r_t, r_{t-k})}{\sqrt{\text{Var}(r_t) \text{Var}(r_{t-k})}} \\ &= \frac{\gamma_k}{\sqrt{\gamma_0 \gamma_0}} = \frac{\gamma_k}{\gamma_0}.\end{aligned}$$

We can estimate ρ_k by

$$\hat{\rho}_k = \frac{\sum_{t=k+1}^T (r_t - \bar{r})(r_{t-k} - \bar{r})}{\sum_{t=1}^T (r_t - \bar{r})^2}.$$

A plot of $\hat{\rho}_k$, for $k \geq 1$, is called autocorrelation function (or ACF) plot. An autocorrelation plot of the simple daily returns of SPX, normalized to have mean 0 and variance 1, is given in Fig. 3.

Note that the two dotted lines in Fig. 3 indicate the 95% significant levels for autocorrelation. More precisely, if $r_t = \mu + a_t$, where a_t is a sequence of i.i.d. random variables with finite mean and variance, then as the total time period $T \rightarrow \infty$, it can be shown that $\hat{\rho}_k$ is asymptotic normal with mean 0 and variance $1/T$. This is what is plotted in Fig. 3 as the two dotted lines, which are $\pm 1.96/\sqrt{T}$, as a 95% c.i. for the autocorrelation functions in the above ACF plot.

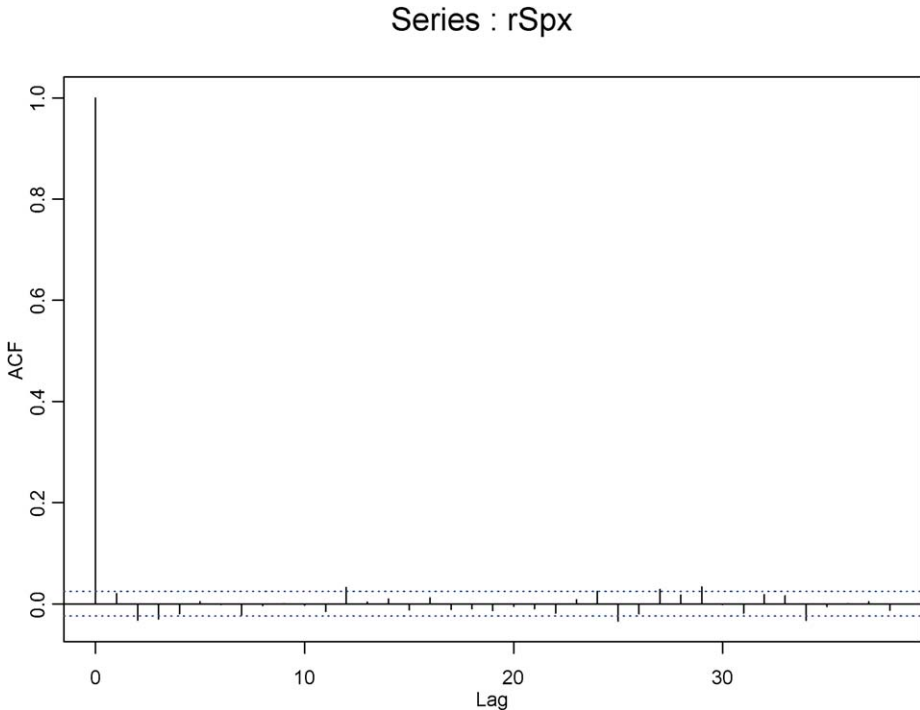


Fig. 3. The ACF plot of the returns of S&P 500 index from Jan 2, 1980 to Dec 31, 2005.

Graphically we can see from the plot that, although the first few autocorrelations significantly exceed the 95% confidence interval, the magnitude of autocorrelations is quite small, only about -0.05 to 0.05 among daily returns; it is even smaller for weekly and monthly returns.

Because of this, many finance models simply ignore the dependent structure, and assume that the stock returns have zero autocorrelations. This is, for example, in the case of Black–Scholes option pricing model, and in the capital asset pricing model, etc. Indeed, most of the classical models assume that the stock prices satisfy “a random walk hypothesis” with independent asset returns. However, starting in 1980’s, researches reveal some fascinating dependent structures among asset returns.

In Figs. 4 and 5 we see the autocorrelations for the absolute values and the squared values of the SPX daily returns are quite large. This suggests that returns distributions are dependent in an interesting way that the volatility of returns (which are related to the squared returns) are correlated, but asset returns themselves have almost no autocorrelation. In the literature this is called “volatility clustering effect.”

This in particular implies that any model for stock returns with independent increments (such as Lévy processes) cannot incorporate the volatility clustering effect. Since jump-diffusion models are special cases of Lévy processes,

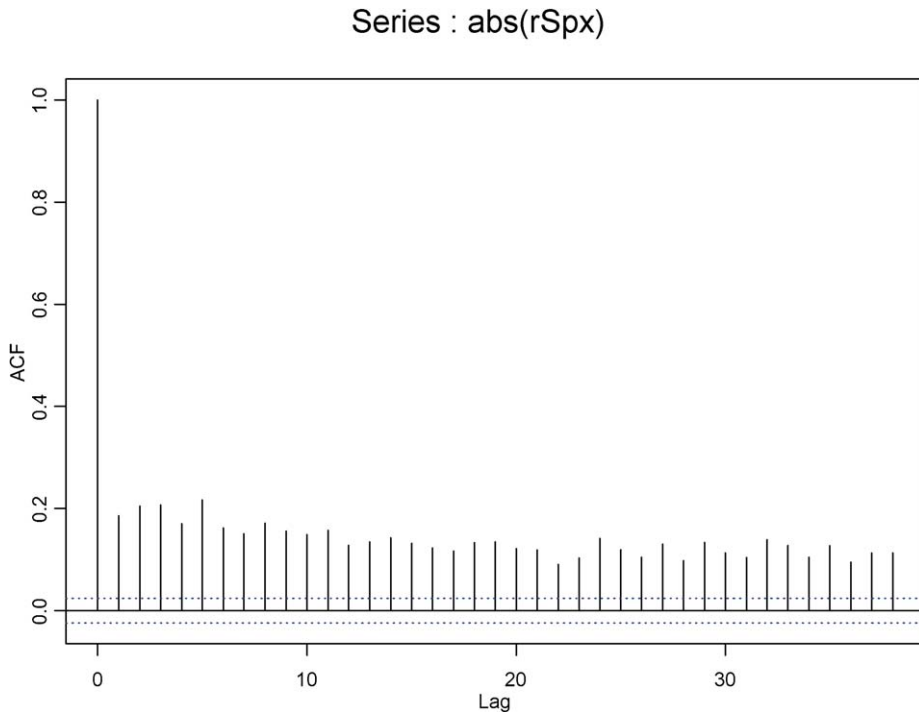


Fig. 4. The ACF plot of the absolute returns of S&P 500 index from Jan 2, 1980 to Dec 31, 2005.

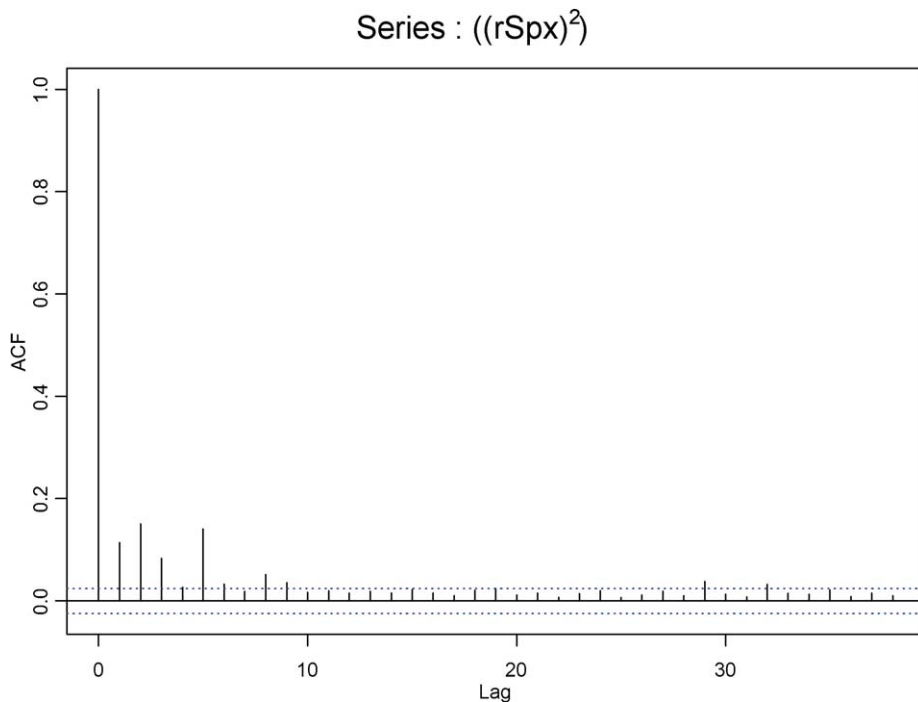


Fig. 5. The ACF plot of the squared returns of S&P 500 index from Jan 2, 1980 to Dec 31, 2005.

they cannot incorporate the volatility clustering effect directly. However, one can combine jump-diffusion processes with other processes (e.g. stochastic volatility) or consider time-changed Lévy processes to incorporate the volatility clustering effect.

2.3 Implied volatility smile

Because in the Black–Scholes formula the call and put option prices are monotone increasing functions of the volatility, we can define an inverse function that maps from a given option price to the volatility parameter, assuming that we know the other parameters in the formula. More precisely, the implied volatility $\sigma(T, K)$ is a parameter associated with a particular strike K and a particular maturity T such that if we use it as the volatility parameter in the Black–Scholes formula for European call and put options, then we should obtain a price that exactly matches the market price of a particular call/put option. In other words $\sigma(T, K)$ is the inverse function of the market option price in terms of volatility.

One immediate question is that whether the above definition is self-consistent. In particular, suppose that one person computes $\sigma(T, K)$ from a call option with maturity T and strike K , and another computes $\sigma(T, K)$ from

a put option with the same maturity T and strike K , will the two people get the same answer? The answer is yes due to the put–call parity, at least in theory. The put–call parity says that no arbitrage implies that the stock price $S(0)$, call price $C(S, K)$, the price $P(S, K)$, and the zero coupon bond price $B(T)$ must satisfy

$$S(0) = C(S, K) - P(S, K) + K \cdot B(T).$$

The relationship is model-free; in other words, no matter what model we use the above put–call parity must hold to prevent arbitrage.

Let $C_{BS}(S, K)$ and $P_{BS}(S, K)$ denote the call and put prices given by the Black–Scholes formula based on the same input variable σ . Then $S(0) = C_{BS}(S, K) - P_{BS}(S, K) + K \cdot B(T)$. Similarly, $S(0) = C_M(S, K) - P_M(S, K) + K \cdot B(T)$, where $C_M(S, K)$ and $P_M(S, K)$ denote the market prices of the call and put. Taking the difference between the two equations, we get

$$C_{BS}(S, K) - C_M(S, K) = P_{BS}(S, K) - P_M(S, K). \quad (1)$$

Now suppose we get the implied volatility $\sigma_c(T, K)$ from the market call option and the implied volatility $\sigma_p(T, K)$ from the market put option. By the definition of implied volatility, if we use $\sigma_c(T, K)$ then we must have $C_{BS}(S, K) - C_M(S, K) = 0$, if $\sigma = \sigma_c(T, K)$. By (1), we must have $P_{BS}(S, K) - P_M(S, K) = 0$, if $\sigma = \sigma_c(T, K)$. Since $\sigma_p(T, K)$ is the unique volatility such that $P_{BS}(S, K) - P_M(S, K) = 0$, we must have $\sigma_p(T, K) = \sigma_c(T, K)$. This shows that the implied volatilities from otherwise identical call and put options must be the same. Of course, in practice, we do have bid–ask spreads for options. So the implied volatility will be different depending whether you use a bid price, an ask price or the average of the bid–ask prices. Therefore, the implied volatilities from otherwise identical call and put options may also be somewhat different.

When one uses the implied volatilities from call and put options to price other options not traded in exchanges, effectively we want to do extrapolation from prices of liquidated options to get prices of less liquidated options. Many practitioners think that implied volatilities are better than the historical volatilities for the purpose of option pricing, as historical volatilities may not reflect the current situation. For example, suppose an extreme event happens to the Wall Street, e.g. a financial crisis, a terrorist attack, etc., then it is hard to find similar events in the historical database, thus making historical volatilities unsuitable.

We can calculate implied volatilities from the market prices of options with different strike prices and maturities. If the geometric Brownian motion assumption is correct, then the implied volatilities should be the same for all the options on the same underlying asset. However, empirically options on the same underlying asset but with different strike prices or maturities tend to have different implied volatilities.

In particular, it is widely recognized that if we plot implied volatilities against strike prices, then the implied volatility curve resembles a “smile,”

meaning the implied volatility is a convex curve of the strike price. In addition, the “smile” curve changes for different maturities. While mispricing exists and statistically significant, the implied volatility smile was not economically significant in early tests before the 1987 market crash (e.g. MacBeth and Merville, 1979; Rubinstein, 1985). However, after the 1987 crash, the implied volatility smile becomes economically significant and the performance of the Black–Scholes model deteriorated.

It is worth mentioning that the leptokurtic features under a risk-neutral measure lead to the “volatility smiles” in option prices; and the volatility clustering effect may lead to implied volatility smile across maturities, especially for long maturity options.

3 Motivation for jump-diffusion models

3.1 Alternative models to the Black–Scholes

Many studies have been conducted to modify the Black–Scholes model to explain the above three empirical stylized facts, namely the leptokurtic feature, volatility clustering effect, and implied volatility smile. Below is a list of some of them.

(a) Chaos theory and fractal Brownian motions. In these models, one typically replaces the Brownian motion by a fractal Brownian motion which has dependent increments (rather than independent increments); see, for example, Mandelbrot (1963). However, as Rogers (1997) pointed out these models may lead to arbitrage opportunities.

(b) Generalized hyperbolic models, including log t model and log hyperbolic model, and stable processes. These models replace the normal distribution assumption by some other distributions; see, for example, Barndorff-Nielsen and Shephard (2001), Samorodnitsky and Taqqu (1994), Blattberg and Gonedes (1974).

(c) Models based on Lévy processes; see, for example, Cont and Tankov (2004) and reference therein.

(d) Stochastic volatility and GARCH models; see, for example, Hull and White (1987), Engle (1995), Fouque et al. (2000), Heston (1993). These models are mainly designed to capture the volatility clustering effect. A typical example of these models is

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \mu dt + \sigma(t) dW_1(t), \\ d\sigma(t) &= -\alpha(\sigma(t) - \beta) dt + \gamma\sqrt{\sigma(t)} dW_2(t),\end{aligned}$$

where $W_1(t)$ and $W_2(t)$ are two correlated Brownian motions.

(e) Constant elasticity of variance (CEV) model; see, for example, Cox and Ross (1976) and Davydov and Linetsky (2001). In this model

$$dS(t) = \mu S(t) dt + \sigma(t) S^\alpha(t) dW_1(t), \quad 0 < \alpha \leq 1.$$

(f) Jump-diffusion models proposed by Merton (1976) and Kou (2002).

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)} \prod_{i=1}^{N(t)} e^{Y_i},$$

where $N(t)$ is a Poisson process. In Merton (1976) model, Y has a normal distribution, and in Kou (2002) it has a double exponential distribution. The double exponential distribution enables us to get analytical solutions for many path-dependent options, including barrier and lookback options, and analytical approximations for American options, as we will see later.

(g) A numerical procedure called “implied binomial trees”; see, for example, Derman and Kani (1994) and Dupire (1994).

There are models combining several features, such as stochastic volatility, jumps, and time changes. Below are two examples of them.

(h) Time changed Brownian motions and time changed Lévy processes. In these models, the asset price $S(t)$ is modeled as

$$S(t) = G(M(t)),$$

as G is a either geometric Brownian motion or a Lévy process, and $M(t)$ is a nondecreasing stochastic process modeling the stochastic activity time in the market. The activity process $M(t)$ may link to trading volumes. See, for example, Clark (1973), Madan and Seneta (1990), Madan et al. (1998), Heyde (2000), Carr et al. (2003).

(i) Affine stochastic-volatility and affine jump-diffusion models; see, for example, Duffie et al. (2000), which combines both stochastic volatilities and jump-diffusions.

3.2 Jump-diffusion models

In jump-diffusion models under the physical probability measure \mathbb{P} the asset price, $S(t)$, is modeled as

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right), \quad (2)$$

where $W(t)$ is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ , and $\{V_i\}$ is a sequence of independent identically distributed (i.i.d.) nonnegative random variables. In the model, all sources of randomness, $N(t)$, $W(t)$, and Y 's, are assumed to be independent. Solving the stochastic differential equation (2) gives the dynamics of the asset price:

$$S(t) = S(0) \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\} \prod_{i=1}^{N(t)} V_i. \quad (3)$$

In Merton (1976) model, $Y = \log(V)$ has a normal distribution. In Kou (2002) $Y = \log(V)$ has an asymmetric double exponential distribution with the density

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y < 0\}}, \quad \eta_1 > 1, \eta_2 > 0,$$

where $p, q \geq 0, p+q = 1$, represent the probabilities of upward and downward jumps. The requirement $\eta_1 > 1$ is needed to ensure that $E(V) < \infty$ and $E(S(t)) < \infty$; it essentially means that the average upward jump cannot exceed 100%, which is quite reasonable. For notational simplicity and in order to get analytical solutions for various option pricing problems, the drift μ and the volatility σ are assumed to be constants, and the Brownian motion and jumps are assumed to be one-dimensional. Ramezani and Zeng (2002) independently propose the double exponential jump-diffusion model from an econometric viewpoint as a way of improving the empirical fit of Merton's normal jump-diffusion model to stock price data.

There are two interesting properties of the double exponential distribution that are crucial for the model. First, it has the leptokurtic feature; see Johnson et al. (1995). The leptokurtic feature of the jump size distribution is inherited by the return distribution. Secondly, a unique feature, also inherited from the exponential distribution, of the double exponential distribution is the memoryless property. This special property explains why the closed-form solutions (or approximations) for various option pricing problems, including barrier, lookback, and perpetual American options, are feasible under the double exponential jump-diffusion model, while it seems difficult for many other models, including the normal jump-diffusion model.

3.3 Why jump-diffusion models

Since essentially all models are “wrong” and rough approximations of reality, instead of arguing the “correctness” of a particular model we shall evaluate jump-diffusion models by four criteria.

- (1) A model must be internally self-consistent. In the finance context, it means that a model must be arbitrage-free and can be embedded in an equilibrium setting. Note that some of the alternative models may have arbitrage opportunities, and thus are not self-consistent (e.g. the arbitrage opportunities for fractal Brownian motions as shown by Rogers, 1997). In this regard, both the Merton's normal jump-diffusion model and the double exponential jump-diffusion model can be embedded in a rational expectations equilibrium setting.
- (2) A model should be able to capture some important empirical phenomena. However, we should emphasize that empirical tests should not be used as the only criterion to judge a model good or bad. Empirical tests tend to favor models with more parameters. However, models with many parameters tend to make calibration more difficult (the calibration may involve high-dimensional numerical optimization with many

local optima), and tend to have less tractability. This is a part of the reason why practitioners still like the simplicity of the Black–Scholes model. Jump-diffusion models are able to reproduce the leptokurtic feature of the return distribution, and the “volatility smile” observed in option prices (see Kou, 2002). The empirical tests performed in Ramezani and Zeng (2002) suggest that the double exponential jump-diffusion model fits stock data better than the normal jump-diffusion model, and both of them fit the data better than the classical geometric Brownian motion model.

- (3) A model must be simple enough to be amenable to computation. Like the Black–Scholes model, the double exponential jump-diffusion model not only yields closed-form solutions for standard call and put options, but also leads to a variety of closed form solutions for path-dependent options, such as barrier options, lookback options, perpetual American options (see Kou and Wang, 2003, 2004; Kou et al., 2005), as well as interest rate derivatives (see Glasserman and Kou, 2003).
- (4) A model must have some (economical, physical, psychological, etc.) interpretation. One motivation for the double exponential jump-diffusion model comes from behavioral finance. It has been suggested from extensive empirical studies that markets tend to have both overreaction and underreaction to various good news or bad news (see, for example, Fama, 1998 and Barberis et al., 1998, and references therein). One may interpret the jump part of the model as the market response to outside news. More precisely, in the absence of outside news the asset price simply follows a geometric Brownian motion. Good or bad news arrive according to a Poisson process, and the asset price changes in response according to the jump size distribution. Because the double exponential distribution has both a high peak and heavy tails, it can be used to model both the overreaction (attributed to the heavy tails) and underreaction (attributed to the high peak) to outside news. Therefore, the double exponential jump-diffusion model can be interpreted as an attempt to build a simple model, within the traditional random walk and efficient market framework, to incorporate investors’ sentiment. Interestingly enough, the double exponential distribution has been widely used in mathematical psychology literature, particularly in vision cognitive studies; see, for example, papers by David Mumford and his authors at the computer vision group, Brown University.

Incidentally, as a by product, the model also suggests that the fact of markets having both overreaction and underreaction to outside news can lead to the leptokurtic feature of asset return distribution.

There are many alternative models that can satisfy at least some of the four criteria listed above. A main attraction of the double exponential jump-diffusion model is its simplicity, particularly its analytical tractability for path-dependent options and interest rate derivatives. Unlike the original Black–Scholes model, many alternative models can only compute prices for standard

call and put options, and analytical solutions for other equity derivatives (such as path-dependent options) are unlikely. Even numerical methods for interest rate derivatives and path-dependent options are not easy, as the convergence rates of binomial trees and Monte Carlo simulation for path-dependent options are typically much slower than those for call and put options (for a survey, see Boyle et al., 1997). This makes it harder to persuade practitioners to switch from the Black–Scholes model to more realistic alternative models. The double exponential jump-diffusion model attempts to improve the empirical implications of the Black–Scholes model, while still retaining its analytical tractability.

3.4 Shortcoming of jump-diffusion models

The main problem with jump-diffusion models is that they cannot capture the volatility clustering effects, which can be captured by other models such as stochastic volatility models. Jump-diffusion models and the stochastic volatility model complement each other: the stochastic volatility model can incorporate dependent structures better, while the double exponential jump-diffusion model has better analytical tractability, especially for path-dependent options and complex interest rate derivatives. For example, one empirical phenomenon worth mentioning is that the daily return distribution tends to have more kurtosis than the distribution of monthly returns. As Das and Foresi (1996) point out, this is consistent with models with jumps, but inconsistent with stochastic volatility models. More precisely, in stochastic volatility models (or essentially any models in a pure diffusion setting) the kurtosis decreases as the sampling frequency increases; while in jump models the instantaneous jumps are independent of the sampling frequency. This, in particular, suggests that jump-diffusion models may capture short-term behavior better, while stochastic volatility may be more useful to model long term behavior.

More general models combine jump-diffusions with stochastic volatilities resulting in “affine jump-diffusion models,” as in Duffie et al. (2000) which can incorporate jumps, stochastic volatility, and jumps in volatility. Both normal and double exponential jump diffusion models can be viewed as special cases of their model. However, because of the special features of the exponential distribution, the double exponential jump-diffusion model leads to analytical solutions for path-dependent options, which are difficult for other affine jump-diffusion models (even numerical methods are not easy). Furthermore, jump-diffusion models are simpler than general affine jump-diffusion models; in particular jump-diffusion model have fewer parameters that makes calibration easier. Therefore, jump-diffusion models attempt to strike a balance between reality and tractability, especially for short maturity options and short term behavior of asset pricing.

In summary, many alternative models may give some analytical formulae for standard European call and put options, but analytical solutions for interest rate derivatives and path-dependent options, such as perpetual American options, barrier and lookback options, are difficult, if not impossible. In the dou-

ble exponential jump-diffusion model analytical solution for path-dependent options are possible. However, the jump-diffusion models cannot capture the volatility clustering effect. Therefore, jump-diffusion models are more suitable for pricing short maturity options in which the impact of the volatility clustering effect is less pronounced. In addition jump-diffusion models can provide a useful benchmark for more complicated models (for which one perhaps has to resort to simulation and other numerical procedures).

4 Equilibrium for general jump-diffusion models

4.1 Basic setting of equilibrium

Consider a typical rational expectations economy (Lucas, 1978) in which a representative investor tries to solve a utility maximization problem $\max_c E[\int_0^\infty U(c(t), t) dt]$, where $U(c(t), t)$ is the utility function of the consumption process $c(t)$. There is an exogenous endowment process, denoted by $\delta(t)$, available to the investor. Also given to the investor is an opportunity to invest in a security (with a finite liquidation date T_0 , although T_0 can be very large) which pays no dividends. If $\delta(t)$ is Markovian, it can be shown (see, for example, Stokey and Lucas, 1989, pp. 484–485) that, under mild conditions, the rational expectations equilibrium price (also called the “shadow” price) of the security, $p(t)$, must satisfy the Euler equation

$$p(t) = \frac{E(U_c(\delta(T), T)p(T) | \mathcal{F}_t)}{U_c(\delta(t), t)}, \quad \forall T \in [t, T_0], \quad (4)$$

where U_c is the partial derivative of U with respect to c . At this price $p(t)$, the investor will never change his/her current holdings to invest in (either long or short) the security, even though he/she is given the opportunity to do so. Instead, in equilibrium the investor find it optimal to just consume the exogenous endowment, i.e. $c(t) = \delta(t)$ for all $t \geq 0$.

In this section we shall derive explicitly the implications of the Euler equation (4) when the endowment process $\delta(t)$ follows a general jump-diffusion process under the physical measure P :

$$\frac{d\delta(t)}{\delta(t-)} = \mu_1 dt + \sigma_1 dW_1(t) + d \left[\sum_{i=1}^{N(t)} (\tilde{V}_i - 1) \right], \quad (5)$$

where the $\tilde{V}_i \geq 0$ are any independent identically distributed, nonnegative random variables. In addition, all three sources of randomness, the Poisson process $N(t)$, the standard Brownian motion $W_1(t)$, and the jump sizes \tilde{V} , are assumed to be independent.

Although it is intuitively clear that, generally speaking, the asset price $p(t)$ should follow a similar jump-diffusion process as that of the dividend process $\delta(t)$, a careful study of the connection between the two is needed. This is

because $p(t)$ and $\delta(t)$ may not have similar jump dynamics; see (15). Furthermore, deriving explicitly the change of parameters from $\delta(t)$ to $p(t)$ also provides some valuable information about the risk premiums embedded in jump diffusion models.

Naik and Lee (1990) consider the special case that \tilde{V}_i has a lognormal distribution is investigated. In addition, Naik and Lee (1990) also require that the asset pays continuous dividends, and there is no outside endowment process; while here the asset pays no dividends and there is an outside endowment process. Consequently, the pricing formulae here are different even in the case of lognormal jumps.

For simplicity, we shall only consider the utility function of the special forms $U(c, t) = e^{-\theta t} \frac{c^\alpha}{\alpha}$ if $\alpha < 1$, and $U(c, t) = e^{-\theta t} \log(c)$ if $\alpha = 0$, where $\theta > 0$ (although most of the results below hold for more general utility functions), where θ is the discount rate in utility functions. Under these types of utility functions, the rational expectations equilibrium price of (4) becomes

$$p(t) = \frac{\mathbb{E}(e^{-\theta T} (\delta(T))^{\alpha-1} p(T) \mid \mathcal{F}_t)}{e^{-\theta t} (\delta(t))^{\alpha-1}}. \quad (6)$$

4.2 Choosing a risk-neutral measure

We shall assume that the discount rate θ should be large enough so that

$$\theta > -(1 - \alpha)\mu_1 + \frac{1}{2}\sigma_1^2(1 - \alpha)(2 - \alpha) + \lambda\zeta_1^{(\alpha-1)},$$

where the notation $\zeta_1^{(a)}$ means

$$\zeta_1^{(a)} := \mathbb{E}[(\tilde{V})^a - 1].$$

This assumption guarantees that in equilibrium the term structure of interest rates is positive.

Suppose $\zeta_1^{(\alpha-1)} < \infty$. The following result in Kou (2002) justifies risk-neutral pricing by choosing a particular risk-neutral measure for option pricing:

- (1) Letting $B(t, T)$ be the price of a zero coupon bond with maturity T , the yield $r := -\frac{1}{T-t} \log(B(t, T))$ is a constant independent of T ,

$$r = \theta + (1 - \alpha)\mu_1 - \frac{1}{2}\sigma_1^2(1 - \alpha)(2 - \alpha) - \lambda\zeta_1^{(\alpha-1)} > 0. \quad (7)$$

- (2) Let $Z(t) := e^{rt} U_c(\delta(t), t) = e^{(r-\theta)t} (\delta(t))^{\alpha-1}$. Then $Z(t)$ is a martingale under \mathbb{P} ,

$$\frac{dZ(t)}{Z(t-)} = -\lambda \zeta_1^{(\alpha-1)} dt + \sigma_1(\alpha-1) dW_1(t) + d \left[\sum_{i=1}^{N(t)} (\tilde{V}_i^{\alpha-1} - 1) \right]. \quad (8)$$

Using $Z(t)$, one can define a new probability measure \mathbb{P}^* : $\frac{d\mathbb{P}^*}{d\mathbb{P}} := Z(t)/Z(0)$. The Euler equation (6) holds if and only if the asset price satisfies

$$S(t) = e^{-r(T-t)} \mathbb{E}^*(S(T) | \mathcal{F}_t), \quad \forall T \in [t, T_0]. \quad (9)$$

Furthermore, the rational expectations equilibrium price of a (possibly path-dependent) European option, with the payoff $\psi_S(T)$ at the maturity T , is given by

$$\psi_S(t) = e^{-r(T-t)} \mathbb{E}^*(\psi_S(T) | \mathcal{F}_t), \quad \forall t \in [0, T]. \quad (10)$$

4.3 The dynamic under the risk-neutral measure

Given the endowment process $\delta(t)$, it must be decided what stochastic processes are suitable for the asset price $S(t)$ to satisfy the equilibrium requirement (6) or (9). Now consider a special jump-diffusion form for $S(t)$,

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma \{ \rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t) \} + d \left(\sum_{i=1}^{N(t)} (V_i - 1) \right),$$

$$V_i = \tilde{V}_i^\beta, \quad (11)$$

where $W_2(t)$ is a Brownian motion independent of $W_1(t)$. In other words, the same Poisson process affects both the endowment $\delta(t)$ and the asset price $S(t)$, and the jump sizes are related through a power function, where the power $\beta \in (-\infty, \infty)$ is an arbitrary constant. The diffusion coefficients and the Brownian motion part of $\delta(t)$ and $S(t)$, though, are totally different. It remains to determine what constraints should be imposed on this model, so that the jump-diffusion model can be embedded in the rational expectations equilibrium requirement (6) or (9).

Suppose $\zeta_1^{(\alpha+\beta-1)} < \infty$ and $\zeta_1^{(\alpha-1)} < \infty$. It can be shown (Kou, 2002) that the model (11) satisfies the equilibrium requirement (9) if and only if

$$\begin{aligned} \mu &= r + \sigma_1 \sigma \rho (1 - \alpha) - \lambda (\zeta_1^{(\alpha+\beta-1)} - \zeta_1^{(\alpha-1)}) \\ &= \theta + (1 - \alpha) \left\{ \mu_1 - \frac{1}{2} \sigma_1^2 (2 - \alpha) + \sigma_1 \sigma \rho \right\} - \lambda \zeta_1^{(\alpha+\beta-1)}. \end{aligned} \quad (12)$$

If (12) is satisfied, then under \mathbf{P}^*

$$\frac{dS(t)}{S(t-)} = r dt - \lambda^* \mathbf{E}^* (\tilde{V}_i^\beta - 1) dt + \sigma dW^*(t) + d \left[\sum_{i=1}^{N(t)} (\tilde{V}_i^\beta - 1) \right]. \quad (13)$$

Here, under \mathbf{P}^* , $W^*(t)$ is a new Brownian motion, $N(t)$ is a new Poisson process with jump rate $\lambda^* = \lambda \mathbf{E}(\tilde{V}_i^{\alpha-1}) = \lambda(\zeta_1^{(\alpha-1)} + 1)$, and $\{\tilde{V}_i\}$ are independent identically distributed random variables with a new density under \mathbf{P}^* :

$$f_{\tilde{V}}^*(x) = \frac{1}{\zeta_1^{(\alpha-1)} + 1} x^{\alpha-1} f_{\tilde{V}}(x). \quad (14)$$

A natural question is under what conditions all three dynamics, $\delta(t)$ and $S(t)$ under \mathbf{P} and $S(t)$ under \mathbf{P}^* , have the same jump-diffusion form, which is very convenient for analytical calculation. Suppose the family \mathcal{V} of distributions of the jump size \tilde{V} for the endowment process $\delta(t)$ satisfies that, for any real number a ,

$$\text{if } \tilde{V}^a \in \mathcal{V} \text{ then } \text{const} \cdot x^a f_{\tilde{V}}(x) \in \mathcal{V}, \quad (15)$$

where the normalizing constant, const , is $\{\zeta_1^{(a-1)} + 1\}^{-1}$ (provided that $\zeta_1^{(a-1)} < \infty$). Then the jump sizes for the asset price $S(t)$ under \mathbf{P} and the jump sizes for $S(t)$ under the rational expectations risk-neutral measure \mathbf{P}^* all belong to the same family \mathcal{V} . The result follows immediately from (5), (11), and (14).

The condition (15) essentially requires that the jump size distribution belongs to the exponential family. It is satisfied if $\log(V)$ has a normal distribution or a double exponential distribution. However, the log power-type distributions, such as log t -distribution, do not satisfy (15).

5 Basic setting for option pricing

In the rest of the chapter, we shall focus on option pricing under jump-diffusion models. To do this we shall fix some notations. For a jump-diffusion process, the log-return $X(t) = \ln(S(t)/S(0))$ will be a process such that

$$X(t) = \tilde{\mu}t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i, \quad X_0 \equiv 0. \quad (16)$$

Here $\{W_t; t \geq 0\}$ is a standard Brownian motion with $W_0 = 0$, $\{N_t; t \geq 0\}$ is a Poisson process with rate λ , constants $\tilde{\mu}$ and $\sigma > 0$ are the drift and volatility of the diffusion part, respectively, and the jump sizes $\{Y_1, Y_2, \dots\}$ are independent identically distributed random variables. We also assume that the random processes $\{W_t; t \geq 0\}$, $\{N_t; t \geq 0\}$, and random variables $\{Y_1, Y_2, \dots\}$ are independent representing $Y_i = \log(\tilde{V}_i)$.

The infinitesimal generator of the jump-diffusion process (16) is given by

$$\mathcal{L}u(x) = \frac{1}{2}\sigma^2 u''(x) + \tilde{\mu}u'(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)]f_Y(y) dy, \quad (17)$$

for all twice continuously differentiable functions $u(x)$. In addition, suppose $\theta \in (-\eta_2, \eta_1)$. The moment generating function of $X(t)$ can be obtained as

$$\mathbb{E}[e^{\theta X(t)}] = \exp\{G(\theta)t\}, \quad (18)$$

where

$$G(x) := x\tilde{\mu} + \frac{1}{2}x^2\sigma^2 + \lambda(\mathbb{E}[e^{xY}] - 1).$$

In the case of Merton's normal jump-diffusion model, Y has a normal density

$$f_Y(y) \sim \frac{1}{\sigma'\sqrt{2\pi}} \exp\left\{-\frac{(y - \mu')^2}{2\sigma'^2}\right\},$$

where μ' and σ' are the mean and standard deviation for Y . Thus,

$$G(x) = x\tilde{\mu} + \frac{1}{2}x^2\sigma^2 + \lambda\left\{\mu'x + \frac{(\sigma')^2x^2}{2} - 1\right\}.$$

In the case of double exponential jump-diffusion model

$$f_Y(y) \sim p \cdot \eta_1 e^{-\eta_1 y} \mathbf{1}_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} \mathbf{1}_{\{y < 0\}}, \quad \eta_1 > 1, \quad \eta_2 > 0,$$

and the function $G(x)$ is

$$G(x) = x\tilde{\mu} + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right). \quad (19)$$

Kou and Wang (2003) show that for $\alpha > 0$ in the case of double exponential jump-diffusion model the equation $G(x) = \alpha$ has exactly four roots $\beta_{1,\alpha}, \beta_{2,\alpha}, -\beta_{3,\alpha}, -\beta_{4,\alpha}$, where

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty, \quad 0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty. \quad (20)$$

The analytical formulae for the four roots of the equation $G(x) = \alpha$, which is essentially a quartic equation, are given in Kou et al. (2005). The explicit formulae of β 's are useful for the Euler algorithm in Laplace inversion.

Under the risk-neutral probability \mathbb{P}^* in (13), we have

$$\tilde{\mu} = r - \frac{1}{2}\sigma^2 - \lambda\zeta,$$

where $\zeta := \mathbb{E}^*[e^Y] - 1$. Similarly, if the underlying asset pays continuous dividend at the rate δ , then under \mathbb{P}^*

$$\tilde{\mu} = r - \delta - \frac{1}{2}\sigma^2 - \lambda\zeta.$$

In the Merton's model

$$\zeta = \mathbb{E}^*[e^Y] - 1 = \mu' + \frac{(\sigma')^2}{2} - 1,$$

and in the double exponential jump-diffusion model

$$\zeta = \mathbb{E}^*[e^Y] - 1 = p\eta_1/(\eta_1 - 1) + q\eta_2/(\eta_2 + 1) - 1.$$

6 Pricing call and put option via Laplace transforms

Laplace transforms have been widely used in valuing financial derivatives. For example, Carr and Madan (1999) propose Fourier transforms with respect to log-strike prices; Geman and Yor (1993), Fu et al. (1999) use Laplace transforms to price Asian options in the Black–Scholes setting; Laplace transforms for double-barrier and lookback options under the CEV model are given in Davydov and Linetsky (2001); Petrella and Kou (2004) use a recursion and Laplace transforms to price discretely monitored barrier and lookback options. For a survey of Laplace transforms in option pricing, see Craddock et al. (2000).

Kou et al. (2005) adapted the method in Carr and Madan (1999), which is based on a change of the order of integration, to price European call and put option via Laplace transforms. In principle, the Laplace transforms for the prices of European call and European put options can also be obtained by using standard results from Fourier transforms for general Lévy processes (see Cont and Tankov, 2004, pp. 361–362).

To fix the notation, the price of a European call with maturity T and strike K , is given by

$$\begin{aligned} C_T(k) &= e^{-rT} \mathbb{E}^*[(S(T) - K)^+] \\ &= e^{-rT} \mathbb{E}^*[(S(0)e^{X(T)} - e^{-k})^+], \end{aligned} \quad (21)$$

where $k = -\log(K)$, and the price of a European put is

$$P_T(k') = e^{-rT} \mathbb{E}^*[(K - S(T))^+] = e^{-rT} \mathbb{E}^*[(e^{k'} - S(0)e^{X(T)})^+],$$

where $k' = \log(K)$. The Laplace transform with respect to k of $C_T(k)$ in (21) is given by

$$\begin{aligned} \hat{f}_C(\xi) &:= \int_{-\infty}^{\infty} e^{-\xi k} C_T(k) dk \\ &= e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} \exp(G(\xi+1)T), \quad \xi > 0, \end{aligned} \quad (22)$$

and the Laplace transform with respect to k' for the put option $P_T(k')$ is

$$\begin{aligned}\hat{f}_P(\xi) &:= \int_{-\infty}^{\infty} e^{-\xi k'} P_T(k') dk' \\ &= e^{-rT} \frac{S(0)^{-(\xi-1)}}{\xi(\xi-1)} \exp(G(-(\xi-1)T)), \quad \xi > 1,\end{aligned}\quad (23)$$

in the notation of (18).

To show this, note that by (21) the Laplace transform for the call options is

$$\hat{f}_C(\xi) = e^{-rT} \int_{-\infty}^{\infty} e^{-\xi k} \mathbf{E}^*[(S(0)e^{X(T)} - e^{-k})^+] dk.$$

Applying the Fubini theorem yields for every $\xi > 0$,

$$\begin{aligned}\hat{f}_C(\xi) &= e^{-rT} \mathbf{E}^* \left[\int_{-\infty}^{\infty} e^{-\xi k} (S(0)e^{X(T)} - e^{-k})^+ dk \right] \\ &= e^{-rT} \mathbf{E}^* \left[\int_{-X(T) - \log S(0)}^{\infty} e^{-\xi k} (S(0)e^{X(T)} - e^{-k}) dk \right] \\ &= e^{-rT} \mathbf{E}^* \left[S(0)e^{X(T)} e^{\xi(X(T) + \log S(0))} \frac{1}{\xi} \right. \\ &\quad \left. - e^{(\xi+1)(X(T) + \log S(0))} \frac{1}{\xi+1} \right] \\ &= e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} \mathbf{E}^* [e^{(\xi+1)X(T)}],\end{aligned}$$

from which (22) follows readily from (18). The proof of (23) is similar.

The Laplace transforms can be inverted numerically in the complex plane, using the two-sided extension of the Euler algorithm as described and implemented in Petrella (2004). To check the accuracy of the inversion, Kou et al. (2005) compare the inversion results with the prices of call and put options under the double exponential jump-diffusion model obtained by using the closed-form formulae using Hh function as in Kou (2002). They found that the results from the Laplace inversion method agree to the fifth decimal with the analytical solutions for European call and put options. Because of the difficulty in precise calculation of the normal distribution function and the $Hh(x)$ function for very positive and negative x , it is possible that for very large values of the return variance $\sigma^2 T$ and for very high jump rate λ (though perhaps not within the typical parameter ranges seen in finance applications) the closed-form formulae may not give accurate results. In such cases, the inversion method still performs remarkably well.

It is also possible to compute the sensitivities of the option, such as Delta, Gamma, Theta, Vega, etc., by inverting the derivatives of the option's Laplace transform in (22). For example, the delta is given by

$$\begin{aligned}\Delta(C_T(k)) &= \frac{\partial}{\partial S(0)} C_T(k) \\ &= \mathcal{L}_\xi^{-1} \left(e^{-rT} \frac{S(0)^\xi}{\xi} \exp(G(\xi + 1)T) \right) \Big|_{k=-\log K},\end{aligned}$$

where \mathcal{L}_ξ^{-1} means the Laplace inversion with respect to ξ .

7 First passage times

To price perpetual American options, barrier options, and lookback options for general jump-diffusion processes, it is crucial to study the first passage time of a jump-diffusion process $X(t)$ to a flat boundary:

$$\tau_b := \inf\{t \geq 0; X(t) \geq b\}, \quad b > 0,$$

where $X(\tau_b) := \limsup_{t \rightarrow \infty} X(t)$, on the set $\{\tau_b = \infty\}$.

7.1 The overshoot problem

Without the jump part, the process $X(t)$ simply becomes a Brownian motion with drift $\tilde{\mu}$. The distributions of the first passage times can be obtained either by a combination of a change of measure (Girsanov theorem) and the reflection principle, or by calculating the Laplace transforms via some appropriate martingales and the optional sampling theorem. Details of both methods can be found in many classical textbooks on stochastic analysis, e.g. Karlin and Taylor (1975), Karatzas and Shreve (1991). With the jump part, however, it is very difficult to study the first passage times for general jump-diffusion processes. When a jump-diffusion process crosses boundary level b , sometimes it hits the boundary exactly and sometimes it incurs an "overshoot," $X(\tau_b) - b$, over the boundary. See Fig. 6 for an illustration.

The overshoot presents several problems for option pricing. First, one needs to get the exact distribution of the overshoot. It is well known from stochastic renewal theory that this is in general difficult unless the jump size Y has an exponential-type distribution, thanks to the special memoryless property of the exponential distribution. Second, one needs to know the dependent structure between the overshoot and the first passage time. The two random variables are conditionally independent, given that the overshoot is bigger than 0, if the jump size Y has an exponential-type distribution, thanks to the memoryless property. This conditionally independent structure seems to be very special to the exponential-type distribution, and does not hold for other distributions,

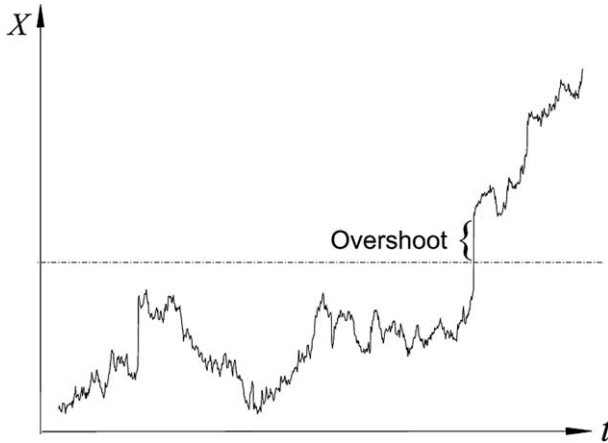


Fig. 6. A simulated sample path with the overshoot problem.

such as the normal distribution. Third, if one wants to use the reflection principle to study the first passage times, the dependent structure between the overshoot and the terminal value X_t is also needed. This is not known to the best of our knowledge, even for the double exponential jump-diffusion process.

Consequently, we can derive closed form solutions for the Laplace transforms of the first passage times for the double exponential jump-diffusion process, yet cannot give more explicit calculations beyond that, as the correlation between $X(t)$ and $X(\tau_b) - b$ is not available. However, for other jump-diffusion processes, even analytical forms of the Laplace transforms seem to be quite difficult. See [Asmussen et al. \(2004\)](#), [Boyarchenko and Levendorskiĭ \(2002\)](#), and [Kyprianou and Pistorius \(2003\)](#) for some representations (though not explicit calculations) based on the Wiener–Hopf factorization related to the overshoot problems for general Lévy processes; and see also [Avram et al. \(2004\)](#) and [Rogers \(2000\)](#) for first passage times with one-sided jumps.

7.2 Conditional independence

The following result shows that the memoryless property of the random walk of exponential random variables leads to the conditional memoryless property of the jump-diffusion process. For any $x > 0$,

$$\mathbb{P}(\tau_b \leq t, X(\tau_b) - b \geq x) = e^{-\eta_1 x} \mathbb{P}(\tau_b \leq t, X(\tau_b) - b > 0), \quad (24)$$

$$\mathbb{P}(X(\tau_b) - b \geq x \mid X(\tau_b) - b > 0) = e^{-\eta_1 x}. \quad (25)$$

Furthermore, conditional on $X_{\tau_b} - b > 0$, the stopping time τ_b and the overshoot $X_{\tau_b} - b$ are independent; more precisely, for any $x > 0$,

$$\mathbb{P}(\tau_b \leq t, X(\tau_b) - b \geq x \mid X(\tau_b) - b > 0)$$

$$= \mathbb{P}(\tau_b \leq t \mid X(\tau_b) - b > 0) \mathbb{P}(X(\tau_b) - b \geq x \mid X(\tau_b) - b > 0). \quad (26)$$

It should be pointed out that τ_b and the overshoot $X(\tau_b) - b$ are dependent even in the case of double exponential jump diffusion, although they are conditionally independent.

7.3 Distribution of the first passage times

For any $\alpha \in (0, \infty)$, let $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$ be the only two positive roots for the equation $\alpha = G(\beta)$, where $0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$. Then Kou and Wang (2003) give the following results regarding the Laplace transform of τ_b

$$\begin{aligned} \mathbb{E}[e^{-\alpha\tau_b}] &= \frac{\eta_1 - \beta_{1,\alpha}}{\eta_1} \cdot \frac{\beta_{2,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\eta_1} \\ &\quad \cdot \frac{\beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{2,\alpha}}, \\ \mathbb{E}^*[e^{-\alpha\tau_b} \mathbf{1}_{\{X(\tau_b) > b\}}] &= \frac{(\eta_1 - \beta_{1,\alpha})(\beta_{2,\alpha} - \eta_1)}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})} [e^{-b\beta_{1,\alpha}} - e^{-b\beta_{2,\alpha}}], \\ \mathbb{E}^*[e^{-\alpha\tau_b} \mathbf{1}_{\{X(\tau_b) = b\}}] &= \frac{\eta_1 - \beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{2,\alpha}}. \end{aligned} \quad (27)$$

The results for the down-crossing barrier problem, i.e. $b < 0$, will involve the other two roots, $\beta_{3,\alpha}$ and $\beta_{4,\alpha}$.

For simplicity, we will focus on (27). It is easy to give a heuristic argument for (27). Let $u(x) = \mathbb{E}^x[e^{-\alpha\tau_b}]$, $b > 0$, we expect from a heuristic application of the Feymann–Kac formula that u satisfies the integro-differential equation

$$-\alpha u(x) + \mathcal{L}u(x) = 0, \quad \forall x < b, \quad (28)$$

and $u(x) = 1$ if $x \geq b$. This equation can be explicitly solved at least heuristically. Indeed, consider a solution taking form

$$u(x) = \begin{cases} 1, & x \geq b, \\ A_1 e^{-\beta_1(b-x)} + B_1 e^{-\beta_2(b-x)}, & x < b, \end{cases} \quad (29)$$

where constants A_1 and B_1 are yet to be determined. Plug in to obtain, after some algebra, that $(-\alpha u + \mathcal{L}u)(x)$ for all $x < b$ is equal to

$$\begin{aligned} &A_1 e^{-(b-x)\beta_1} f(\beta_1) + B_1 e^{-(b-x)\beta_2} f(\beta_2) \\ &- \lambda p e^{-\eta_1(b-x)} \left(\frac{A_2 \eta_1}{\eta_1 - \beta_1} + \frac{B_2 \eta_1}{\eta_1 - \beta_2} - e^{-\eta_1 y} \right), \end{aligned} \quad (30)$$

where $f(\beta) = G(\beta) - \alpha$.

To set $(-\alpha u + \mathcal{L}u)(x) = 0$ for all $x < b$, we can first have $f(\beta_1) = f(\beta_2) = 0$, which means that we shall choose β_1 and β_2 to be two roots of $G(\beta) = \alpha$, although it is not clear which two roots among the four are needed.

Afterward it is enough to set the third term in (30) to be zero by choosing A_1 and B_1 so that

$$A_1 \frac{\eta_1}{\eta_1 - \beta_1} + B_1 \frac{\eta_1}{\eta_1 - \beta_2} = e^{-\eta_1 y}.$$

Furthermore, the continuity of u at $x = b$ implies that

$$A_1 + B_1 = 1.$$

Solve the equations to obtain A_1 and B_1 ($A_1 = 1 - B_1$), which are exactly the coefficients in (27). However, the above heuristic argument has several difficulties.

7.4 Difficulties

7.4.1 Nonsmoothness

Because the function $u(x)$ in (29) is continuous, but not C^1 at $x = b$, we cannot apply the Itô formula and the Feymann–Kac formula directly to the process $e^{-\alpha t} u(X_t); t \geq 0$. Furthermore, even if we can use the Feymann–Kac formula, it is not clear whether the solution to the integro-differential equation (28) is well defined and unique. Therefore, Kou and Wang (2003) have to use some approximation of $u(x)$ so that Itô formula can be used, and then they used a martingale method to solve the integro-differential equation (28) directly. In addition, the martingale method also helps to identify which two roots are needed in the formulae. Note that a heuristic argument based on the Feymann–Kac formula for double barrier options (with both upper and lower barriers) is given in Sepp (2004) by extending (28) and (29), and ignoring the nonsmoothness issue.

7.4.2 Explicit calculation

It should be mentioned that the special form of double exponential density function enables us to explicitly solve the integro-differential equation (28) associated with the Laplace transforms using martingale methods, thanks to the explicit calculation in (30). This is made possible as the exponential function has some good properties such as the product of two exponential functions is still an exponential function, and the integral of an exponential function is again an exponential function. For general jump-diffusion processes, however, such explicit solution will be very difficult to obtain.

7.4.3 Nonuniqueness in renewal integral equations

We have used martingale and differential equations to derive closed form solutions of the Laplace transforms for the first-passage-time probabilities. Another possible and popular approach to solving the problems is to set up some integral equations by using renewal arguments. For simplicity, we shall only consider the case of overall drift being nonnegative, i.e. $\bar{u} \geq 0$, in which $\tau_b < \infty$ almost surely. For any $x > 0$, define $P(x)$ as the probability that

no overshoot occurs for the first passage time τ_x with $X(0) \equiv 0$, that is $P(x) = \mathbb{P}(X(\tau_x) = x)$. It is easy to see that $P(x)$ satisfies the following renewal-type integral equation:

$$P(x+y) = P(y)P(x) + (1-P(x)) \int_0^y P(y-z) \cdot \eta_1 e^{-\eta_1 z} dz.$$

However, the solution to this renewal equation is not unique. Indeed, for every $\xi \geq 0$, the function

$$P_\xi(x) = \frac{\eta_1}{\eta_1 + \xi} + \frac{\xi}{\eta_1 + \xi} e^{-(\eta_1 + \xi)x}$$

satisfies the integral equation with the boundary condition $P_\xi(0) = 1$.

This shows that, in the presence of two-sided jumps, the renewal-type integral equations may not have unique solutions, mainly because of the difficulty of determining enough boundary conditions based on renewal arguments alone. It is easy to see that $\xi = -P'_\xi(0)$. Indeed, it is possible to use the infinitesimal generator and martingale methods to determine ξ . The point here is, however, that the renewal-type integral equations cannot do the job by themselves.

8 Barrier and lookback options

Barrier and lookback options are among the most popular path-dependent derivatives traded in exchanges and over-the-counter markets worldwide. The payoffs of these options depend on the extrema of the underlying asset. For a complete description of these and other related contracts we refer the reader to Hull (2005). In the standard Black–Scholes setting, closed-form solutions for barrier and lookback options have been derived by Merton (1973) and Gatto et al. (1979).

8.1 Pricing barrier options

We will focus on the pricing of an up-and-in call option (UIC, from now on); other types of barrier options can be priced similarly and using the symmetries described in the Appendix of Petrella and Kou (2004) and Haug (1999). The price of an UIC is given by

$$UIC(k, T) = \mathbb{E}^*[e^{-rT}(S(T) - e^{-k})^+ \mathbf{1}_{\{\tau_b < T\}}], \quad (31)$$

where $H > S(0)$ is the barrier level, $k = -\log(K)$ the transformed strike and $b = \log(H/S(0))$. Using a change of numéraire argument, Kou and Wang (2004) show that under another probability, defined as $\tilde{\mathbb{P}}$, $X(T)$ still has a dou-

ble exponential distribution with drift $r - \delta + \frac{1}{2}\sigma^2 - \lambda\zeta$ and jump parameters

$$\begin{aligned} \tilde{\lambda} &= \lambda(\zeta + 1), & \tilde{p} &= \frac{p\eta_1}{(\zeta + 1)(\eta_1 - 1)}, \\ \tilde{\eta}_1 &= \eta_1 - 1, & \tilde{\eta}_2 &= \eta_2 + 1. \end{aligned}$$

The moment generating function of $X(t)$ under the alternative probability measure $\tilde{\mathbb{P}}$ is given by $\tilde{\mathbb{E}}[e^{\theta X(t)}] = \exp(\tilde{G}(\theta)t)$, with

$$\begin{aligned} \tilde{G}(x) &:= x\left(r - \delta + \frac{1}{2}\sigma^2 - \tilde{\lambda}\tilde{\zeta}\right) \\ &\quad + \frac{1}{2}x^2\sigma^2 + \tilde{\lambda}\left(\frac{\tilde{p}\tilde{\eta}_1}{\tilde{\eta}_1 - x} + \frac{\tilde{q}\tilde{\eta}_2}{\tilde{\eta}_2 + x} - 1\right). \end{aligned}$$

Kou and Wang (2004) further show that

$$UIC(k, T) = S(0)\tilde{\Psi}_{UI}(k, T) - Ke^{-rT}\Psi_{UI}(k, T), \tag{32}$$

where

$$\begin{aligned} \Psi_{UI}(k, T) &= \mathbb{P}^*(S(T) \geq e^{-k}, M_{0,T} > H), \\ \tilde{\Psi}_{UI}(k, T) &= \tilde{\mathbb{P}}(S(T) \geq e^{-k}, M_{0,T} > H), \end{aligned} \tag{33}$$

and show how to price an UIC option by inverting the one-dimensional Laplace transforms for the joint distributions in (32) as in Kou and Wang (2003).

Kou et al. (2005) present an alternative approach that relies on a two-dimensional Laplace transform for both the option price in (31) and the probabilities in (32). The formulae after doing two-dimensional transforms become much simpler than the one-dimensional formulae in Kou and Wang (2003), which involve many special functions.

In particular Kou et al. (2005) show that for ξ and α such that $0 < \xi < \eta_1 - 1$ and $\alpha > \max(G(\xi + 1) - r, 0)$ (such a choice of ξ and α is possible for all small enough ξ as $G(1) - r = -\delta < 0$), the Laplace transform with respect to k and T of $UIC(k, T)$ is given by

$$\begin{aligned} \hat{f}_{UIC}(\xi, \alpha) &= \int_0^\infty \int_{-\infty}^\infty e^{-\xi k - \alpha T} UIC(k, T) dk dT \\ &= \frac{H^{\xi+1}}{\xi(\xi + 1)} \frac{1}{r + \alpha - G(\xi + 1)} \\ &\quad \times \left(A(r + \alpha) \frac{\eta_1}{\eta_1 - (\xi + 1)} + B(r + \alpha) \right), \end{aligned} \tag{34}$$

where

$$\begin{aligned} A(h) &:= \mathbf{E}^* \left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b) > b\}} \right] \\ &= \frac{(\eta_1 - \beta_{1,h})(\beta_{2,h} - \eta_1)}{\eta_1(\beta_{2,h} - \beta_{1,h})} \left[e^{-b\beta_{1,h}} - e^{-b\beta_{2,h}} \right], \end{aligned} \quad (35)$$

$$\begin{aligned} B(h) &:= \mathbf{E}^* \left[e^{-h\tau_b} \mathbf{1}_{\{X(\tau_b) = b\}} \right] \\ &= \frac{\eta_1 - \beta_{1,h}}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{1,h}} + \frac{\beta_{2,h} - \eta_1}{\beta_{2,h} - \beta_{1,h}} e^{-b\beta_{2,h}}, \end{aligned} \quad (36)$$

with $b = \log(H/S(0))$. If $0 < \xi < \eta_1$ and $\alpha > \max(G(\xi), 0)$ (again this choice of ξ and α is possible for all ξ small enough as $G(0) = 0$), then the Laplace transform with respect to k and T of $\Psi_{UI}(k, T)$ in (33) is

$$\begin{aligned} \hat{f}_{\Psi_{UI}}(\xi, \alpha) &= \int_{-\infty}^{\infty} \left(\int_0^{\infty} e^{-\xi k - \alpha T} \Psi_{UI}(k, T) dT \right) dk \\ &= \frac{H^\xi}{\xi} \frac{1}{\alpha - G(\xi)} \left(A(\alpha) \frac{\eta_1}{\eta_1 - \xi} + B(\alpha) \right). \end{aligned} \quad (37)$$

The Laplace transforms with respect to k and T of $\tilde{\Psi}_{UI}(k, T)$ is given similarly with \tilde{G} replacing G and the functions \tilde{A} and \tilde{B} defined similarly.

Kou et al. (2005) price up-and-in calls using the two-dimensional Laplace transform (using the two-dimensional Euler algorithm developed by Choudhury et al., 1994 and Petrella, 2004) and compare the results with the one-dimensional transform in Kou and Wang (2003) (based on the Gaver–Stehfest algorithm). The two-dimensional Laplace inversion matches to the fourth digit the ones obtained by the one-dimensional Gaver–Stehfest algorithm, and are all within the 95% confidence interval obtained via Monte Carlo simulation.

The two-dimensional Laplace inversion algorithms have three advantages compared to the one-dimensional algorithm: (1) The formulae for the two-dimensional transforms Euler are much easier to compute, simplifying the implementation of the methods. (2) Although we are inverting two-dimensional transforms, the Laplace transform methods are significantly faster, mainly because of the simplicity in the Laplace transform formulae. (3) High-precision calculation (with about 80 digit accuracy) as required by the Gaver–Stehfest inversion is no longer needed in the Euler inversion, which is made possible mainly because of the simplicity of the two-dimensional inversion formulae as no special functions are involved and all the roots of $G(x)$ are given in analytical forms.

8.2 Pricing lookback options via Euler inversion

For simplicity, we shall focus on a standard lookback put option, while the derivation for a standard lookback call is similar. The price of a standard look-

back put is given by

$$\begin{aligned} LP(T) &= \mathbb{E}^* \left[e^{-rT} \left\{ \max \left\{ M, \max_{0 \leq t \leq T} S(t) \right\} - S(t) \right\} \right] \\ &= \mathbb{E}^* \left[e^{-rT} \max \left\{ M, \max_{0 \leq t \leq T} S(t) \right\} \right] - S(0), \end{aligned}$$

where $M \geq S(0)$ is the prefixed maximum at time 0. For any $\xi > 0$, the Laplace transform of the lookback put with respect to the time to maturity T is given by (see Kou and Wang, 2004)

$$\begin{aligned} \int_0^{\infty} e^{-\alpha T} LP(T) dT &= \frac{S(0)A_{\alpha}}{C_{\alpha}} \left(\frac{S(0)}{M} \right)^{\beta_{1,\alpha+r}-1} \\ &\quad + \frac{S(0)B_{\alpha}}{C_{\alpha}} \left(\frac{S(0)}{M} \right)^{\beta_{2,\alpha+r}-1} + \frac{M}{\alpha+r} - \frac{S(0)}{\alpha}, \quad (38) \end{aligned}$$

where

$$\begin{aligned} A_{\alpha} &= \frac{(\eta_1 - \beta_{1,\alpha+r})\beta_{2,\alpha+r}}{\beta_{1,\alpha+r} - 1}, & B_{\alpha} &= \frac{(\beta_{2,\alpha+r} - \eta_1)\beta_{1,\alpha+r}}{\beta_{2,\alpha+r} - 1}, \\ C_{\alpha} &= (\alpha + r)\eta_1(\beta_{2,\alpha+r} - \beta_{1,\alpha+r}), \end{aligned}$$

and $\beta_{1,\alpha+r}$, $\beta_{2,\alpha+r}$ are the two positive roots of the equation $G(x) = \alpha + r$, as in (20).

The transform in (38) can be inverted in the complex domain by using the one-dimensional Euler inversion (EUL) algorithm developed by Abate and Whitt (1992), rather than in the real domain by the Gaver–Stehfest (GS) algorithm as in Kou and Wang (2004). The main reason for this is that the EUL inversion (which is carried out in the complex-domain) does not require the high numerical precision of the GS algorithm: a precision of 12 digits will suffice for the EUL, compared with the 80 digits accuracy required by the GS. The EUL algorithm is made possible partly due to an explicit formula for the roots of $G(x)$ given. Kou et al. (2005) show that the difference between the EUL and GS results are small. Ultimately, the EUL implementation is preferable, since it is simple to implement, and it converges fast without requiring high numerical precision as in the GS.

9 Analytical approximations for American options

Most of call and put options traded in the exchanges in both US and Europe are American-type options. Therefore, it is of great interest to calculate the prices of American options accurately and quickly. The price of a finite-horizon American option is the solution of a finite horizon free boundary problem. Even within the classical geometric Brownian motion model, except

in the case of the American call option with no dividend, there is no analytical solution available. To price American options under general jump-diffusion models, one may consider numerically solving the free boundary problems via lattice or differential equation methods; see, e.g., Amin (1993), d'Halluin et al. (2003), Feng and Linetsky (2005), Feng et al. (2004), and the book by Cont and Tankov (2004).

9.1 Quadratic approximation

Extending the Barone-Adesi and Whaley (1987) approximation for the classical geometric Brownian motion model, Kou and Wang (2004) considered an alternative approach that takes into consideration of the special structure of the double exponential jump-diffusions. One motivation for such an extension is its simplicity, as it yields an analytic approximation that only involves the price of a European option. The numerical results in Kou and Wang (2004) suggest that the approximation error is typically less than 2%, which is less than the typical bid-ask spread (about 5% to 10%) for American options in exchanges. Therefore, the approximation can serve as an easy way to get a quick estimate that is perhaps accurate enough for many practical situations. The extension of Barone-Adesi and Whaley's quadratic approximation method works nicely for double exponential jump-diffusion models mainly because explicit solutions are available to a class of relevant integro-differential free boundary problems.

To simplify notation, we shall focus only on the finite horizon American put option without dividends, as the methodology is also valid for the finite horizon American call option with dividends. Also, we shall only consider finite-time horizon American put options. Related American calls can be priced by exploiting the symmetric relationship in Schröder (1999).

The analytic approximation involves two quantities, $\text{EuP}(v, t)$ which denotes the price of the European put option with initial stock price v and maturity t , and $\mathbb{P}^v[S(t) \leq K]$ which is the probability that the stock price at t is below K with initial stock price v . Both $\text{EuP}(v, t)$ and $\mathbb{P}^v[S(t) \leq K]$ can be computed fast by using either the closed form solutions in Kou (2002) or the Laplace transforms in Kou et al. (2005).

We need some notations. Let $z = 1 - e^{-rt}$, $\beta_3 \equiv \beta_{3, \frac{z}{2}}$, $\beta_4 \equiv \beta_{4, \frac{z}{2}}$, $C_\beta = \beta_3\beta_4(1 + \eta_2)$, $D_\beta = \eta_2(1 + \beta_3)(1 + \beta_4)$, in the notation of Eq. (20). Define $v_0 \equiv v_0(t) \in (0, K)$ as the unique solution to the equation

$$\begin{aligned} C_\beta K - D_\beta [v_0 + \text{EuP}(v_0, t)] \\ = (C_\beta - D_\beta) K e^{-rt} \cdot \mathbb{P}^{v_0}[S(t) \leq K]. \end{aligned} \quad (39)$$

Note that the left-hand side of (39) is a strictly decreasing function of v_0 (because $v_0 + \text{EuP}(v_0, t) = e^{-rt} \mathbb{E}^*[\max(S(t), K) \mid S(0) = v_0]$), and the right hand side of (39) is a strictly increasing function of v_0 [because $C_\beta - D_\beta = \beta_3\beta_4 - \eta_2(1 + \beta_3 + \beta_4) < 0$]. Therefore, v_0 can be obtained easily by using, for example, the bisection method.

Approximation. The price of a finite horizon American put option with maturity t and strike K can be approximated by $\psi(S(0), t)$, where the value function ψ is given by

$$\psi(v, t) = \begin{cases} \text{EuP}(v, t) + Av^{-\beta_3} + Bv^{-\beta_4}, & \text{if } v \geq v_0, \\ K - v, & \text{if } v \leq v_0, \end{cases} \quad (40)$$

with v_0 being the unique root of Eq. (39) and the two constants A and B given by

$$A = \frac{v_0^{\beta_3}}{\beta_4 - \beta_3} \{ \beta_4 K - (1 + \beta_4)[v_0 + \text{EuP}(v_0, t)] + Ke^{-rt} \mathbf{P}^{v_0}[S(t) \leq K] \} > 0, \quad (41)$$

$$B = \frac{v_0^{\beta_4}}{\beta_3 - \beta_4} \{ \beta_3 K - (1 + \beta_3)[v_0 + \text{EuP}(v_0, t)] + Ke^{-rt} \mathbf{P}^{v_0}[S(t) \leq K] \} > 0. \quad (42)$$

In the numerical examples showed by Kou and Wang (2004) the maximum relative error is only about 2.6%, while in most cases the relative errors are below 1%. The approximation runs very fast, taking only about 0.04 s to compute one price on a Pentium 1500 PC, irrespective to the parameter ranges; while the lattice method in Amin (1993) works much slower, taking about over one hour to compute one price.

9.2 Piecewise exponential approximation

A more accurate approximation can be obtained by extending the piecewise exponential approximation in Ju (1998). Extending previous work by Carr et al. (1992), Gukhal (2001) and Pham (1997) show that under jump-diffusion models the value at time t of an American put option with maturity $T > t$ on an asset with value S_t at time t ($P_A(S_t, t, T)$ from now on) is given by

$$\begin{aligned} P_A(S_t, t, T) = & P_E(S_t, t, T) + \int_t^T e^{-r(s-t)} r K \mathbf{E}^*[\mathbf{1}_{\{S_s \leq S_s^*\}} | S_t] ds \\ & - \delta \int_t^T e^{-r(s-t)} \mathbf{E}^*[S_s \mathbf{1}_{\{S_s \leq S_s^*\}} | S_t] ds \\ & - \lambda \int_t^T e^{-r(s-t)} \mathbf{E}^*[\{P_A(VS_{s-}, s, T) - (K - VS_{s-})\} \\ & \times \mathbf{1}_{\{S_{s-} \leq S_{s-}^*\}} \mathbf{1}_{\{VS_{s-} > S_{s-}^*\}} | S_t] ds, \end{aligned} \quad (43)$$

where $P_E(S_t, t, T)$ is the value of the corresponding European put option, $\log(V) = Y$ with an independent double exponential distribution, and S_s^* is the early exercise boundary at time s , such that if the stocks price S_s goes below S_s^* at time s , then it is optimal to exercise immediately. Gukhal (2001) provides an interpretation of the four terms in (43): the value of an American put is given by the corresponding European put option value $P_E(X, t, T)$ to which we add the present value of interest accrued on the strike price in the exercise region (IA , from now), subtract the present value of dividends lost in the exercise region (DL , from now on), and subtract the last term in (43), to be denoted by $RCJ(t, T)$, which represents the rebalancing costs due to jumps from the early exercise region to the continuation region and is absent in the case of pure-diffusion.

The term $RCJ(t, T)$ takes into account of the possibility of an upward jump that will move the asset price from the early exercise to the continuation region. Consequently, this term diminishes when the upward jump rate λp is small. Furthermore, intuitively this term should also be very small whenever a jump from the early exercise to the continuation region only causes minimal changes in the American option value, which in particular requires that the overshoot over the exercise boundary is not too large. This happens if the overshoot jump size has small mean, which in the double exponential case is $1/\eta_1$; in most practical cases $\eta_1 > 10$. In other words, the term $RCJ(t, T)$ should be negligible for either small λp or large η_1 . Indeed, Kou et al. (2005) show that for $T > t$, under the double exponential jump-diffusion model,

$$RCJ(t, T) \leq \lambda p \frac{\eta_1}{\eta_1 - 1} K \cdot U(t, T),$$

$$U(t, T) = \int_t^T \mathbb{E}^* \left[\left(\frac{S_{s^-}^*}{S_{s^-}} \right)^{-(\eta_1-1)} \mathbf{1}_{\{S_{s^-} \leq S_{s^-}^*\}} \middle| S_t \right] ds.$$

Thus, we can conclude that the term $RCJ(t, T)$ may be neglected when we have small upside jump rate λp or when the parameter η_1 is large [in which case the integrand inside $U(t, T)$ will be small], and we can ignore the term $RCJ(t, T)$ in Eq. (43) for practical usage.

Observing that at the optimal exercise boundary S_t^* , $P_A(S_t^*, t, T) = K - S_t^*$, we obtain an integral equation for S_t^*

$$K - S_t^* = P_E(S_t^*, t, T) + \int_t^T e^{-r(s-t)} r K \mathbb{E}^* [\mathbf{1}_{\{S_s \leq S_s^*\}} | S_t = S_t^*] ds$$

$$- \int_t^T e^{-r(s-t)} \delta \mathbb{E}^* [S_s \mathbf{1}_{\{S_s \leq S_s^*\}} | S_t = S_t^*] ds,$$

ignoring the term $RCJ(t, T)$. To solve this integral equation, we shall use a piecewise exponential function representation for the early exercise boundary as in Ju (1998).

More precisely, with n intervals of size $\Delta T = T/n$ we approximate the optimal boundary S_t^* by an n -piece exponential function $\tilde{S}_t = \exp(s_i^* + \alpha_i t)$ for $t \in [(i-1)\Delta T, i\Delta T)$ with $i = 1, \dots, n$. In our numerical experiments, even $n = 3$ or 5 will give sufficient accuracy in most cases. To determine the value of the constants s_i^* and α_i in each interval, we make use of the “value-matching” and “smoothing-pasting” conditions (requiring the slope at the contacting point to be -1 to make the curve smooth). Thus, starting from $i = n$ going backwards to $i = 1$ we solve recursively at $t_i = (i-1)\Delta T$ the two unknowns s_i^* and α_i in terms of the system of two equations, i.e., the value matching equation

$$K - \tilde{S}_i = P_E(\tilde{S}_i, t_i, T) + \sum_{j=i}^n IA_j(\tilde{S}_i, t_j) - \sum_{j=i}^n DL_j(\tilde{S}_i, t_j), \quad (44)$$

and the smoothing pasting equation

$$-1 = \frac{\partial}{\partial \tilde{S}_i} P_E(\tilde{S}_i, t_i, T) + \sum_{j=i}^n \frac{\partial}{\partial \tilde{S}_i} IA_j(\tilde{S}_i, t_j) - \sum_{j=i}^n \frac{\partial}{\partial \tilde{S}_i} DL_j(\tilde{S}_i, t_j), \quad (45)$$

where $\tilde{S}_i \equiv \tilde{S}_{t_i} = \exp\{s_i^* + \alpha_i t_i\}$,

$$IA_j(S_t, u) = rK \int_u^{t_{j+1}} e^{-r(s-t)} \mathbf{E}^*[\mathbf{1}_{\{S_s \leq \tilde{S}_s\}} | S_t] ds,$$

$$t \leq u, \quad u \in [t_j, t_{j+1}),$$

$$DL_j(S_t, u) = \delta \int_u^{t_{j+1}} e^{-r(s-t)} \mathbf{E}^*[S_s \mathbf{1}_{\{S_s \leq \tilde{S}_s\}} | S_t] ds,$$

$$t \leq u, \quad u \in [t_j, t_{j+1}).$$

This system of equations can be solved numerically via an iterative procedure to be specified shortly, if the right-hand sides of (44) and (45) can be computed. To this end, Kou et al. (2005) derive the Laplace transforms with respect to s_i^* of IA_j and DL_j , and the Laplace transforms of $\frac{\partial}{\partial S_t} IA_j$ and $\frac{\partial}{\partial S_t} DL_j$.

In summary, we have the following algorithm.

The Algorithm.

1. Compute the approximation exercise boundary \tilde{S} by letting i going backwards from n to 1 while, at each time point t_i one solves the system of two equations in (44) and (45) to get s_i^* and α_i , with the right-hand side of (44)

and (45) being obtained by inverting their Laplace transforms. The system of two equations can be solved, for example, by using the multi-dimensional secant method by Broydn (as implemented in Press et al., 1993).

2. After the boundary \tilde{S} is obtained, at any time $t \in [t_i, t_{i+1})$, the value of the American put option is given by

$$P_E(S_t, t, T) + IA_i(S_t, t) + \sum_{j=i+1}^n IA_j(S_t, t_j) - DL_i(S_t, t) - \sum_{j=i+1}^n DL_j(S_t, t_j).$$

In the numerical implementation, one can use the two-sided Euler algorithm in Petrella (2004) to do inversion in Step 1. The initial values for the secant method is obtained by setting $\alpha_i = 0$ and using the critical value in the approximation given by Kou and Wang (2004) as an initial value of S_i^* .

Kou et al. (2005) report the prices using a 3- and 5-piece exponential approximation of the boundary (3EXP and 5EXP, respectively), and compare the results with (i) the “true” values computed using the tree method as in Amin (1993), and (ii) the prices obtained by the analytic approximation in Kou and Wang (2004). The running time of the new algorithm is less than 2 s for 3EXP and 4 s for 5EXP, compared to more than an hour required by the Amin’s tree method. In most cases 3EXP provides an estimate of the option price more accurate than the quadratic approximation in Kou and Wang (2004), and, as we would expect, 5EXP has even better accuracy.

In summary, the quadratic approximation is easier in terms of programming effort, as it is an analytical approximation, and is faster in terms of computation time. However, the piecewise exponential approximation is more accurate.

10 Extension of the jump-diffusion models to multivariate cases

Many options traded in exchanges and in over-the-counter markets, such as two-dimensional barrier options and exchange options, have payoffs depending on more than one assets. An exchange option gives the holder the right to exchange one asset to another asset. More precisely, the payoff of an exchange option is $(S_1(T) - e^{-k}S_2(T))^+$, where e^{-k} is the ratio of the shares to be exchanged. A two-dimensional barrier option has a regular call or put payoff from one asset while the barrier crossing is determined by another asset. For example, in late 1993 Bankers Trust issued a call option on a basket of Belgian stocks which would be knocked out if the Belgian franc appreciated by more than 30% (Zhang, 1998); in this case we have a up-and-out call option. There are eight types of two-dimensional barrier options: up (down)-and-in (out) call (put) options. Mathematically, the payoff of a two-dimensional up-an-in put

barrier option is $(K - S_1(T))^+ 1_{\{\max_{0 \leq t \leq T} S_2(t) \geq H\}}$, where $S_i(t)$, $i = 1, 2$, are prices of two assets, $K > 0$ is the strike price of the put option and H is the barrier level. To price this option, it is crucial to compute the joint distribution of the first passage time

$$\mathbb{P}\left(X_T^{(1)} \leq a, \max_{0 \leq s \leq T} X_s^{(2)} \geq b\right) = \mathbb{P}\left(X_T^{(1)} \leq a, \tau_b \leq T\right),$$

where the first passage time τ_b is defined to be $\tau_b \equiv \tau_b^{(2)} := \inf\{t \geq 0: X_t^{(2)} \geq b\}$, $b > 0$. Here $X_T^{(i)} = \log(S_i(T)/S_i(0))$ is the return process for the i th asset, $i = 1, 2$.

Analytical solutions for these options are available under the classical Brownian models; see, e.g., the books by Hull (2005) and Zhang (1998). However, it becomes difficult to retain analytical tractability after jumps being introduced, partly because of the “overshoot” problem due to the possibility of jumping over the barrier. For example, it is difficult to get analytical solutions for two-dimensional barrier options under Merton’s normal jump-diffusion model.

Huang and Kou (2006) extends the previous one-dimensional double exponential jump-diffusion models by providing a multivariate jump-diffusion model with both correlated common jumps and individual jumps is proposed. The jump sizes have a multivariate asymmetric Laplace distribution (which is related but not equal to the double exponential distribution). The model not only provides a flexible framework to study correlated jumps but also is amenable for computation, especially for barrier options. Analytical solutions for the first passage time problem in two dimension are given, and analytical solutions for barrier and exchange options and other related options are also given. Compared to the one-dimensional case the two-dimensional problem poses some technical challenges. First, with both common jumps and individual jumps, the generator of the two-dimensional process becomes more involved. Second, because the joint density of the asymmetric Laplace distribution has no analytical expression, the calculation related to the joint density and generator becomes complicated. Third, one has to use several uniform integrability arguments to substantiate a martingale argument, as Itô’s formula cannot be applied directly due to discontinuity.

10.1 Asymmetric Laplace distribution

The common jumps in the multivariate jump-diffusion model to be introduced next will have a multivariate asymmetric Laplace distribution. An n -dimensional asymmetric Laplace random vector Y , denoted by $Y \sim \mathcal{AL}_n(m, J)$, is defined via its characteristic function

$$\Psi_Y(\theta) = \mathbb{E}[e^{i\theta'Y}] = \frac{1}{1 + \frac{1}{2}\theta'J\theta - \text{im}'\theta}, \quad (46)$$

where $m \in R^n$ and J is an $n \times n$ positive definite symmetric matrix. The requirement of the matrix J being positive definite is postulated to guarantee that the n -dimensional distribution is nondegenerate; otherwise, the dimension of the distribution may be less than n . The vector m is the mean $E[Y] = m$ and the matrix J plays a role similar to that of the variance and covariance matrix.

In the case of the univariate Laplace distribution, the characteristic function in (46) becomes

$$\Psi_Y(\theta) = \frac{1}{1 + \frac{1}{2}v^2\theta^2 - \text{im } \theta}, \quad (47)$$

where v^2 is the equivalence of J in (46). For further information about the asymmetric Laplace distribution, see Kotz et al. (2001).

The asymmetric Laplace distribution has many properties similar to those of the multivariate normal distribution. This can be easily seen from the fact that

$$Y \stackrel{d}{=} mB + B^{1/2}Z, \quad (48)$$

where $Z \sim N_n(0, J)$ is a multivariate normal distribution with mean 0 and covariance matrix J , and B is a one-dimensional exponential random variable with mean 1, independent of Z . For example, for the k th component of Y we have $Y^{(k)} \stackrel{d}{=} m_k B + B^{1/2}Z_k$ with $B \sim \exp(1)$ and $Z_k \sim N(0, J_{kk})$, which implies that the marginal distribution of $Y^{(k)}$ has a univariate asymmetric Laplace distribution. Furthermore, the difference between any two components,

$$Y^{(k)} - Y^{(j)} \stackrel{d}{=} (m_k - m_j)B + B^{1/2}(Z_k - Z_j), \quad 1 \leq k, j \leq n, \quad (49)$$

is again a univariate Laplace distribution. However, it is worth mentioning $Y + a$ does not have the asymmetric Laplace distribution, for $a \neq 0$.

The univariate asymmetric Laplace distribution defined by its characteristic function in (47) is a special case of the double exponential distribution, because the univariate asymmetric Laplace distribution has the density function

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y < 0\}},$$

$$p > 0, q > 0, p + q = 1,$$

but with $p\eta_1 = q\eta_2$ and the parameters given by

$$\eta_1 = \frac{2}{\sqrt{m^2 + 2v^2 + m}}, \quad \eta_2 = \frac{2}{\sqrt{m^2 + 2v^2 - m}},$$

$$p = \frac{\sqrt{m^2 + 2v^2 + m}}{2\sqrt{m^2 + 2v^2}}. \quad (50)$$

Asymmetric Laplace distribution can also be viewed as a special case of the generalized hyperbolic distribution introduced by Barndorff-Nielsen (1977). In

fact, a generalized hyperbolic random variable X is defined as $X \stackrel{d}{=} \mu + m\zeta + \zeta^{1/2}Z$, where Z is a multivariate normal distribution, ζ is a generalized inverse Gaussian distribution. Since the exponential random variable belongs to generalized inverse Gaussian distribution, the asymmetric Laplace distribution is a special case of the generalized hyperbolic distribution. For more details on applications of the generalized hyperbolic distribution in finance, see Eberlein and Prause (2002).

10.2 A multivariate jump-diffusion model

We propose a multivariate jump-diffusion model in which the asset prices $S(t)$ have two parts, a continuous part driven by a multivariate geometric Brownian motion, and a jump part with jump events modeled by a Poisson process. In the model, there are both common jumps and individual jumps. More precisely, if a Poisson event corresponds to a common jump, then all the asset prices will jump according to the multivariate asymmetric Laplace distribution; otherwise, if a Poisson event corresponds to an individual jump of the j th asset, then only the j th asset will jump. In other words, the model attempts to capture various ways of correlated jumps in asset prices.

Mathematically, under the physical measure \mathbb{P} the following stochastic differential equation is proposed to model the asset prices $S(t)$:

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right), \tag{51}$$

where $W(t)$ is an n -dimensional standard Brownian motion, $\sigma \in R^{n \times n}$ with the covariance matrix $\Sigma = \sigma\sigma^T$. The rate of the Poisson process $N(t)$ process is $\lambda = \lambda_c + \sum_{k=1}^n \lambda_k$; in other words, there are two types of jumps, common jumps for all assets with jump rate λ_c and individual jumps with rate λ_k , $1 \leq k \leq n$, only for the k th asset.

The logarithms of the common jumps have an m -dimensional asymmetric Laplace distribution $\mathcal{AL}_n(m_c, J_c)$, where $m_c = (m_{1,c}, \dots, m_{n,c})' \in R^n$ and $J_c \in R^{n \times n}$ is positive definite. For the individual jumps of the k th asset, the logarithms of the jump sizes follow a one-dimensional asymmetric Laplace distribution, $\mathcal{AL}_1(m_k, v_k^2)$. In summary

$$Y = \log(V) \sim \begin{cases} \mathcal{AL}_n(m_c, J_c), & \text{with prob. } \lambda_c/\lambda, \\ \underbrace{(0, \dots, 0)}_{k-1}, \mathcal{AL}_1(m_k, v_k^2), \underbrace{0, \dots, 0}_{n-k}, & \\ \text{with prob. } \lambda_k/\lambda, & 1 \leq k \leq n. \end{cases}$$

The sources of randomness, $N(t)$, $W(t)$ are assumed to be independent of the jump sizes V_i s. Jumps at different times are assumed to be independent.

Note that in the univariate case the above model degenerates to the double exponential jump-diffusion model (Kou, 2002) but with $p\eta_1 = q\eta_2$.

Solving the stochastic differential equation in (51) gives the dynamic of the asset prices:

$$S(t) = S(0) \exp \left[\left(\mu - \frac{1}{2} \Sigma_{\text{diag}} \right) t + \sigma W(t) \right] \prod_{i=1}^{N(t)} V_i, \tag{52}$$

where Σ_{diag} denotes the diagonal vector of Σ . Note that $\forall 1 \leq k \leq n$,

$$E(V^{(k)}) = E(e^{Y^{(k)}}) = \frac{\lambda_c/\lambda}{1 - m_{k,c} - J_{c,kk}/2} + \frac{\lambda_k/\lambda}{1 - m_k - v_k^2/2}. \tag{53}$$

The requirements $m_{k,c} + J_{c,kk}/2 < 1$ and $m_k + v_k^2/2 < 1$ are needed to ensure $E(V^{(k)}) < \infty$ and $E(S_k(t)) < \infty$, i.e. the stock price has finite expectation.

In the special case of two-dimension, the asset prices can be written as

$$\begin{aligned} S_1(t) &= S_1(0) \exp \left[\mu_1 t + \sigma_1 W_1(t) + \sum_{i=1}^{N(t)} Y_i^{(1)} \right], \\ S_2(t) &= S_2(0) \exp \left[\mu_2 t + \sigma_2 [\rho W_1(t) + \sqrt{1 - \rho^2} W_2(t)] + \sum_{i=1}^{N(t)} Y_i^{(2)} \right]. \end{aligned} \tag{54}$$

Here all the parameters are risk-neutral parameters, $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motions, and $N(t)$ is a Poisson process with rate $\lambda = \lambda_c + \lambda_1 + \lambda_2$. The distribution of the logarithm of the jump sizes Y_i is given by

$$Y_i = (Y_i^{(1)}, Y_i^{(2)})' \sim \begin{cases} \mathcal{AL}_2(m_c, J_c), & \text{with prob. } \lambda_c/\lambda, \\ (\mathcal{AL}_1(m_1, v_1^2), 0)', & \text{with prob. } \lambda_1/\lambda, \\ (0, \mathcal{AL}_1(m_2, v_2^2))', & \text{with prob. } \lambda_2/\lambda, \end{cases} \tag{55}$$

where the parameters for the common jumps are

$$m_c = \begin{pmatrix} m_{1,c} \\ m_{2,c} \end{pmatrix}, \quad J_c = \begin{pmatrix} v_{1,c}^2 & cv_{1,c}v_{2,c} \\ cv_{1,c}v_{2,c} & v_{2,c}^2 \end{pmatrix}.$$

Since $S(t)$ is a Markov process, an alternative characterization of $S(t)$ is to use the generator of $X(t) = \log S(t)/S(0)$. The two-dimensional jump-diffusion return process $(X_1(t), X_2(t))$ in (54) is given by

$$\begin{aligned} X_1(t) &= \mu_1 t + \sigma_1 W_1(t) + \sum_{i=1}^{N(t)} Y_i^{(1)}, \\ X_2(t) &= \mu_2 t + \sigma_2 [\rho W_1(t) + \sqrt{1 - \rho^2} W_2(t)] + \sum_{i=1}^{N(t)} Y_i^{(2)}, \end{aligned}$$

with the infinitesimal generator

$$\begin{aligned}
 \mathcal{L}u &= \mu_1 \frac{\partial u}{\partial x_1} + \mu_2 \frac{\partial u}{\partial x_2} + \frac{1}{2} \sigma_1^2 \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2} \sigma_2^2 \frac{\partial^2 u}{\partial x_2^2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \\
 &+ \lambda_c \int_{y_2=-\infty}^{\infty} \int_{y_1=-\infty}^{\infty} [u(x_1 + y_1, x_2 + y_2) - u(x_1, x_2)] \\
 &\times f_{(Y^{(1)}, Y^{(2)})}^c(y_1, y_2) dy_1 dy_2 \\
 &+ \lambda_1 \int_{y_1=-\infty}^{\infty} [u(x_1 + y_1, x_2) - u(x_1, x_2)] f_{Y^{(1)}}(y_1) dy_1 \\
 &+ \lambda_2 \int_{y_2=-\infty}^{\infty} [u(x_1, x_2 + y_2) - u(x_1, x_2)] f_{Y^{(2)}}(y_2) dy_2, \quad (56)
 \end{aligned}$$

for all continuously twice differentiable function $u(x_1, x_2)$, where $f_{(Y^{(1)}, Y^{(2)})}^c(y_1, y_2)$ is the joint density of correlated common jumps $\mathcal{AL}_2(m_c, J_c)$, and $f_{Y^{(i)}}(y_i)$ is the individual jump density of $\mathcal{AL}_1(m_i, J_i)$, $i = 1, 2$.

One difficulty in studying the generator is that the joint density of the asymmetric Laplace distribution has no analytical expression. Therefore, the calculation related to the joint density and generator becomes complicated. See [Huang and Kou \(2006\)](#) for change of measures from a physical measure to a risk-neutral measure, analytical solutions for the first passage times, and pricing formulae for barrier options and exchange options.

References

- Abate, J., Whitt, W. (1992). The Fourier series method for inverting transforms of probability distributions. *Queueing Systems* 10, 5–88.
- Amin, K. (1993). Jump-diffusion option valuation in discrete time. *Journal of Finance* 48, 1833–1863.
- Artzner, P., Delbaen, F., Eber, J.-M., Heath, D. (1999). Coherent measures of risk. *Mathematical Finance* 9, 203–228.
- Asmussen, S., Avram, F., Pistorius, M.R. (2004). Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and Their Applications* 109, 79–111.
- Avram, F., Kyprianou, A.E., Pistorius, M.R. (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Annals of Applied Probability* 14, 215–238.
- Barberis, N., Shleifer, A., Vishny, R. (1998). A model of investor sentiment. *Journal of Financial Economics* 49, 307–343.
- Barone-Adesi, G., Whaley, R.E. (1987). Efficient analytic approximation of American option values. *Journal of Finance* 42, 301–320.
- Barndorff-Nielsen, O. (1977). Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society of London Series A* 353, 409–419.
- Barndorff-Nielsen, O.E., Shephard, N. (2001). Non-Gaussian Ornstein–Uhlenbeck based models and some of their uses in financial economics (with discussion). *Journal of Royal Statistical Society Series B* 63, 167–241.

- Blattberg, R.C., Gonedes, N.J. (1974). A comparison of the stable and Student distributions as statistical models for stock prices. *Journal of Business* 47, 244–280.
- Boyarchenko, S., Levendorskiĭ, S. (2002). Barrier options and touch-and-out options under regular Lévy processes of exponential type. *Annals of Applied Probability* 12, 1261–1298.
- Boyle, P., Broadie, M., Glasserman, P. (1997). Simulation methods for security pricing. *Journal of Economic Dynamics and Control* 21, 1267–1321.
- Carr, P., Madan, D.B. (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance* 2, 61–73.
- Carr, P., Jarrow, R., Myneni, R. (1992). Alternative characterizations of American puts. *Mathematical Finance* 2, 87–106.
- Carr, P., Geman, H., Madan, D., Yor, M. (2003). Stochastic volatility for Lévy processes. *Mathematical Finance* 13, 345–382.
- Chen, N., Kou, S.G., 2005. Credit spreads, optimal capital structure, and implied volatility with endogenous default and jump risk. *Mathematical Finance*, in press.
- Choudhury, G.L., Lucantoni, D.M., Whitt, W. (1994). Multidimensional transform inversion with applications to the transient M/M/1 queue. *Annals of Applied Probability* 4, 719–740.
- Clark, P.K. (1973). A subordinated stochastic process model with finite variance for speculative prices. *Econometrica* 41, 135–155.
- Cont, R., Tankov, P. (2004). *Financial Modelling with Jump Processes*, second printing. Chapman and Hall/CRC Press, London.
- Cont, R., Voltchkova, E. (2005). Finite difference methods for option pricing in jump-diffusion and exponential Lévy models. *SIAM Journal of Numerical Analysis* 43, 1596–1626.
- Cox, J.T., Ross, S.A. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics* 3, 145–166.
- Craddock, M., Heath, D., Platen, E. (2000). Numerical inversion of Laplace transforms: A survey of techniques with applications to derivative pricing. *Journal of Computational Finance* 4, 57–81.
- Das, S.R., Foresi, S. (1996). Exact solutions for bond and option prices with systematic jump risk. *Review of Derivatives Research* 1, 7–24.
- Davydov, D., Linetsky, V. (2001). Pricing and hedging path-dependent options under the CEV process. *Management Science* 47, 949–965.
- Derman, E., Kani, I. (1994). Riding on a smile. *RISK* Feb. 32–39.
- d’Halluin, Y., Forsyth, P.A., Vetzal, K.R. (2003). Robust numerical methods for contingent claims under jump-diffusion processes. *Working paper*, University of Waterloo.
- Duffie, D., Pan, J., Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* 68, 1343–1376.
- Dupire, B. (1994). Pricing with a smile. *RISK* Feb. 18–20.
- Eberlein, E., Prause, K. (2002). The generalized hyperbolic model: Financial derivatives and risk measures. In: Geman, H., Madan, D., Pliska, S., Vorst, T. (Eds.), *Mathematical Finance-Bachelor Congress 2000*. Springer-Verlag, New York, pp. 245–267.
- Engle, R. (1995). *ARCH, Selected Readings*. Oxford Univ. Press, Oxford, UK.
- Fama, E. (1998). Market efficiency, long-term returns, and behavioral finance. *Journal of Financial Economics* 49, 283–306.
- Feng, L., Linetsky, V. (2005). Pricing options in jump-diffusion models: An extrapolation approach. *Operations Research*, in press.
- Feng, L., Linetsky, V., Marozzi, M. (2004). On the valuation of options in jump-diffusion models by variational methods. *Preprint*, Northwestern University.
- Fouque, J.-P., Papanicolaou, G., Sircar, K.R. (2000). *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge Univ. Press, Cambridge, UK.
- Fu, M., Madan, D., Wang, T. (1999). Pricing continuous Asian options: A comparison of Monte Carlo and Laplace transform inversion methods. *Journal of Computational Finance* 2, 49–74.
- Gatto, M., Goldman, M.B., Sosin, H. (1979). Path dependent options: “Buy at the low, sell at the high”. *Journal of Finance* 34, 1111–1127.
- Geman, H., Yor, M. (1993). Bessel processes, Asian options and perpetuities. *Mathematical Finance* 3, 349–375.

- Glasserman, P., Kou, S.G. (2003). The term structure of simple forward rates with jump risk. *Mathematical Finance* 13, 383–410.
- Gukhal, C.R. (2001). Analytical valuation of American options on jump-diffusion processes. *Mathematical Finance* 11, 97–115.
- Haug, E.G. (1999). Barrier put–call transformations. *Working paper*, Tempus Financial Engineering.
- Heston, S. (1993). A closed-form solution of options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* 6, 327–343.
- Heyde, C.C. (2000). A risky asset model with strong dependence through fractal activity time. *Journal of Applied Probability* 36, 1234–1239.
- Heyde, C.C., Kou, S.G. (2004). On the controversy over tailweight of distributions. *Operations Research Letters* 32, 399–408.
- Heyde, C.C., Kou, S.G., Peng, X.H. (2006). What is a good risk measure: Bridging the gaps between robustness, subadditivity, prospect theory, and insurance risk measures. *Preprint*, Columbia University.
- Huang, Z., Kou, S.G. (2006). First passage times and analytical solutions for options on two assets with jump risk. *Preprint*, Columbia University.
- Hull, J. (2005). *Options, Futures, and Other Derivatives Securities*, sixth ed. Prentice Hall, New Jersey.
- Hull, J., White, A. (1987). The pricing of options on assets with stochastic volatilities. *Journal of Finance* 42, 281–300.
- Johnson, N., Kotz, S., Balakrishnan, N. (1995). *Continuous Univariate Distribution*, vol. 2, second ed. Wiley, New York.
- Ju, N. (1998). Pricing an American option by approximating its early exercise boundary as a multipiece exponential function. *Review of Financial Studies* 11, 627–646.
- Karatzas, I., Shreve, S. (1991). *Brownian Motion and Stochastic Calculus*. Springer-Verlag, New York.
- Karlin, S., Taylor, H. (1975). *A First Course in Stochastic Processes*, second ed. Academic Press, New York.
- Kijima, M. (2002). *Stochastic Processes with Applications to Finance*. Chapman and Hall, London.
- Kotz, S., Kozubowski, T.J., Podgórski, K. (2001). *The Laplace Distribution and Generalization: A Revisit with Applications to Communications, Economics, Engineering and Finance*. Birkhäuser, Boston.
- Kou, S.G. (2002). A jump-diffusion model for option pricing. *Management Science* 48, 1086–1101.
- Kou, S.G., Wang, H. (2003). First passage time of a jump diffusion process. *Advances in Applied Probability* 35, 504–531.
- Kou, S.G., Wang, H. (2004). Option pricing under a double exponential jump-diffusion model. *Management Science* 50, 1178–1192.
- Kou, S.G., Petrella, G., Wang, H. (2005). Pricing path-dependent options with jump risk via Laplace transforms. *Kyoto Economic Review* 74, 1–23.
- Kyprianou, A., Pistorius, M. (2003). Perpetual options and Canadization through fluctuation theory. *Annals of Applied Probability* 13, 1077–1098.
- Lucas, R.E. (1978). Asset prices in an exchange economy. *Econometrica* 46, 1429–1445.
- MacBeth, J., Merville, L. (1979). An empirical examination of the Black–Scholes call option pricing model. *Journal of Finance* 34, 1173–1186.
- Madan, D.B., Seneta, E. (1990). The variance gamma (V.G.) model for share market returns. *Journal of Business* 63, 511–524.
- Madan, D.B., Carr, P., Chang, E.C. (1998). The variance gamma process and option pricing. *European Finance Review* 2, 79–105.
- Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business* 36, 394–419.
- Merton, R.C. (1973). The theory of rational option pricing. *Bell Journal of Economics and Management Science* 4, 141–183.
- Merton, R.C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3, 125–144.
- Naik, V., Lee, M. (1990). General equilibrium pricing of options on the market portfolio with discontinuous returns. *Review of Financial Studies* 3, 493–521.
- Petrella, G. (2004). An extension of the Euler Laplace transform inversion algorithm with applications in option pricing. *Operations Research Letters* 32, 380–389.

- Petrella, G., Kou, S.G. (2004). Numerical pricing of discrete barrier and lookback options via Laplace transforms. *Journal of Computational Finance* 8, 1–37.
- Pham, H. (1997). Optimal stopping, free boundary and American option in a jump-diffusion model. *Applied Mathematics and Optimization* 35, 145–164.
- Press, W., Teukolsky, S., Vetterling, W., Flannery, B. (1993). *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge Univ. Press, New York.
- Ramezani, C.A., Zeng, Y. (2002). Maximum likelihood estimation of asymmetric jump-diffusion process: Application to security prices. *Working paper*, University of Missouri at Kansas City, Department of Mathematics and Statistics.
- Rogers, L.C.G. (1997). Arbitrage from fractional Brownian motion. *Mathematical Finance* 7, 95–105.
- Rogers, L.C.G. (2000). Evaluating first-passage probabilities for spectrally one-sided Lévy processes. *Journal of Applied Probability* 37, 1173–1180.
- Rubinstein, M. (1985). Nonparametric tests of alternative option pricing models using all reported trades and quotes on the 30 most active CBOE option classes from August 23, 1976 through August 31, 1978. *Journal of Finance* 40, 455–480.
- Samorodnitsky, G., Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman and Hall, New York.
- Schröder, M. (1999). Changes of numéraire for pricing futures, forwards and options. *Review of Financial Studies* 12, 1143–1163.
- Sepp, A. (2004). Analytical pricing of double-barrier options under a double exponential jump diffusion process: Applications of Laplace transform. *International Journal of Theoretical and Applied Finance* 7, 151–175.
- Singleton, K. (2006). *Empirical Dynamic Asset Pricing*. Princeton Univ. Press, Princeton, NJ.
- Stokey, N.L., Lucas, R.E. (1989). *Recursive Methods in Economic Dynamics*. Harvard Univ. Press, Cambridge, MA.
- Zhang, P.G. (1998). *Exotic Options*, second ed. World Scientific, Singapore.