Chapter IV
Thermal Instability in the Expanding Universe

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In the expanding universe, the thermal instability, if it occurs, can develop much more quickly than the gravitational instability and, therefore, may provide a possible mechanism to initiate the formation of galaxies. The general characteristics of thermal instability are discussed for non-equilibrium media with due regard to the ionization change and the optical depth effect of fluctuations.

If the matter can be brought to a temperature high enough to ionize hydrogen after 10^7 years of the cosmic age, the energy loss from the matter becomes mainly due to the bound-free processes rather than the Compton scattering and the thermal instability will be set up. Such a high temperature ($\sim 10^4$ °K) is excluded in the thermal history of the expanding universe even if the effect of hydrodynamic turbulence is taken into account, since the inverse Compton loss it extremely large just after the epoch of the recombination of hydrogen when the turbulence decays rather rapidly. There is, however, a possibility that the turbulent energy is stored in the magnetic field and in cosmic-ray particles at the epoch of hydrogen recombination and released later ($\sim 10^7$ years) when the inverse Compton loss becomes less efficient.

§1. Significance of thermal instability

Galaxies may be formed from fluctuations in the expanding universe. Statistical density fluctuations of thermal origin, however, are very small, for example, 10^{-35} for the scale of galaxies.[1] Moreover, the growth rate of fluctuations for the case of gravitational instability in the flat-Friedmann universe is proportional to $t^{2/3}$ (t is the age of the universe) and is very slow.[2] If the nonlinear acceleration is considered, density fluctuations at 10^5 years with the relative amplitude larger than 10^{-3} are estimated to reach distinct condensations within the cosmic age of 10^{10} years.[3),4]

Two types of origins may be considered for these initial fluctuations; one is the primordial one and the other is due to the thermal instability which amplifies fluctuations. The primordial density fluctuation, however, cannot survive until the epoch of the recombination of hydrogen if the mass is less than $10^{12}M_\odot$, and in addition they suffer strong damping at the stage of the recombination of hydrogen.[5),6] This means that some mechanism of excitating fluctuations is needed after the recombination for the formation of galaxies. Then, the thermal instability may provide a suitable mechanism,

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because the growth rate of the amplification is possibly large when the energy exchange between matter and radiation exists.

Now, we consider that the gravitational contraction proceeds at the time scale of free fall if disturbances of finite amplitude are triggered by the thermal instability in larger sizes than the Jeans wave length. On the other hand, since the time scale of thermal instability is characterized by the length scale and the speed of sound, a disturbance of smaller size has larger growth rate.[7] Therefore, we expect that clouds or grobules first formed by thermal instability may be much less massive than galaxies, and that statistical fluctuation of an ensemble of grobules may be large enough for larger masses than the Jeans mass to condense into proto-galaxies. If grobules are well stirred by the existing turbulence, the root-mean-square density fluctuation of an ensemble of N grobules amounts to be $N^{-1/2}$ which should be larger than 10^{-3} at the cosmic age of 10^5 years.

The above considerations lead us to conclude that

$$N \leq 10^6 \tag{1·1}$$

and

$$NM_{\mathrm{gr}} \geq M_J\,, \tag{1·2}$$

where M_{gr} and M_J denote the mass of a grobule and the Jeans mass, respectively.

Thus, we have

$$M_{\mathrm{gr}} \geq 10^{-6} M_J\,. \tag{1·3}$$

The value of M_J is provided by the balance between the gravitational energy and the turbulent kinetic energy of grobules so that

$$\frac{GM_J}{\left(M_J \Big/ \frac{4\pi}{3}\rho\right)^{1/3}} = \frac{1}{2} v_{\mathrm{gr}}^2\,, \tag{1·4}$$

where v_{gr} denotes the random turbulent velocity of grobule.[8] The value v_{gr} depends in general on the length scale concerned. Referring to the peculiar velocities of galaxies, we may take v_{gr} to be 10^4 km/sec at 10^5 years. Then, the Jeans mass M_J is estimated to be about $10^{12} M_\odot$ which is of the order of mass of a galaxy, if ρ in Eq. (1·4) is put to 10^{-23} g/cm^3. From inequality (1·3), we have $M_{\mathrm{gr}} \geq 10^6 M_\odot$.

In the following sections, we shall study whether grobules of mass larger than $10^6 M_\odot$ could be formed by the action of the thermal instability in the expanding universe.

§2. Thermal instability in the expanding universe

Radiative processes provide interactions between matter and radiation which consist the universe. If fluctuations of macroscopic thermodynamical quantities of matter are excited by the action of radiative processes, the phenomena are called thermal instability. The most important factor characterising thermal instability is the functional dependence of the energy loss or gain of matter through radiative processes on the thermodynamical quantities.[9] However, the expansion of the universe gives rise to a number of effects in the problem to which we shall now turn to study.

To begin with, we consider the general equations which matter and radiation must be obeyed under the Newtonian approach. In the case of the size and density of galaxies, the Newtonian treatment is sufficient. Thermodynamical quantities of matter are governed by the following equations of continuity of mass, momentum and energy, the equations of state and the degree of ionization.

$$\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})=0, \tag{2·1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}+\frac{1}{\rho}\boldsymbol{\nabla}p+\boldsymbol{\nabla}\phi=-\mathscr{P}, \tag{2·2}$$

$$\left[\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\right]U+p\left[\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\right]\left(\frac{1}{\rho}\right)$$
$$=-\mathscr{L}+\boldsymbol{\nabla}(K\cdot\boldsymbol{\nabla}T), \tag{2·3}$$

$$p=N_A(1+x)\rho kT \tag{2·4}$$

and

$$N_A\frac{dx}{dt}=\mathscr{I}, \tag{2·5}$$

where ρ, p, T, U, $\boldsymbol{v}$, ϕ, $\mathscr{L}$, $\mathscr{P}$, K, x, $\mathscr{I}$ and N_A are, respectively, density, pressure, matter temperature, internal energy per unit mass, velocity, gravitational potential, energy and momentum loss of matter per unit mass, thermal conductivity, degree of ionization, the function of ionization rate and the Avogadro number. Here, the internal energy U is composed of thermal energy and ionization energy:

$$U=N_A\left\{\frac{1+x}{\gamma-1}kT+\chi x\right\}=\frac{1}{\gamma-1}\frac{p}{\rho}+N_A\chi x, \tag{2·6}$$

where χ is the ionization potential of the atom.

The energy loss of matter $\mathscr{L}$ is due to the radiative processes and mechanical ones. If the former is denoted by $\mathscr{R}$ and the latter by $\mathscr{H}$,

$$\mathscr{L}=\mathscr{R}+\mathscr{H}. \tag{2·7}$$

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$\mathcal{R}$ now should be an integral of energy loss of matter $\mathcal{R}_\nu$ due to the radiative processes at the frequency of ν over the whole range of frequency of photon,

$$\mathcal{R}=\int_0^\infty \mathcal{R}_\nu d\nu, \tag{2·8}$$

and $\mathcal{R}_\nu$ is given by

$$\mathcal{R}_\nu=-\kappa_\nu J_\nu+\varepsilon_\nu\,, \tag{2·9}$$

where J_ν is the mean intensity at the frequency of ν, given by $J_\nu=1/4\pi\cdot\int I_\nu d\Omega$ (I_ν is the intensity of a beam in the direction of $\boldsymbol{\mu}$ and Ω is the solid angle), κ_ν is the mass absorption coefficient and ε_ν the emission rate per unit mass.

The momentum loss of $\mathcal{P}$ is mainly due to radiative processes in the expanding universe, so that $\mathcal{P}$ is the integral of momentum exchange between matter and radiation $\mathcal{P}_\nu$ over the whole range of frequency of photon,

$$\mathcal{P}=\int_0^\infty \mathcal{P}_\nu d\nu, \tag{2·10}$$

and $\mathcal{P}_\nu$ is given by

$$\mathcal{P}_\nu=-\frac{\kappa_\nu+\sigma_\nu}{c}\boldsymbol{F}_\nu\,, \tag{2·11}$$

where $\boldsymbol{F}_\nu$ is the radiative flux given by $1/\pi\cdot\int\boldsymbol{\mu}I_\nu d\Omega$, σ_ν the scattering coefficient, and c the light velocity.

And then, we consider the radiation field at the frequency of ν. The mean intensity J_ν and the radiative flux $\boldsymbol{F}_\nu$ are determined by the following equations:[10]

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+\frac{1}{a}\frac{da}{dt}\right)J_\nu+\frac{1}{4}\boldsymbol{\nabla}\cdot\boldsymbol{F}_\nu=\rho\mathcal{R}_\nu \tag{2·12}$$

and

$$\frac{1}{c}\left(\frac{\partial}{\partial t}+\frac{1}{a}\frac{da}{dt}\right)\boldsymbol{F}_\nu+4\boldsymbol{\nabla}\cdot\mathbf{K}_\nu=-\rho(\kappa_\nu+\sigma_\nu)\boldsymbol{F}_\nu\,, \tag{2·13}$$

where $\mathbf{K}_\nu$ is the radiative stress tensor defined by $1/4\pi\cdot\int\boldsymbol{\mu}\boldsymbol{\mu}I_\nu d\Omega$. We now use the Eddington approximation which is known to be very optional;[11]

$$\mathbf{K}_\nu=\frac{1}{3}J_\nu\boldsymbol{I}, \tag{2·14}$$

where $\boldsymbol{I}$ is the unit tensor.

Next, we rewrite Eqs. (2·1)～(2·5), (2·12) and (2·13), in terms of normalized variables by unperturbed quantities. We assume the unperturbed

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state of the universe to be isotropic and homogeneous. In the matter dominant stage, we have the relation that $\rho_0 a^3=$const., where a is a scale factor and ρ_0 is the unperturbed density. From the standpoint of the Newtonian model, the medium expands from the center of an observer at the speed of $\boldsymbol{v}_0=(1/a)(da/dt)\boldsymbol{x}$ where $\boldsymbol{x}$ is a position vector. The time scale of the expansion is defined as $\tau_{\rm ex}=(3/a\cdot da/dt)^{-1}$ which is related to the time scale of free fall $\tau_{\rm f}^{(0)}=(4\pi G\rho_0)^{-1/2}$ as $\tau_{\rm ex}=1/\sqrt{6}\cdot\tau_{\rm f}^{(0)}$ for the case of the flat space.[12] The radiation field of the unperturbed state is expressed by $\boldsymbol{F}_\nu^{(0)}=0$.

We now measure time and distance in units of $\tau_{\rm f}^{(0)}$ and a, respectively. Then we obtain the following equations which are satisfied by the physical quantities normalized by that of the unperturbed state:[13]

$$\frac{\partial\rho^*}{\partial\tau}+\boldsymbol{\nabla}\cdot(\rho^*\boldsymbol{v}^*)=0, \tag{2·15}$$

$$\left[\frac{\partial}{\partial t}+(\boldsymbol{v}^*\cdot\boldsymbol{\nabla})\right]\boldsymbol{v}^*+\alpha\boldsymbol{v}^*+\frac{1}{k_T^2\rho^*}\boldsymbol{\nabla}p^*+\boldsymbol{\nabla}\phi^*=\frac{v_{\rm f}}{ck_T^2}(\kappa^*+\sigma^*)\boldsymbol{F}^*, \tag{2·16}$$

$$\frac{1}{\gamma-1}\left[\frac{\partial}{\partial\tau}+(\boldsymbol{v}^*\cdot\boldsymbol{\nabla})\right]\ln\frac{p^*}{\rho^{*\gamma}}=-\frac{\rho^*}{p^*}\{\Lambda^*+\boldsymbol{\nabla}(K^*\boldsymbol{\nabla}T^*)\}+\Lambda_0^*, \tag{2·17}$$

$$p^*=\mu^*\rho^*T^*, \tag{2·18}$$

$$\left[\frac{\chi}{kT_0}\frac{d}{d\tau}+\mathcal{J}_0^*\right]\mu^*=\mathcal{J}^*, \tag{2·19}$$

$$\frac{1}{c}\left[\frac{\partial}{\partial\tau}-(\gamma-1)(\sqrt{6}+\Lambda_0^*)\right]J_\nu^*+\frac{1}{4v_{\rm f}}\boldsymbol{\nabla}\cdot\boldsymbol{F}_\nu^*=\rho^*\mathcal{R}_\nu^* \tag{2·20}$$

and

$$\frac{1}{c}\left[\frac{\partial}{\partial\tau}-(\gamma-1)(\sqrt{6}+\Lambda_0^*)\right]\boldsymbol{F}_\nu^*+\frac{4}{v_{\rm f}}\boldsymbol{\nabla}J_\nu^*=-\rho^*(\kappa^*+\sigma^*)\boldsymbol{F}_\nu^*, \tag{2·21}$$

where

$$\mathcal{R}_\nu^*=-\kappa_\nu^*J_\nu^*+\varepsilon_\nu^*. \tag{2·22}$$

The differentiations with time and space are modified as follows:

$$d\tau=\frac{dt}{\tau_{\rm f}^{(0)}}\quad\text{and}\quad\boldsymbol{\nabla}=a\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right). \tag{2·23}$$

The variable with asterisk are the physical quantities relative to the unperturbed quantities designated by the suffix 0 or to the free-fall velocity $v_{\rm f}$ and $v_{\rm f}^2$;

$$\rho^*=\rho/\rho_0,\quad p^*=p/p_0,\quad T^*=T/T_0,\quad \boldsymbol{v}^*=(\boldsymbol{v}-\boldsymbol{v}_0)/v_{\rm f},$$

$$\phi^*=(\phi-\phi_0)/v_{\rm f}^2,\quad \mu^*=\frac{1+x}{1+x_0},\quad J_\nu^*=J_\nu/p_0$$

and

$$\boldsymbol{F}_\nu^*=\boldsymbol{F}_\nu/p_0. \tag{2·24}$$

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In addition, the following functions with asterisk are defined,

$$\Lambda^* = \mathcal{L}^* + \mathcal{I}^*, \quad \Lambda_0^* = \mathcal{L}_0^* + \mathcal{I}_0^*,$$

$$\mathcal{L}^* = \tau_{\rm f}^{(0)} \frac{\rho_0}{p_0} \mathcal{L}, \quad \mathcal{I}^* = \tau_{\rm f}^{(0)} \frac{\chi \rho_0}{p_0} \mathcal{I}, \quad \mathcal{L}_0^* = \tau_{\rm f}^{(0)} \frac{\rho_0}{p_0} \mathcal{L}_0, \quad \mathcal{I}_0^* = \tau_{\rm f}^{(0)} \frac{\chi \rho_0}{p_0} \mathcal{I}_0,$$

$$\mathcal{R}^* = \tau_{\rm f}^{(0)} \frac{\rho_0}{p_0} \mathcal{R}, \quad K^* = \frac{\tau_{\rm f}^{(0)}}{N_{\rm A}(1+x)ka^2} K,$$

$$\varepsilon_\nu^* = \tau_{\rm f}^{(0)} \frac{\rho_0}{p_0} \varepsilon, \quad \kappa_\nu^* = \rho_0 \tau_{\rm f}^{(0)} \kappa \quad \text{and} \quad \sigma_\nu^* = \rho_0 \tau_{\rm f}^{(0)} \sigma_\nu . \tag{2·25}$$

In the momentum equation of (2·16), the term $\alpha \boldsymbol{v}^*$ representing the drag force appears. The resistance coefficient α is derived from two effects, the expansion of the universe and the slipping of matter through radiation field.[14] The former is given by $\alpha_m = \tau_{\rm f}^{(0)}/6\tau_{\rm ex} = 1/\sqrt{6}$ [3] and the latter by $\alpha_r = \tau_{\rm f}^{(0)}/\tau_{\rm TS}$, where $\tau_{\rm TS}$ is the mean-free time of the Thomson scattering $(4\sigma a_r T_r^4 x_e / 3m_{\rm H} c)^{-1}$ (σ is the Thomson scattering cross section, a_r the radiation constant, x_e the fraction of electrons, c the light velocity), and so $\alpha = \alpha_m + \alpha_r$. The other effect of expansion is the time variation of the isothermal Jeans wave length $k_T = (p_0/4\pi G \rho_0^2 a^2)^{1/2}$, appearing in the momentum equation (2·16);

$$\frac{d}{d\tau} \ln k_T^2 = \sqrt{6}(\gamma - 4/3) + (\gamma - 1)\Lambda_0^* . \tag{2·26}$$

The first term of R.H.S. is due to the expansion and the second to the non-adiabaticity of the unperturbed state.

The equation of energy conservation does not include explicitly the term of the adiabatic cooling of the unperturbed state, but that of non-adiabaticity. As Λ^* includes the ionizational energy loss of internal energy, (2·17) expresses the conservation of thermal energy of matter.[15] Denoting the time scales of energy loss due to radiative, mechanical and ionizational processes by $\tau_{\rm R}$, $\tau_{\rm H}$ and $\tau_{\rm I}$, $\mathcal{R}^*$, $\mathcal{H}^*$ and $\mathcal{I}^*$ are rewritten as follows:

$$\mathcal{R}^* = \tau_{\rm f}^{(0)}/\tau_{\rm R}, \quad \mathcal{H}^* = \tau_{\rm f}^{(0)}/\tau_{\rm H} \quad \text{and} \quad \mathcal{I}^* = \tau_{\rm f}^{(0)}/\tau_{\rm I}, \tag{2·27}$$

where $\tau_{\rm R}$, $\tau_{\rm H}$ and $\tau_{\rm I}$ are given by

$$\tau_{\rm R} = \frac{N_{\rm A}(1+x)kT_0}{\mathcal{R}}, \quad \tau_{\rm H} = \frac{N_{\rm A}(1+x)kT_0}{\mathcal{H}} \quad \text{and} \quad \tau_{\rm I} = \frac{N_{\rm A}(1+x)kT_0}{\chi \mathcal{I}}. \tag{2·28}$$

Moreover, denoting the time scale of energy loss by $\tau_{\rm E}$, we have

$$\Lambda^* = \tau_{\rm f}^{(0)}/\tau_{\rm E} \quad \text{and} \quad \frac{1}{\tau_{\rm E}} = \frac{1}{\tau_{\rm R}} + \frac{1}{\tau_{\rm H}} + \frac{1}{\tau_{\rm I}} . \tag{2·29}$$

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In Eq. (2·18) determining the ionization degree, the term of $\mathcal{J}_0^*$ is included because of the time variation of x_0. Determining the radiation field, J_ν^* and $\boldsymbol{F}_\nu^*$ are the ratio of J_ν and $\boldsymbol{F}_\nu$ to the pressure of matter. In (2·20) and (2·21), the time derivatives of J_ν^* and $\boldsymbol{F}_\nu^*$ appear in the form of $[(\partial/\partial\tau)-(\gamma-1)(\sqrt{6}+\Lambda_0^*)]$, because the term of $-(\gamma-1)(\sqrt{6}+\Lambda_0^*)$ is derived from the time variation of material pressure.

Next, we shall study the stability from the behavior of infinitesimal fluctuations. Without loss of generality, a perturbation δf of any physical quantity is assumed to have a wave form defined with a wave vector $\boldsymbol{k}$ so that $\delta f=f_1(\tau)e^{i\boldsymbol{k}\cdot\boldsymbol{x}^*}$, where $\boldsymbol{x}^*=(1/a)\boldsymbol{x}$. From Eqs. (2·15)~(2·21), we then have the following linearized equations:

$$\frac{d\rho_1}{d\tau}+u_1=0, \tag{2·30}$$

$$\frac{du_1}{d\tau}+\alpha u_1-\left(\frac{k}{k_T}\right)^2 p_1+\rho_1=\frac{v_f}{c}\frac{\kappa_0^*+\sigma_0^*}{k_T^2}F_1, \tag{2·31}$$

$$\left[\frac{d}{d\tau}+(\gamma-1)\bar{\Lambda}_T^*\right]p_1-\gamma\left[\frac{d}{d\tau}+\frac{\gamma-1}{\gamma}(\bar{\Lambda}_T^*-\Lambda_\rho^*)\right]\rho_1+(\gamma-1)(\Lambda_\mu^*-\Lambda_T^*)\mu_1$$
$$=(\gamma-1)\kappa_0^*J_1, \tag{2·32}$$

$$\left[\frac{\chi}{kT_0}\frac{d}{d\tau}+\mathcal{J}_0^*\right]\mu_1=(\mathcal{J}_\rho^*-\mathcal{J}_T^*)\rho_1+\mathcal{J}_T^*p_1+\mathcal{J}_\mu^*\mu_1+\mathcal{J}_J^*J_1, \tag{2·33}$$

$$\frac{1}{c}\left[\frac{d}{d\tau}-(\gamma-1)(\sqrt{6}+\Lambda_0^*)\right]J_1+\frac{1}{4v_f}F_1$$
$$=-\kappa_0^*J_1+(\mathcal{R}_\rho^*-\mathcal{R}_T^*)\rho_1+\mathcal{R}_T^*p_1+(\mathcal{R}_\mu^*-\mathcal{R}_T^*)\mu_1 \tag{2·34}$$

and

$$\frac{1}{c}\left[\frac{d}{d\tau}-(\gamma-1)(\sqrt{6}+\Lambda_0^*)\right]F_1-\frac{4k^2}{v_f}J_1=-(\kappa_0^*+\sigma_0^*)F_1, \tag{2·35}$$

where

$$u_1=i\boldsymbol{k}\cdot\boldsymbol{v}_1,\quad J_1=\int_0^\infty \delta J_\nu d\nu \quad\text{and}\quad F_1=i\boldsymbol{k}\cdot\int_0^\infty \delta\boldsymbol{F}_\nu d\nu. \tag{2·36}$$

Here,

$$\Lambda_\rho^*=-\frac{\tau_f^{(0)}}{\tau_E^{(0)}}\left(\frac{\partial\ln|\tau_E|}{\partial\ln\rho}\right)_0,\quad \Lambda_T^*=-\frac{\tau_f^{(0)}}{\tau_E^{(0)}}\left(\frac{\partial\ln|\tau_E|}{\partial\ln T}\right)_0,$$

$$\bar{\Lambda}_T^*=\Lambda_T^*-\Lambda_0^*+k^2K_0^*,$$

$$\Lambda_\mu^*=-\frac{\tau_f^{(0)}}{\tau_E^{(0)}}\left(\frac{\partial\ln|\tau_E|}{\partial\ln\mu}\right)_0,$$

$$\mathcal{R}_\rho^*=-\frac{\tau_f^{(0)}}{\tau_R^{(0)}}\left(\frac{\partial\ln|\tau_R|}{\partial\ln\rho}\right)_0,\quad \mathcal{R}_T^*=-\frac{\tau_f^{(0)}}{\tau_R^{(0)}}\left(\frac{\partial\ln|\tau_R|}{\partial\ln T}\right)_0,\quad \mathcal{R}_\mu^*=-\frac{\tau_f^{(0)}}{\tau_R^{(0)}}\left(\frac{\partial\ln|\tau_R|}{\partial\ln\mu}\right)_0,$$

$$\mathcal{J}_\rho^*=-\frac{\tau_f^{(0)}}{\tau_I^{(0)}}\left(\frac{\partial\ln|\tau_I|}{\partial\ln\rho}\right)_0,\quad \mathcal{J}_T^*=-\frac{\tau_f^{(0)}}{\tau_I^{(0)}}\left(\frac{\partial\ln|\tau_I|}{\partial\ln T}\right)_0,\quad \mathcal{J}_\mu^*=-\frac{\tau_f^{(0)}}{\tau_I^{(0)}}\left(\frac{\partial\ln|\tau_I|}{\partial\ln\mu}\right)_0$$

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and

$$\mathcal{J}_J^* = -\frac{p_0}{J_0}\frac{\tau_f^{(0)}}{\tau_I^{(0)}}\left(\frac{\partial \ln|\tau_I|}{\partial \ln J_1}\right)_0. \tag{2·37}$$

Besides, κ_0^* and σ_0^* are given by

$$\kappa_0^* = \frac{\int_0^\infty \kappa_\nu^* \delta J_\nu^* d\nu}{\int_0^\infty \delta J_\nu d\nu} \quad \text{and} \quad \sigma_0^* = \frac{\int_0^\infty \sigma_\nu^* \delta J_\nu d\nu}{\int_0^\infty \delta J_\nu d\nu}. \tag{2·38}$$

Then, Eqs. (2·30)~(2·35) are rewritten as follows:

$$\frac{d\boldsymbol{U}}{d\tau} = \mathbf{A}\boldsymbol{U}, \tag{2·39}$$

where

$$\boldsymbol{U} = \begin{pmatrix} \rho_1 \\ u_1 \\ p_1 \\ \mu_1 \\ J_1 \\ F_1 \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0, & -1, & 0, & 0, & 0, & 0, \\ -1, & -\alpha, & (k/k_T)^2, & 0, & 0, & (v_f/c)(\kappa_0^* + \sigma_0^*)/k_T^2, \\ (\gamma-1)(\bar{\Lambda}_T^* - \Lambda_\rho^*), & -\gamma, & -(\gamma-1)\Lambda_T^*, & -(\gamma-1)(\Lambda_\mu^* - \Lambda_T^*), & (\gamma-1)\kappa_0^*, & 0, \\ (1/\theta)(\mathcal{J}_\rho^* - \mathcal{J}_T^*), & 0, & (1/\theta)\mathcal{J}_T^*, & (1/\theta)\mathcal{J}_\mu^*, & (1/\theta)\mathcal{J}_J^*, & 0, \\ c(\mathcal{R}_\rho^* - \mathcal{R}_T^*), & 0, & c\mathcal{R}_T^*, & c(\mathcal{R}_\mu^* - \mathcal{R}_T^*), & -c\kappa_0^* + (\gamma-1)(\sqrt{6} + \Lambda_0^*), & -1/4v_f, \\ 0, & 0, & 0, & 0, & 4k^2/v_f, & -c(\kappa_0^* + \sigma_0^*) + (\gamma-1)(\sqrt{6} + \Lambda_0^*). \end{pmatrix} \tag{2·40}$$

where

$$\theta = \chi/kT_0 \quad \text{and} \quad \bar{\mathcal{J}}_\mu^* = \mathcal{J}_\mu^* - \mathcal{J}_0.$$

If the eigenvalues and left-eigenvectors of $\mathbf{A}$ are represented by n_i and $\boldsymbol{l}_i$ respectively, $\boldsymbol{U}$ is given as a superposition of the normal modes,

$$\boldsymbol{U} = \sum_{i=1}^{6} c_i \boldsymbol{l}_i e^{n_i \tau}. \tag{2·41}$$

In this case, $e^{n_i\tau}$ is also represented as $t^{(\sqrt{6}/3)n_i}$ in terms of the proper time t. It shows that the temporal changes of fluctuations obey the power law, which is an important consequence of the expansion of the universe. Then, the eigenvalues are the roots of the characteristic equation, which is formally a 6th order algebraic equation.

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$$n^2+\alpha n-1$$
$$=-x^2\frac{\{1+(1-1/\gamma)\mathcal{P}_T+(1/\gamma)\mathcal{P}_\rho\}n+1/\gamma\{(1+\mathcal{P}_\rho)\mathfrak{L}_T-(1+\mathcal{P}_T)\mathfrak{L}_\rho\}}{n+\mathfrak{L}_T}\,. \tag{2·42}$$

Here, $x=k/k_J$ (k_J is the Jeans wave number of $(1/\sqrt{\gamma})k_T$. Moreover, $\mathcal{P}_\rho$, $\mathcal{P}_T$, $\mathfrak{L}_\rho$ and $\mathfrak{L}_T$ are the third order algebraic function of n and defind as follows:

$$\mathcal{P}_\rho=\frac{4(\kappa_0^*+\sigma_0^*)}{n+c(\bar{\kappa}_0^*+\sigma_0^*)}\frac{\eta}{1-\eta\zeta\mathcal{J}_J^*\mathcal{R}_\mu'}(\mathcal{R}_\rho^*+\zeta\mathcal{J}_\rho^*\mathcal{R}_\mu'), \tag{2·43}$$

$$\mathcal{P}_T=\frac{4(\kappa_0^*+\sigma_0^*)}{n+c(\bar{\kappa}_0^*+\sigma_0^*)}\frac{\eta}{1-\eta\zeta\mathcal{J}_J^*\mathcal{R}_\mu'}(\mathcal{R}_T^*+\zeta\mathcal{J}_T^*\mathcal{R}_\mu'), \tag{2·44}$$

$$\mathfrak{L}_\rho=\Lambda_\rho^*+\Lambda_\mu'\frac{\zeta(\mathcal{J}_\rho^*+\eta\mathcal{J}_J^*\mathcal{R}_\rho^*)}{1-\eta\zeta\mathcal{J}_J^*\mathcal{R}_\mu'}-\frac{\kappa_0^*\eta(\mathcal{R}_\rho^*+\zeta\mathcal{J}_\rho^*\mathcal{R}_\mu')}{1-\eta\zeta\mathcal{J}_J^*\mathcal{R}_\mu'} \tag{2·45}$$

and

$$\mathfrak{L}_T=\Lambda_T^*+\Lambda_\mu'\frac{\zeta(\mathcal{J}_T^*+\eta\mathcal{J}_J^*\mathcal{R}_T^*)}{1-\eta\zeta\mathcal{J}_J^*\mathcal{R}_\mu'}-\frac{\kappa_0^*\eta(\mathcal{R}_T^*+\zeta\mathcal{J}_T^*\mathcal{R}_\mu')}{1-\eta\zeta\mathcal{J}_J^*\mathcal{R}_\mu'}, \tag{2·46}$$

where

$$\eta=c\frac{n+c(\bar{\kappa}_0^*+\sigma_0^*)}{(n+c\bar{\kappa}_0^*)(n+c\bar{\kappa}_0^*+c\sigma_0^*)+(ck_J/v_t)^2x^2},$$

$$\zeta=\frac{1}{\theta n-\bar{\mathcal{J}}_\mu^*},\quad \bar{\kappa}_0^*=\kappa_0^*-\frac{1}{c}(\gamma-1)(\sqrt{6}+\Lambda_0^*),$$

$$\bar{\mathcal{J}}_\mu^*=\mathcal{J}_\mu^*-\mathcal{J}_0^* \quad\text{and}\quad \mathcal{R}_\mu'=\mathcal{R}_\mu^*-\mathcal{R}_T^*\,. \tag{2·47}$$

η expresses the influence of radiative retardation and optical thickness of a fluctuation and is the 2nd order rational function of n. Then, ζ expresses the influence of the ionizational change. The exact expression of (2·42) is given in Appendix A.

Now, we shall consider the characteristic behaviors of the eigenvalues. The characteristic equation (2·42) has six roots, the three of which express the thermodynamical modes, the one the ionizational mode and the two the radiational modes. And then these three modes are coupled with each other in different ways at various wave numbers.

Then, we are interested in the thermodynamical modes composed of three roots, because they are accompanied with density changes. To observe the nature of three modes, we shall consider the simple case that these modes are isolated from the others.

The dispersion relation for the thermodynamical modes is given by

$$n^2+\alpha n-1=-x^2\frac{n+(1/\gamma)(L_T-L_\rho)}{n+L_T}\,. \tag{2·48}$$

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Here L_T and L_ρ do not include ionization effect and represent the dependence of energy loss on temperature and density. The examples of the solid expression of L_T and L_ρ are given in Appendix A.

If $L_\rho = L_T = 0$, the characteristic equation becomes that of the adiabatic Jeans instability;[12)]

$$n^2 + \alpha n - 1 + x^2 = 0, \tag{2·49}$$

where the term αn is the difference from the case of static medium.

The characteristic behaviors of the eigenvalues can be analyzed conveniently in terms of the eigenvalues at the limiting wave numbers.[13)]

For $x=0$, the three roots corresponds to the growth rates of free fall, free expansion and thermal change:

$$n_f = \sqrt{1+\left(\frac{\alpha}{2}\right)^2} - \frac{\alpha}{2}, \tag{2·50}$$

$$n_e = -\sqrt{1+\left(\frac{\alpha}{2}\right)^2} - \frac{\alpha}{2} \tag{2·51}$$

and

$$n_T = -[L_T]_{x=0}. \tag{2·52}$$

On the other hand, for $x=\infty$, we have

$$n_c = -\frac{1}{\gamma}[L_T - L_\rho]_{x=\infty} \tag{2·53}$$

and

$$n_s = -\frac{1}{2\gamma}[(\gamma-1)L_T + L_\rho] - \frac{\alpha}{2} \pm xi. \tag{2·54}$$

Here, n_c is the growth rate of the condensation mode while n_s gives the growth rate and the frequency of the sound mode.

In the case of the flat Friedmann universe, neglecting the radiation drag force, we have

$$n_f = \frac{2}{\sqrt{6}} \quad \text{and} \quad n_e = -\frac{3}{\sqrt{6}}, \tag{2·55}$$

while $n_f = 1$ and $n_e = -1$ in the case of static medium.[3)] In terms of the characteristic time scale of τ, n_f and n_e are represented by $\tau_f^{(0)}/\tau_f$ and $\tau_f^{(0)}/\tau_e$, respectively. Therefore, $\tau_f = (\sqrt{6}/2)\tau_f^{(0)} = 3\tau_{ex}$, that is, the free fall in the expanding universe takes treble of the expansion time.

The nature of the three modes may be observed from the ratio Γ of the relative amplitude of perturbation in pressure and density,

$$\Gamma \equiv \frac{p_1}{\rho_1} = \gamma\frac{n-n_c}{n-n_T} = -\gamma\frac{(n-n_f)(n-n_e)}{x^2}. \tag{2·56}$$

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Table I. The ratio Γ of the pressure amplitude to the density amplitude of fluctuations at the limiting wave numbers of $x=0$ and $x=\infty$.

$x=0$		$x=\infty$	
n	Γ	n	Γ
n_f	$\gamma\frac{n_f-n_c}{n_f-n_T}$	n_c	0
n_e	$\gamma\frac{n_e-n_c}{n_e-n_T}$	n_s	γ
n_T	$-\infty$		

The value of Γ at the limit of $x=0$ and $x=\infty$ are tabulated in Table I.

In the case of $n=n_T$, density change does not occur, and so this case corresponds to the thermal mode where temperature and pressure vary proportionally. The case of $n=n_c$ corresponds to the condensation mode because equilibrium of pressure is preserved. Further the case of $n=n_s$ corresponds to the sound mode because $\Gamma=\gamma$ and the phase velocity of this mode agrees with the sound velocity which is expressed by k_J^{-1}.

Among the growth rates in the limits of large and small wave numbers, n_f, n_e and n_s belong to the modes of mechanical nature, while n_T and n_c belong to the modes of thermal nature. At $x=0$, the eigenvalues n of the three modes are real (2 modes are gravitational and 1 mode thermal), while, at $x=\infty$, only one eigenvalue is real and the other two are complex conjugate (the condensation mode and the sound modes). Over the whole range of wave number, therefore, one of the three modes has a real eigenvalue, and the other two have real eigenvalues at larger wave lengths than a critical one, but they have complex conjugate eigenvalues at smaller wave lengths. In the adiabatic case, the critical wave length corresponds to the Jeans one. The behavior of the three eigenvalues with increasing wave number depends on the magnitudes of the growth rates at $x=0$ and $x=\infty$, and typical cases are illustrated in Fig. 1.

Now, Γ of each mode varies with wave length. For example, Γ of the thermal mode which has a real n value over the whole wave lengths changes from $-\infty$ to 0 as the wave length increases from 0 to ∞ (Fig. 1(c)), reflecting the fact that the compression of matter propagates at the speed of sound and that for shorter wave length of a disturbance the compression rate becomes larger.

We are now interested in the mode with a large growth rate, and we shall consider the condition that $\mathcal{R}_e n > n_f$. This condition means that the scale of growth time should be shorter than three times of the expansion time of the universe.

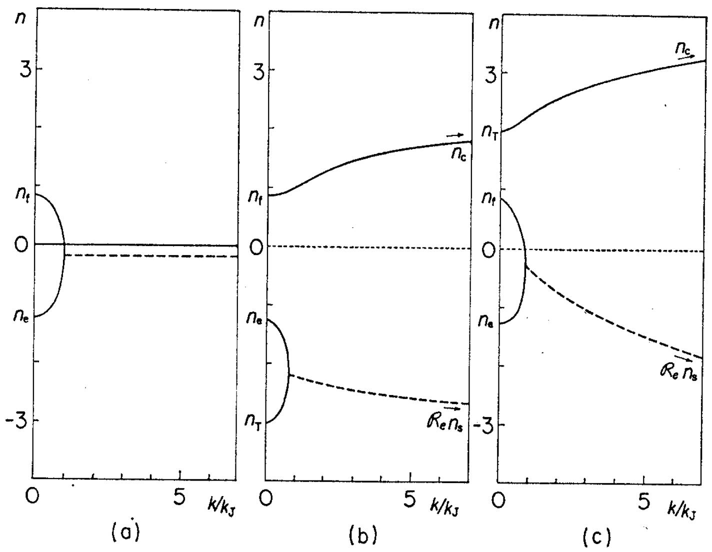

Fig. 1. The dependence of the growth rates on the wave number. The solid lines show the values of real root n of the dispersion relation, and the dotted lines show the real part of the complex root n. The case (a) is that of $L_\rho = L_T = 0$, the case (b) $L_\rho = 2 \times L_T > 0$, and the case (c) $L_\rho = -2L_T > 0$.

We see from Eq. (2·56) that $\Gamma < 0$, whenever $\mathcal{R}_e n > n_f$. Then, from the dispersion relation of (2·48), the condition that $\mathcal{R}_e n > n_f$ is satisfied by one of the following inequalities,

$$L_T - L_\rho + \gamma n_f < 0, \tag{2·57}$$

$$(L_T + n_f + \sqrt{\alpha^2+4})\{\sqrt{\alpha^2+4}(L_T+n_f) + x^2\} - \frac{x^2}{\gamma}(L_T - L_\rho + \gamma n_f) < 0, \tag{2·58}$$

$$L_T + n_f + \sqrt{\alpha^2+4} < 0 \quad \text{or} \quad \sqrt{\alpha^2+4}(L_T+n_f) + x^2 < 0. \tag{2·59}$$

If L_ρ and L_T are independent of x, we can exhibit the region of $\mathcal{R}_e n > n_f$ in the $L_T - L_\rho$ plane, and this region is partitioned by the boundaries given below;

from Eq. (2·57),

$$n_c > n_f, \tag{2·60}$$

from Eq. (2·58),

$$n_T > n_f \tag{2·61}$$

and from the same equation,

$$\mathcal{R}_e n_s > n_f. \tag{2·62}$$

In Fig. 2, this region is shown. The shaded part exhibits the region where $\mathcal{R}_e n < n_f$ over the whole range of x.

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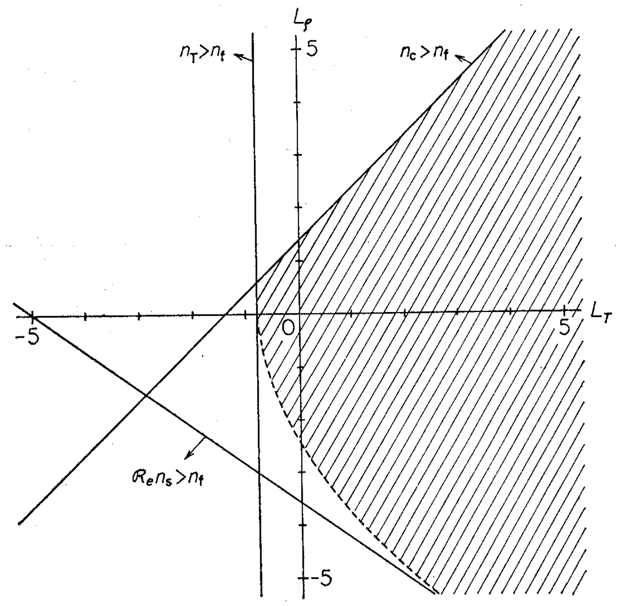

Fig. 2. The region of $\mathcal{R}_e n > n_t$ on the $L_T - L_\rho$ plane. The shaded area exhibits the region where $\mathcal{R}_e n > n_t$ over the whole range of x.

§3. Thermal conditions in the expanding universe

The spectrum of the background radiation first observed by Penzias and Wilson fits with the Planck curve of 2.89°K.[16] This background radiation is attributed widely to the cosmic black-body radiation in the Big-Bang model of the Universe as was predicted by Gamow[17,18] and as evidenced by the observed high isotropy,[19,20] although other interpretations may not be ruled out. The thermal history of the expanding universe may be divided into two stages depending on the degree of ionization of hydrogen. In the earlier stage, hydrogen is practically fully ionized and matter and radiation couple strongly through the Compton scattering. In the later stage, the recombination of hydrogen is almost completed and the universe becomes transparent for significant frequency range of radiation. The circumstance of the hydrogen recombination and the later thermal history, however, may depend very much on whether the dissipation of the existing turbulence could be strong enough to heat up the matter to a high temperature or not. In this section, we shall briefly summarize possible thermal conditions with and without the turbulence in connection with the possibility of the thermal instability.

If the dissipation of turbulence is disregarded, protons and electrons begin to recombine as the radiation temperature decreases to about 4500°K, and the Lyman photons fill the universe. The Ly-α photons strongly interact with neutral hydrogen atoms so that the ionization which occurs most efficiently from the first excited level is directly influenced by the rate of the reduction of Ly-α photons which are subject to the reddening

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and to the two-photon emission from the metastable 2*s* state. The kinetic temperature of matter does not separate from the radiation temperature until the degree of ionization becomes very small, owing to the coupling due to the Compton scattering. Later, the degree of ionization decreases asymptotically to a value of 10^{-5} when the temperature considerably decreased.[21),22)] Meanwhile, hydrogen molecules are formed by the reaction between neutral atoms and negative ions of hydrogen which are produced by the relic free electrons.[23),24)] The number fraction of hydrogen molecules amounts to 10^{-11} at a radiation temperature of about 500°K and to 10^{-6} at about 300°K.[25)]

Throughout the thermal history discussed above (without turbulence), the main process of energy exchange is the Compton scattering. At the stage of hydrogen recombination, the two-photon emission takes away the Ly-α radiation for which the medium is optically thick, but this process of cooling cannot be comparable to the heating due to the Compton scattering. Even after the temperatures of matter and of radiation split off at the stage of neutral hydrogen, the Compton effect works through the relic free electrons. Although the rotational levels of H_2 molecules provide efficient emission processes, the population of H_2 molecules is too small.

The heat loss function (which is negative because the adiabatic cooling of the matter temperature is more rapid than that of the radiative temperature; $T<T_r$) is then given by[26)]

$$\mathcal{L}^*_{\rm comp}=\tau_{\rm f}^{(0)}\frac{\rho_0}{p_0}\frac{\sigma_T k a_r T_r^4}{m_{\rm H}m_e c}x_e(T-T_r)$$
$$=-\kappa_c^* J^*,$$

where

$$\kappa_c^*=\tau_{\rm f}^{(0)}\cdot\rho_0\frac{4\pi\sigma_T k}{m_{\rm H}m_e c^2}x_e(T_r-T)$$

and x_e in the electron fraction. Then we have

$$\mathcal{L}^*_\rho=0 \quad\text{and}\quad \mathcal{L}^*_T=\frac{\mathcal{L}^*_{\rm comp}}{T-T_r}\,.$$

In this case, ionizational effect can be neglected.

Since in this case $\mathcal{L}^*_\rho=0$ and $\mathcal{L}^*_T$ is positive regardless of the optical thickness of disturbances, the medium is thermally stable. The only unstable mode exhibits the gravitational instability for large wave lengths as shown in Fig. 3, but the growth

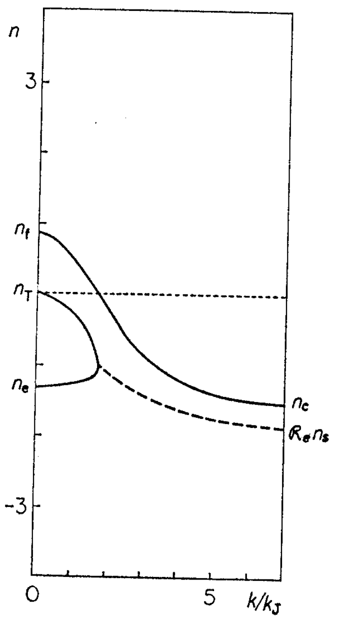

Fig. 3. The wave number dependence of the growth rate under the Compton scattering.

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rate is small. The thermal effect lengthens the critical wave length.

§4. Possible models of thermal instability in the expanding universe

In the ordinary thermal history, thermal instability cannot occur because of the Compton scattering. We shall consider the turbulent universe in which thermal instability may possibly occur. This state exists after the recombination, when the temperature of the matter and that of the radiation split off from each other. Now, as the heat capacity of matter is much smaller than that of the radiation field so that the matter temperature alone is possibly increased by the dissipation of the turbulent kinetic energy.

If the matter is heated up, the thermal history will proceed in three steps: At the first step, the mechanical energy gain suppresses the energy loss, so that the matter temperature continues to increase with the larger value than the radiative temperature. At this time, the matter is thermally stable, for the mechanical heating rate per unit mass is almost independent of density and temperature.[27] At the second step, the rate of the mechanical heating is balanced with that of energy loss, and the matter temperature attains the maximum value. The radiative cooling results from the relaxation between the matter temperature and the radiative one. At the third step, the energy loss dominates the heating so that the matter temperature declines rapidly. This third step is not suitable for thermal instability, because the unperturbed state passes through the thermal unstable state before fluctuations sufficiently grow. After all, thermal instability may suitably occur at the second step. From now on, we consider the thermal instability under the condition that $\Lambda_0^* = 0$.

It is to be noted that the radiative temperature is increased by the energy gain of radiation field from the thermal field in the process of thermal instability, but the change is very small for the radiation field because the number of photons is much larger than that of particles. Therefore, we can reasonably assume that the radiation temperature is determined by the adiabatic expansion.

Next, we consider the radiative energy exchange between matter and radiation. The radiative processes are composed of the Compton scattering, free-bound and bound-bound transitions, whose detailed expressions are given in Appendix B. It is important that they depend on the ionization degree, determined by the photoelectric and collisional processes.[28] However, the ionizational process is almost due to collision, because the matter is heated up by the mechanical processes after the epoch of the decoupling. In this case, the mean-free time of photons and the characteristic time of the ionizational change are much shorter than that of radiative energy loss, over the wide range of matter temperatures. Therefore, in this situation, L_ρ and

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L_T appeared in thermodynamical modes are given by

$$L_\rho = \eta_0 \mathcal{R}_\rho^* \quad \text{and} \quad L_T = \eta_0 \mathcal{R}_T^* , \tag{4·1}$$

where

$$\eta_0 = \frac{(\bar{\kappa}_0^* + \sigma_0^*)(\bar{\kappa}_0^* - \kappa_0^* - \mathcal{J}_J^*) + (k_J/v_f \cdot x)^2}{\bar{\kappa}_0^*(\bar{\kappa}_0^* + \sigma_0^*) + (k_J/v_f \cdot x)^2} . \tag{4·2}$$

This is shown in (A·34) in Appendix A. Here, η_0 means the reduction factor due to the optical thickness of $\tau_p = \kappa_0^*/(k_J/v_f \cdot x)$: If $\tau_p \gg 1$ (the optically thick case), $\eta_0 \sim (\bar{\kappa}_0^* - \kappa_0^* - \mathcal{J}_J)/\bar{\kappa}_0^*$, on the other hand, if $\tau_p \ll 1$ (the transparent case), $\eta_0 \sim 1$. And $\mathcal{J}_J^*$ exhibits the influence of photo-ionization whether the ionization is determined by collisional process or not.

Next, we show the curves of constant $\mathcal{R}_0^*$ in the $\mathfrak{N} - T$ plane in Fig. 4, where $\mathfrak{N}$ is the number density of nucleons. At higher temperatures, the Compton scattering is most efficient, then free-bound and Ly-α of singly ionized helium, free-bound and Ly-α of hydrogen become efficient in order of decreasing temperature.[29] As the photons of Ly-α are optically thick, Ly-α does not contribute to energy loss and is neglected in Fig. 4: Even if they are optically thick, however, the photons of Ly-α can diffuse out by incoherent scattering[30] or leak out by their conversion into 2 photons.[31] But these processes have little contribution to energy-loss rate.

Then, in Fig. 5, we exhibit the free-bound absorption cross section.

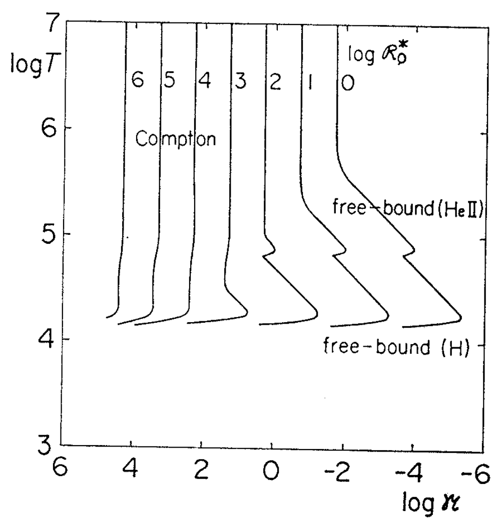

Fig. 4. The equi-energy-loss curves in the lon $\mathfrak{N}$—log T plane under the radiative processes of the Compton scattering and free-bound. The matter is composed of hydrogen (90%) and helium (10%).

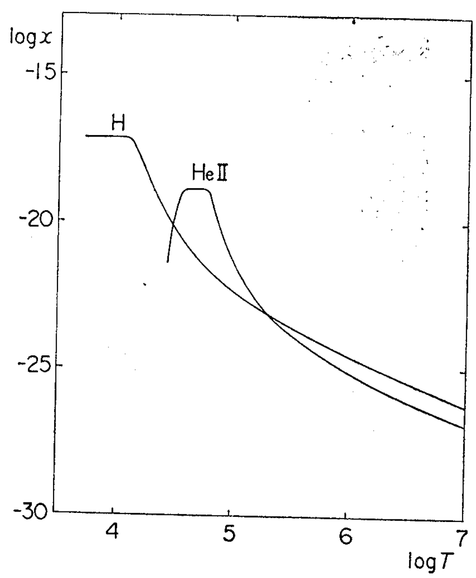

Fig. 5. The absorption coefficients for bound-free and Ly-α of hydrogen and helium. The ratio of numbers of hydrogen and helium is 9 : 1.

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The wave number for a mass of M is given by $0.954\times10^{-18}\mathfrak{N}^{1/3}(M/M_{\odot})^{-1/3}$, and so the range of matter temperature exists where free-bound photons are optically thin for fluctuations with larger masses than $10^{6}M_{\odot}$.

For the cases of disturbances with the scales corresponding to the masses of $10^{12}M_{\odot}$, $10^{9}M_{\odot}$ and $10^{6}M_{\odot}$, we show curves of constant $\eta_0\mathcal{R}_0^*$ values and the growth rate of the condensation mode in the $\mathfrak{N}-T$ plane in Fig. 6. Thermal instability occurs in the two regions in the $\mathfrak{N}-T$ plane, one due to free-bound processes of singly ionized helium, and the other to that of hydrogen. For, at the state where the effect of ionizational change can be neglected, the free-bound process satisfies the following relation:

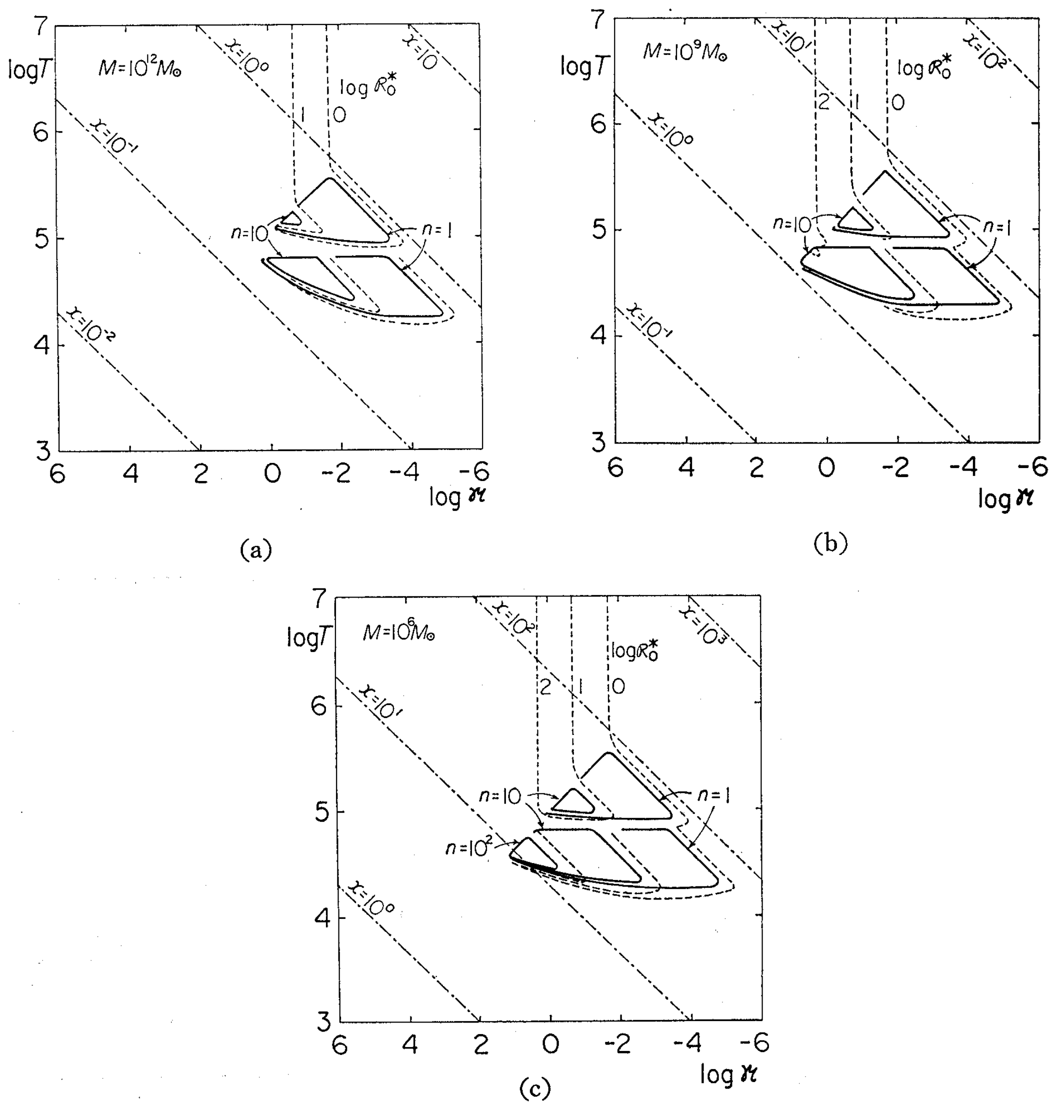

Fig. 6. The equi-energy-loss curves and the equi-growth-rate curves in the $\log\mathfrak{N}-\log T$ plane for the cases of the fluctuations with the scales of $10^{12}M_{\odot}$, $10^{9}M_{\odot}$ and $10^{6}M_{\odot}$. The solid line is the equi-growth-rate curve and the dotted line the equi-energy loss. The chain lines show the constant ratio of the wave number to the Jeans wave number for the given masses.

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$$L_\rho = -2L_T \,. \tag{4·3}$$

The condensation mode is unstable if L_ρ is positive, which occurs at the state where

$$\mathcal{G}_f^*(\kappa_0^* + \sigma_0^*) < \left(\frac{k_J}{v_f} x\right)^2 . \tag{4·4}$$

Therefore, the unstable region is wider for fluctuations with smaller mass, as shown in Fig. 6.

Now the unstable region is determined by the three conditions:

1) The first condition is that the free-bound energy loss dominates over that of the Compton scattering. This is satisfied after the epoch of $\mathfrak{R}=10$, which we denote t_b hereafter.
2) The second is inequality (4·4).
3) The third is that ionizational degree is larger than 0.9, for it is necessary that the ionizational change occurs quickly and the ionizational mode is isolated.

Now, we shall discuss the mass range where the fluctuations can be amplified by thermal instability and proceed further to condense by gravitational instability. To do so, we shall derive the value of the growth rate, by which density fluctuations can attain an amplitude $(\delta\rho/\rho)_G$ enough to condense gravitationally within the cosmic age. With due regard to the nonlinear acceleration, $(\delta\rho/\rho)_G$ must be larger than $10^{-8}t_G$, where t_G (measured in year) is the epoch when the gravitational instability occurs. For example, if t_G is 10^5 years, $(\delta\rho/\rho)_G$ is necessarily greater than 10^{-3}. The fluctuation, with the initial amplitude $(\delta\rho/\rho)_0$ at the epoch of t_0, attains the amplitude of $10^{-8}t_G$ at t_G (given by the following equation) after it is amplified by the thermal instability with the growth rate n,

$$\log t_G = \frac{1}{n-(3/\sqrt{6})}\left[n \log t_0 - \frac{3}{\sqrt{6}} \log\left(\frac{\delta\rho}{\rho}\right) - \frac{24}{\sqrt{6}}\right]. \tag{4·5}$$

If $n > 10^2$, the fluctuations with extremely small amplitude, for example $10^{-35} \sim 10^{-15}$, can sufficiently grow to $(\delta\rho/\rho)_G$ within a period comparable to t_0.

Now, the upper limit of the mass range turns out to be $10^{12}M_\odot$, because the larger masses than $10^{12}M_\odot$ do not provide a state of $n > 10^2$. It is also unlikely that the condensation with a mass of $10^{12}M_\odot$ originates directly from thermal instability. The reason is that although the state with the largest growth rate is gravitationally unstable for $10^{12}M_\odot$, density fluctuations are relatively small because $|\Gamma|$, the ratio of the amplitude of pressure to that of density, is very large.

The lower limit may be $10^6M_\odot$: For, lower masses than $10^7M_\odot$ provide almost the same growth rate of thermal instability but are less unstable

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gravitationally. As the unperturbed state cools down to the thermally stable state after the fluctuations are amplified by thermal instability, the fluctuations with smaller wave length undergo more strongly the adiabatic damping by the expansion of the universe. Therefore, only the fluctuations with larger masses than $10^6 M_\odot$ can survive.

After all, the most probable mass of grobule [which originates in thermal instability and proceeds further to condense gravitationally is found to be $10^6 M_\odot$ to $10^7 M_\odot$. The remaining problem is the possibility that the thermal instability occurs in the universe, that is, how the matter temperature can be heated up to a higher temperature than 10^4 °K at the epoch after t_b. We shall discuss the heating due to turbulence and cosmic rays.

Before the recombination of hydrogen, the sound velocity c_s^* is given by $(kT/m_H)^{1/2}(n_r/n_m)^{1/2}$, where n_r and n_m are the number density of photons and particles, since matter and radiation are strongly coupled. But this c_s^* changes to $c_s=(kT/m_H)^{1/2}$ by the recombination of hydrogen, and the large reduction of the sound velocity by a factor of 5×10^3 will turn the primordial turbulence from subsonic to supersonic.[32] Therefore, the shock dissipation of turbulence can increase the matter temperature, provided that the primordial turbulence is strong enough.

If the turbulent dissipation heats up the matter quickly before $t=t_b$, the decay of turbulence will proceed under the condition that turbulent velocity is approximately equal to the sound velocity. Therefore

$$P_m=P_t \quad \text{and} \quad U_m=U_t\,, \tag{4·6}$$

where P_m, P_t are the material and turbulent pressures and U_m, U_t are the internal energy of matter and turbulence respectively.[33] Then, the total energy of matter and turbulence is diminished by the energy loss of the Compton scattering because the epoch is prior to t_b.

$$\frac{d}{dt}(U_m+U_t)+(P_m+P_t)\frac{d}{dt}\left(\frac{1}{\rho}\right)=-c_0 T_r^4(T-T_r), \tag{4·7}$$

where

$$c_0=\frac{\sigma_T k a_r}{m_H m_e c}=1.53\times10^{-13}.$$

From Eqs. (4·6) and (4·7), we derive the equation on T;

$$\frac{d}{dt}\ln\frac{T}{\rho^{2/3}}=-\frac{m_H}{k}c_0 T_r^4\left(1-\frac{T_r}{T}\right). \tag{4·8}$$

As we consider the case that the heated-up temperature T is greater than T_r, we have

$$\frac{T}{T_0}=\left(\frac{\rho}{\rho_0}\right)^{2/3}\exp X_0\left[\left(\frac{t_0}{t}\right)^{5/3}-1\right], \tag{4·9}$$

where the suffix 0 denotes the initial value, and $X_0=(c_0 T_{r0}^4/5k)t_0=3.68\times10^{-21} T_{r0}^4 t_0$. As X_0 is greater than 4×10^5 before the time of t_b, matter temperature and turbulent energy decrease very rapidly by the Compton scattering. Then, in order that the universe passes through the state where thermal instability occurs, the turbulent energy is required to be transformed in different form and be preserved enough to heat up the matter temperature to $10^4\,°K$ after the epoch t_b. Such a possibility may be provided, if the magnetic field exists in the universe. Now, the magnetic and cosmic-ray energies decrease proportionally to $\rho^{4/3}$ with the cosmic expansion, while the turbulent energy decreases proportionally to $\rho^{5/3}$. So, the turbulent energy may well be preserved in the magnetic field and/or in the cosmic-ray particles until 10^7 years.

At the early epoch when the turbulent energy dominates over the magnetic one, the turbulent eddies entangle the magnetic lines of force. Once the magnetic energy becomes superior to the turbulent one, the stretching of magnetic lines of force accelerates the turbulent velocity v_t so that v_t becomes nearly equal to c_A (Alfven velocity). Then the maximum size l_A where the acceleration occurs whithin the cosmic age t is given by $l_A=c_A t$, and the mass M_A with the size of l_A is $3.35\times10^{11}(H_0/10^{-7})^3 M_\odot$, where H_0 is the magnetic intensity at present. If H_0 is 10^{-7}, $M_A=3.35\times10^{11}M_\odot$.

Cosmic ray particles are likely to be generated by the abrupt stretching of magnetic lines of force due to the shock turbulence after the recombination of hydrogen before the turbulence decays. In this way a substantial fraction of turbulent energy can be transformed in the magnetic and cosmic-ray energies. Then, while the magnetic eddies of smaller scales and the cosmic ray particles of lower energies are subject to dissipation by counter-winding and ionization loss in relatively short time (10^6 years), the magnetic eddies of larger scales and the relativistic cosmic rays (R.C.R.) of energy higher than 10 GeV/particle can survive until 10^7 years or more. In its earlier stages, the Compton loss is so effective that the matter temperature is hardly raised by the cosmic-ray heating, but then as the universe is diluted the heating due to R.C.R. balances with the cooling due to the Compton scattering at higher temperatures and the matter is re-ionized. The thermal history depends strongly on the fraction ε of R.C.R. per proton. If $1.6\times10^{-6}>\epsilon>1.0\times10^{-6}$, the temperature evolves into the region of the boundfree cooling and is kept constant between $1.6\times10^4\,°K$ to $3\times10^4\,°K$.[3,4] In this case, the thermal instability operates, and grobules will be formed. Meanwhile, counter-winding of large magnetic eddies sets grobules in random motion which produces the required statistical density fluctuation for the Jeans volume containing $10^{12}M_\odot$.

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§5. Conclusion

Through the ordinary history of the universe, the Compton scattering is the most effective process of energy exchange between matter and radiation, so that thermal instability does not occur, but only gravitational instability occurs. However, if the universe is heated up to the ionized state after the recombination stage, there can be the state in which the free-bound photons can escape transparently giving rise to the energy loss more than that of the Compton scattering after the epoch t_b of $\mathfrak{R}=10$, which corresponds to 10^7 years for the flat model of the universe. This state is thermally unstable and the growth rate is so large that very small fluctuations can be excited to an amplitude necessary for the formation of condensation by gravitational instability within 10^{10} years. The most probable mass of a grobule resulting from the thermal instability is found to be $10^6 M_\odot$.

The following process is probable to heat up the matter to a reionized state after t_b. When the universe changes from the plasma state to the state of neutral hydrogen, the large reduction of the sound velocity by a factor of 5×10^3 turns the primordial turbulence from subsonic to supersonic and eddies generates shocks. If the generated shocks heat up the matter rapidly before t_b, not only does the heated matter cool down very quickly by the Compton scattering, but also the turbulent energy decays rapidly. But, if the magnetic field exists in sufficient strength, then the energy is preserved without much exhaustion in the form of twisted magnetic field and in cosmic rays. After the Compton scattering becomes less effective, the energy thus stored will be released with the time scale of 10^7 years for the eddy which includes a mass of $10^6 M_\odot$. The heating due to cosmic rays overcomes the energy loss of the Compton scattering.

The grobules with $10^6 M_\odot$ originated by thermal instability aggregates into galaxies: The velocity of M.H.D. eddies containing the grobules is 10^3 km/sec at 10^7 years, if the magnetic field extraporated to the present amounts to 10^{-7} gauss. Thereupon, the Jeans mass is $10^{12} M_\odot$, provided that the random velocity of grobules is provided by the velocity of M.H.D. turbulent eddies.

Appendix A

—— *The general characteristic equation* ——

In §2, we derive the 6th order characteristic equation (2·42);

$$n^2+\alpha n-1$$
$$=-x^2\frac{\{1+(1-1/\gamma)\mathcal{P}_T+(1/\gamma)\mathcal{P}_\rho\}n+(1/\gamma)\{(1+\mathcal{P}_\rho)\mathfrak{L}_T-(1+\mathcal{P}_T)\mathfrak{L}_\rho\}}{n+\mathfrak{L}_T}, \tag{A·1}$$

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where $\mathscr{P}_\rho$, $\mathscr{P}_T$, $\mathfrak{L}_\rho$ and $\mathfrak{L}_T$ are given by Eqs. (2·43)～(2·46). We shall now cite a few instances for $\mathscr{P}_\rho$, $\mathscr{P}_T$, $\mathfrak{L}_\rho$ and $\mathfrak{L}_T$.

i) The effects of ionizational change and optical thickness being neglected;

$$\mathscr{P}_\rho=\mathscr{P}_T=0, \tag{A·2}$$

$$\mathfrak{L}_\rho=\mathscr{R}_\rho^* \quad \text{and} \quad \mathfrak{L}_T=\mathscr{R}_T^*. \tag{A·3}$$

ii) The ionizational effect only being neglected,

$$\mathscr{P}_\rho=\frac{4(\kappa_0^*+\sigma_0^*)}{(n/c+\bar{\kappa}_0^*)(n/c+\bar{\kappa}_0^*+\sigma_0^*)+(k_J/v_t\cdot x)^2}\frac{\mathscr{R}_\rho^*}{c}, \tag{A·4}$$

$$\mathscr{P}_T=\frac{4(\kappa_0^*+\sigma_0^*)}{(n/c+\bar{\kappa}_0^*)(n/c+\bar{\kappa}_0^*+\sigma_0^*)+(k_J/v_t\cdot x)^2}\frac{\mathscr{R}_T^*}{c}, \tag{A·5}$$

$$\mathfrak{L}_\rho=(1-\kappa_0^*\eta)\mathscr{R}_\rho^* \tag{A·6}$$

and

$$\mathfrak{L}_T=(1-\kappa_0^*\eta)\mathscr{R}_T^*. \tag{A·7}$$

$\mathscr{P}_{\rho,T}$ is increasing as optical depth becomes thicker, on the other hand, $\mathfrak{L}_{\rho,T}$ is decreasing.

iii) Fluctuations being completely transparent for radiations and ionizational effect only working,

$$\mathscr{P}_\rho=\mathscr{P}_T=0, \tag{A·8}$$

$$\mathfrak{L}_\rho=\mathscr{R}_\rho^*+\left(1+\frac{\mathscr{R}'_\mu+\mathscr{I}'_\mu}{\theta n-\mathscr{I}_\mu^*}\right)\mathscr{I}_\rho^* \tag{A·9}$$

and

$$\mathfrak{L}_T=\mathscr{R}_T^*+\left(1+\frac{\mathscr{R}'_\mu+\mathscr{I}'_\mu}{\theta n-\mathscr{I}_\mu^*}\right)\mathscr{I}_T^*. \tag{A·10}$$

Next, we shall give the exact expression of the characteristic equation (A·1) for the case that the heating rate is independent of ρ^*, T^* and μ^*. (A·1) is rewritten as follows:

$$n^2+\alpha n-1=-x^2\frac{Q(n,x)}{P(n,x)}. \tag{A·11}$$

$P(n,x)$ and $Q(n,x)$ are the 4th order algebraic functions of n and are given by

$$P(n,x)=n^4+A_1n^3+A_2n^2+A_3n+A_4 \tag{A·12}$$

and

$$Q(n,x)=n^4+B_1n^3+B_2n^2+B_3n+B_4, \tag{A·13}$$

where

$$A_1=c(2\bar{\kappa}_0^*+\sigma_0^*)+\mathscr{L}_T^*-\frac{\mathscr{I}_\mu^*}{\theta}, \tag{A·14}$$

$$A_2=c^2\bar{\kappa}_0^*(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2+c(2\bar{\kappa}_0^*+\sigma_0^*)\left(\mathcal{L}_T^*-\frac{\mathcal{J}_\mu^*}{\theta}\right)$$
$$-c\bar{\kappa}_0^*\mathcal{R}_T^*+\mathcal{J}_T^*\frac{\mathcal{L}'_\mu}{\theta}-\mathcal{L}_T^*\frac{\overline{\mathcal{J}}_\mu^*}{\theta}-c\mathcal{J}_J^*\frac{\mathcal{R}'_\mu}{\theta}, \tag{A·15}$$

$$A_3=\left\{c^2(\bar{\kappa}_0^*-\kappa_0^*)(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2-c(2\bar{\kappa}^*+\sigma^*-\kappa_0^*)\frac{\overline{\mathcal{J}}_\mu^*}{\theta}+c\mathcal{J}_J^*\frac{\mathcal{J}'_\mu}{\theta}\right\}\mathcal{R}_T^*$$
$$+\left\{c^2\bar{\kappa}_0^*(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2\right.$$
$$\left.-c(2\bar{\kappa}_0^*+\sigma_0^*)\overline{\mathcal{J}}_T^*+c(2\bar{\kappa}_0^*+\sigma_0^*-\kappa_0^*-\mathcal{J}_J)\frac{\mathcal{R}'_\mu}{\theta}\right\}\mathcal{J}_T^*$$
$$-\left\{c^2\bar{\kappa}_0^*(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2\right\}\frac{\overline{\mathcal{J}}_\mu^*}{\theta}-c^2(\bar{\kappa}_0^*+\sigma_0^*)\mathcal{J}_J^*\frac{\mathcal{R}'_\mu}{\theta}, \tag{A·16}$$

$$A_4=\frac{1}{\theta}\left[\left\{c^2(\bar{\kappa}_0^*-\kappa_0^*)(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2\right\}(\mathcal{R}'_\mu\mathcal{J}_T^*-\mathcal{R}_T^*\overline{\mathcal{J}}_\mu^*)\right.$$
$$\left.-\mathcal{J}_T^*\overline{\mathcal{J}}_T^*\left\{c^2\bar{\kappa}_0^*(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2\right\}+c^2(\bar{\kappa}_0^*+\sigma_0^*)\mathcal{J}_J^*(\mathcal{R}_T^*\mathcal{J}'_\mu-\mathcal{R}'_\mu\mathcal{J}_T^*)\right] \tag{A·17}$$

and

$$B_1=c(2\bar{\kappa}_0^*+\sigma_0^*)+\frac{\mathcal{L}_T^*-\mathcal{L}_\rho^*}{\gamma}-\frac{\overline{\mathcal{J}}_\mu^*}{\theta}, \tag{A·18}$$

$$B_2=c^2\bar{\kappa}_0^*(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2+c(2\bar{\kappa}_0^*+\sigma_0^*)\left(\frac{\mathcal{L}_T^*-\mathcal{L}_\rho^*}{\gamma}-\frac{\overline{\mathcal{J}}_\mu^*}{\theta}\right)$$
$$+4c(\kappa_0^*+\sigma_0^*)\left(\mathcal{R}_T^*-\frac{\mathcal{R}_T^*-\mathcal{R}_\rho^*}{\gamma}\right)-c\bar{\kappa}_0^*(\mathcal{R}_T^*-\mathcal{R}_\rho^*)$$
$$+\frac{\mathcal{L}'_\mu}{\theta}(\mathcal{J}_T^*-\mathcal{J}_\rho^*)-\frac{\overline{\mathcal{J}}_\mu^*}{\theta}(\mathcal{L}_T^*-\mathcal{L}_\rho^*)-c\mathcal{J}_J^*\frac{\mathcal{R}'_\mu}{\theta}, \tag{A·19}$$

$$B_3=\left\{c^2\bar{\kappa}_0^*(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2\right\}\left(\frac{\mathcal{L}_T^*-\mathcal{L}_\rho^*}{\gamma}-\frac{\overline{\mathcal{J}}_\mu^*}{\theta}\right)+4c(\kappa_0^*+\sigma_0^*)$$
$$\times\left\{\frac{\mathcal{R}'_\mu}{\theta}\left(\mathcal{J}_T^*-\frac{\mathcal{J}_T^*-\mathcal{J}_\rho^*}{\gamma}\right)-\frac{\overline{\mathcal{J}}_\mu^*}{\theta}\left(\mathcal{R}_T^*-\frac{\mathcal{R}_T^*-\mathcal{R}_\rho^*}{\gamma}\right)+\frac{\mathcal{R}_\rho^*\mathcal{J}_T^*-\mathcal{R}_T^*\mathcal{J}_\rho^*}{\gamma}\right\}$$
$$-\left\{c^2\bar{\kappa}_0^*(\bar{\kappa}_0^*+\sigma_0^*)+c(2\bar{\kappa}_0^*+\sigma_0^*-\kappa_0^*)\frac{\overline{\mathcal{J}}_\mu^*}{\theta}+c\mathcal{J}_J^*\frac{\mathcal{J}'_\mu}{\theta}\right\}\frac{\mathcal{R}_T^*-\mathcal{R}_\rho^*}{\gamma}$$
$$+\left\{c(2\bar{\kappa}_0^*+\sigma_0^*-\kappa_0^*-\mathcal{J}_J^*)\frac{\mathcal{R}'_\mu}{\theta}-c(2\bar{\kappa}_0^*+\sigma_0^*)\mathcal{J}_T^*\right\}\frac{\mathcal{J}_T^*-\mathcal{J}_\rho^*}{\gamma}$$
$$-c^2(\bar{\kappa}_0^*+\sigma_0^*)\mathcal{J}_J^*\frac{\mathcal{R}'_\mu}{\theta}, \tag{A·20}$$

$$B_4=\frac{1}{\gamma\theta}\left[\left\{c^2(\bar{\kappa}_0^*-\kappa_0^*)(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2\right\}\{\mathcal{R}'_\mu(\mathcal{J}_T^*-\mathcal{J}_\rho^*)-\overline{\mathcal{J}}_\mu^*(\mathcal{R}_J^*-\mathcal{R}_\rho^*)\}\right.$$

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$$-\mathcal{J}_T^*\overline{\mathcal{J}}_T^*\left\{c^2\bar{\kappa}_0^*(\bar{\kappa}_0^*+\sigma_0^*)+\left(\frac{ck_J}{v_f}x\right)^2\right\}-4c(\kappa_0^*+\sigma_0^*)\overline{\mathcal{J}}_T^*(\mathcal{R}_\rho^*\mathcal{J}_T^*-\mathcal{R}_T^*\mathcal{J}_\rho^*)$$
$$+c^2(\bar{\kappa}_0^*+\sigma_0^*)\mathcal{J}_J^*\{\mathcal{J}'_\mu(\mathcal{R}_T^*-\mathcal{R}_\rho^*)-\mathcal{R}'_\mu(\mathcal{J}_T^*-\mathcal{J}_\rho^*)\}\Big]. \qquad (\mathrm{A}\cdot 21)$$

Then, to observe the characteristic behaviors of the eigenvalues, we give the eigenvalues at the limiting wave numbers.

For $x=0$, the three of six roots are obtained explicitly, corresponding to the growth rates of free fall, free expansion and the damping rate of radiational mode;

$$n_f=\sqrt{1+\left(\frac{\alpha}{2}\right)^2}-\frac{\alpha}{2},\quad n_e=-\sqrt{1+\left(\frac{\alpha}{2}\right)^2}-\frac{\alpha}{2} \qquad (\mathrm{A}\cdot 22)$$

and

$$n_{r1}=-c(\bar{\kappa}_0^*+\sigma_0^*). \qquad (\mathrm{A}\cdot 23)$$

The other three are the roots of the following third-order equation;

$$n^3+p_1n^2+p_2n+p_3=0, \qquad (\mathrm{A}\cdot 24)$$

where

$$p_1=c\bar{\kappa}_0^*+\mathcal{L}_T^*-\frac{\mathcal{J}_\mu^*}{\theta}, \qquad (\mathrm{A}\cdot 25)$$

$$p_2=c(\bar{\kappa}_0^*-\kappa_0^*)\mathcal{R}_T^*+c\bar{\kappa}_0^*\mathcal{J}_T^*-c\bar{\kappa}_0^*\frac{\overline{\mathcal{J}}_\mu^*}{\theta}-c\mathcal{J}_J^*\frac{\mathcal{R}'_\mu}{\theta}$$
$$+\frac{1}{\theta}(\mathcal{J}_T^*\mathcal{R}'_\mu-\overline{\mathcal{J}}_\mu^*\mathcal{R}_T^*)-\frac{1}{\theta}\mathcal{J}_T^*\overline{\mathcal{J}}_T^* \qquad (\mathrm{A}\cdot 26)$$

and

$$p_3=\frac{1}{\theta}[c(\bar{\kappa}_0^*-\kappa_0^*)(\mathcal{J}_T^*\mathcal{R}'_\mu-\overline{\mathcal{J}}_\mu^*\mathcal{R}_T^*)-c\bar{\kappa}_0^*\mathcal{J}_T^*\overline{\mathcal{J}}_T^*$$
$$-c\mathcal{J}_J^*(\mathcal{J}_T^*\mathcal{R}'_\mu-\mathcal{J}'_\mu\mathcal{R}_T^*)]. \qquad (\mathrm{A}\cdot 27)$$

These roots represents the modes of thermal, ionizational and radiative change, and they are coupled with each other.

For $x=\infty$, however, we have the explicit expression of the six roots;

$$n_c=-\frac{1}{2}\left(\frac{\mathcal{L}_T^*-\mathcal{L}_\rho^*}{\gamma}-\frac{1}{\theta}\overline{\mathcal{J}}_\mu^*\right)$$
$$-\frac{1}{2}\sqrt{\left(\frac{\mathcal{L}_T^*-\mathcal{L}_\rho^*}{\gamma}-\frac{1}{\theta}\overline{\mathcal{J}}_\mu^*\right)^2-\frac{4}{\theta}\{\mathcal{R}'_\mu(\mathcal{J}_T^*-\mathcal{J}_\rho^*)-\overline{\mathcal{J}}_\mu^*(\mathcal{R}_T^*-\mathcal{R}_\rho^*)-\mathcal{J}_T^*\overline{\mathcal{J}}_T^*\}}, \qquad (\mathrm{A}\cdot 28)$$

$$n_I=-\frac{1}{2}\left(\frac{\mathcal{L}_T^*-\mathcal{L}_\rho^*}{\gamma}-\frac{1}{\theta}\overline{\mathcal{J}}_\mu^*\right)$$
$$+\frac{1}{2}\sqrt{\left(\frac{\mathcal{L}_T^*-\mathcal{L}_\rho^*}{\gamma}-\frac{1}{\theta}\overline{\mathcal{J}}_\mu^*\right)^2-\frac{4}{\theta}\{\mathcal{R}'_\mu(\mathcal{J}_T^*-\mathcal{J}_\rho^*)-\overline{\mathcal{J}}_\mu^*(\mathcal{R}_T^*-\mathcal{R}_\rho^*)-\mathcal{J}_T^*\overline{\mathcal{J}}_T^*\}}, \qquad (\mathrm{A}\cdot 29)$$

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$$n_s = \frac{1}{2}\left(\frac{\mathcal{L}_T^* - \mathcal{L}_\rho^*}{\gamma} - \mathcal{L}_T^* - \alpha\right) \pm xi \tag{A·30}$$

and

$$n_r = -\left(\bar{\kappa}_0^* + \frac{1}{2}\sigma_0^*\right) \pm \frac{ck_J}{v_f} xi, \tag{A·31}$$

where n_c and n_I represent the condensation mode and the ionization mode, and then n_s and n_r indicate the growth rate and frequency of the sound mode and the radiative mode.

One of the cases that the thermodynamical modes are isolated from the others is given at the state where $\mathcal{J}_\mu^*/\theta$, $c\bar{\kappa}_0^* \gg \mathcal{R}_0^*$ as appeared in §4. In this case, the three modes can be explicitly expressed as follows:

For the ionization mode,

$$n_I = -\frac{\overline{\mathcal{J}_\mu^*}}{\theta}, \tag{A·32}$$

and for the radiative modes the two conjugate eigenvalues are given by

$$n_r = -c\left(\bar{\kappa}_0^* + \frac{\sigma_0^*}{2}\right) \pm c\sqrt{\left(\frac{\sigma_0^*}{2}\right)^2 - \left(\frac{k_J}{v_f}x\right)^2}. \tag{A·33}$$

Then the eigenvalues of the thermodynamical modes are given by the roots of the following equation:

$$n^2 + \alpha n - 1 = -x^2 \frac{n + \eta_0(\mathcal{R}_\rho^* + \mathcal{R}_T^*)/\gamma}{n + \eta_0 \mathcal{R}_T^*}, \tag{A·34}$$

where

$$\eta_0 = \frac{(\bar{\kappa}_0^* + \sigma_0^*)(\bar{\kappa}_0^* - \kappa_0^* - \mathcal{J}_J^*) + (k_J/v_f \cdot x)^2}{\bar{\kappa}_0^*(\bar{\kappa}_0^* + \sigma_0^*) + (k_J/v_f \cdot x)^2}. \tag{A·35}$$

(A·34) corresponds to the case of (4·1). Here, it must be noted that in this case the thermodynamical modes include the ionizational effect only through $\mathcal{J}_J^*$.

Appendix B

——Ionization process and the energy loss by free-bound process——

We assume that the matter of the universe is composed of hydrogen and helium in the ratio of 9:1 in numbers. The numbers of hydrogen and helium are related to the density as follows:

$$n_H = 0.694\left(\frac{\rho}{m_H}\right) \quad \text{and} \quad n_{He} = 0.086\left(\frac{\rho}{m_H}\right). \tag{B·1}$$

As the energy loss due to neutral helium is small, we consider only the recombination processes to neutral hydrogen H and singly-ionized helium

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HeII. We denote the ionization degrees of H and HeII as x and y, and the ionization potentials as χ_{H} and χ_{HeII}, respectively.

a) *Ionization process*

We shall give only ionization function $\mathcal{I}$. For the case of H, we have

$$\mathcal{I}_{\mathrm{II}}=\frac{0.694}{m_{\mathrm{H}}}\left[(1-x)(\mathcal{R}_{1p}+n_e C_{1p})-x n_e(\mathcal{R}_{p1}+n_e C_{p1})\right]. \tag{B·2}$$

Here,

$$\mathcal{R}_{1p}=7.84\times10^{9}\int_{\theta_r}^{\infty}\frac{d\theta_r}{\theta_r(e^{\theta_r}-1)}, \tag{B·3}$$

$$\mathcal{R}_{p1}=3.25\times10^{-6}e^{\theta}E_1(\theta)\,T^{-3/2}, \tag{B·4}$$

$$C_{1p}=1.23\times10^{-5}\theta^{-1}e^{-\theta}\,T^{-1/2}, \tag{B·5}$$

$$C_{p1}=3.20\times10^{-26}\,T^{-1}, \tag{B·6}$$

where $\mathcal{R}$ and C mean the photoelectric process and the collisional one respectively,

$$\theta_r=\chi_{\mathrm{H}}/kT_r,\quad \theta=\chi_{\mathrm{H}}/kT \quad\text{and}\quad E_1(\theta)=\int_{\theta}^{\infty}z^{-1}e^{-z}dz.$$

The case of HeII, it is different from the case of H in that the free electrons are supplied from fully-ionized hydrogens. The ionization function is given by

$$\mathcal{I}_{\mathrm{HeII}}=\frac{0.086}{m_{\mathrm{H}}}\left[(1-y)(\mathcal{R}_{1\alpha}+n_e C_{1\alpha})-y n_e(\mathcal{R}_{\alpha1}+n_e C_{\alpha1})\right]. \tag{B·7}$$

Here,

$$\mathcal{R}_{1\alpha}=1.25\times10^{11}\int_{\theta_r'}^{\infty}\frac{d\theta_r'}{\theta_r'(e^{\theta_r'}-1)}, \tag{B·8}$$

$$\mathcal{R}_{\alpha1}=5.20\times10^{-5}e^{\theta'}E_1(\theta')\,T^{-3/3}, \tag{B·9}$$

$$C_{1\alpha}=9.80\times10^{-5}\theta'^{-1}e^{-\theta'}\,T^{-1/2}, \tag{B·10}$$

$$C_{\alpha1}=4.04\times10^{-20}\theta'^{-1}\,T^{-2}. \tag{B·11}$$

The number of electron n_e is given by

$$n_e=\begin{cases}0.694x(\rho/m_{\mathrm{H}}) & \text{for H,} \quad (B·12)\\ (0.77+0.086y)(\rho/m_{\mathrm{H}}) & \text{for HeII.} \quad (B·12')\end{cases}$$

Now, we show only $\mathcal{I}_J$ among the quantities of ionization change.

$$\mathcal{I}_J=\begin{cases}4.76\times10^{6}(1-x)\theta e^{\theta}E_1(\theta_r) & \text{for H,} \quad (B·13)\\ 1.02\times10^{5}(1-y)\theta' e^{\theta'}E_1(\theta_r') & \text{for HeII.} \quad (B·13')\end{cases}$$

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b) *The energy loss due to free-bound processes*

For simplicity, the function $\mathcal{R}$ (unnormalized) of the radiative energy loss due to free-bound processes is exhibited;

$$\mathcal{R} = -\kappa J + \varepsilon, \tag{B·14}$$

where

$$\kappa = \begin{cases} 4.76\times10^{6}(1-x) & \text{for H,} \quad (\mathrm{B}\cdot15) \\ 1.02\times10^{5}(1-y) & \text{for HeII,} \quad (\mathrm{B}\cdot15') \end{cases}$$

$$J = \begin{cases} (2\chi_{\mathrm{H}}^{4}/c^{2}h^{3})\theta_r^{-1}e^{-\theta_r} & \text{for H,} \quad (\mathrm{B}\cdot16) \\ (2\chi_{\mathrm{HeII}}^{4}/c^{2}h^{3})\theta_r'^{-1}e^{-\theta_r'} & \text{for HeII,} \quad (\mathrm{B}\cdot16') \end{cases}$$

and

$$\varepsilon = \begin{cases} 7.85\times10^{25}\rho x^{2}T^{-1/2} & \text{for H,} \quad (\mathrm{B}\cdot17) \\ 1.73\times10^{26}\rho y(1+0.112y)\,T^{-1/2} & \text{for HeII.} \quad (\mathrm{B}\cdot17') \end{cases}$$

From (B·17) and (B·17′), we obtain the following relation,

$$\mathcal{R}_T = -\frac{1}{2}\mathcal{R}_\rho\,. \tag{B·18}$$

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