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Character, Character Cycle, Fixed Point Theorem and Group Representations

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§ 0. Introduction

Among many methods to derive Weyl's character formula, there is an application of the fixed point theorem (à la Atiyah Singer) to a line bundle on the flag variety. Namely, any finite-dimensional irreducible representation of a reductive group G is obtained as the cohomology group of an equivariant line bundle on the flag variety. Hence the trace of the action of an element g of G is obtained as the sum of the contributions at each fixed point. When g is a regular element, there are as many fixed points as the order of the Weyl group and each of them gives one of the terms $\operatorname{sgn} w e^{w\lambda}/\prod(e^{\alpha/2}-e^{-\alpha/2})$ in Weyl's character formula.

On the other hand, Harish-Chandra [HC] defined the character of an (infinite-dimensional) representation of a real semisimple group G_R as an invariant eigendistribution. In this paper we shall give a character formula in terms of the geometry of flag manifold as a conjecture and prove it for discrete series. The correspondence of Harish-Chandra modules and K-equivariant sheaves is completed by adding representations of G_R and G_R-equivariant sheaves (See [K_2] and also the articles of W. Schmid and J. Wolf in the same volume). Then the character would be calculated from G_R-equivariant sheaves. We can illustrate this schematically as follows.

$$\begin{array}{ccc}
 & & \{\text{invariant eigendistributions}\} \longleftarrow \\
 & & \updownarrow \\
\{\text{Harish-Chandra modules}\} & \longleftrightarrow & \{\text{representations of } G_R\} \\
\updownarrow & & \updownarrow \\
\{K\text{-equivariant sheaves}\} & \longleftrightarrow & \{G_R\text{-equivariant sheaves}\} \\
\updownarrow & & \updownarrow \\
\{\text{character cycles}\} & & \{\text{character cycles}\} \longleftarrow
\end{array}$$

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§ 1. Formalism around fixed point theorem

1.0. Let X be a compact manifold and f: $X \to X$ a continuous map. Then $\sum(-1)^i \operatorname{tr}(f: H^i(X))$ is calculated as the intersection number of the graph of f and the diagonal set. We shall generalize this fact.

1.1. Notations. In this note, for a topological space X, we denote by $D(X)$ the derived category of the abelian category of sheaves of $\boldsymbol{C}$-vector spaces. If X is a locally compact space with finite cohomological dimension, we denote by ω_X the dualizing sheaf; i.e. $\omega_X = a_X^! \boldsymbol{C}$ where a_X: $X \to \mathrm{pt}$ is the projection from X to the set (pt) consisting of a single element. Let $\boldsymbol{D}$ denote the Verdier dual, i.e. $\boldsymbol{D}(\mathscr{F}) = \boldsymbol{R}\mathscr{H}om(\mathscr{F}, \omega_X)$. For a topological manifold X, let or_X denote the orientation sheaf of X, so that we have $\omega_X = or_X[\dim X]$. For a subanalytic (resp. complex analytic) space X, $\boldsymbol{D}_{\boldsymbol{R}-c}(X)$ (resp. $D_{\boldsymbol{C}-c}(X)$) denotes the full subcategory of $D(X)$ consisting of bounded complexes with $\boldsymbol{R}$-constructible (resp. $\boldsymbol{C}$-constructible) sheaves as cohomology groups. Here an $\boldsymbol{R}$-constructible (resp. $\boldsymbol{C}$-constructible) sheaf is a sheaf $\mathscr{F}$ admitting a subanalytic (resp. complex analytic) stratification such that the restriction of $\mathscr{F}$ to each stratum is a locally constant sheaf of finite rank.

1.2. Let X be a compact real analytic manifold, $\mathscr{F}$ an $\boldsymbol{R}$-constructible complex of sheaves on X. Let $\varphi: f^*\mathscr{F} \to \mathscr{F}$ be a morphism in $D(X)$. We set

$$\operatorname{tr}\varphi = \sum_i (-1)^i \operatorname{tr}(\varphi: H^i(X; \mathscr{F})). \tag{1.2.1}$$

Then $\operatorname{tr}(\varphi)$ is expressed by local contributions as follows. Let s: $X \hookrightarrow X \times X$ denotes the graph map $x \mapsto (x, f(x))$, j: $X \hookrightarrow X \times X$ the diagonal embedding and p_i: $X \times X \hookrightarrow X$ the i-th projection ($i = 1, 2$). Then we have a chain of homomorphisms

$$\begin{aligned} &\boldsymbol{R}\operatorname{Hom}(f^*\mathscr{F}, \mathscr{F}) \rightleftarrows \boldsymbol{R}\Gamma(X; \boldsymbol{R}\mathscr{H}om(s^*p_2^*\mathscr{F}, s^!p_1^!\mathscr{F})) \\ &\rightleftarrows \boldsymbol{R}\Gamma(X \times X; s^!\boldsymbol{R}\mathscr{H}om(p_2^*\mathscr{F}, p_1^!\mathscr{F})) \rightleftarrows \boldsymbol{R}\Gamma_{s(X)}(X \times X; \mathscr{F} \boxtimes \boldsymbol{D}\mathscr{F}) \\ &\to \boldsymbol{R}\Gamma_{j^{-1}s(X)}(X; j^*(\mathscr{F} \boxtimes \boldsymbol{D}\mathscr{F})) \to \boldsymbol{R}\Gamma_{j^{-1}s(X)}(X; \mathscr{F} \otimes \boldsymbol{D}\mathscr{F}) \\ &\to \boldsymbol{R}\Gamma_{j^{-1}s(X)}(X; \omega_X), \end{aligned} \tag{1.2.2}$$

and

$$\boldsymbol{R}\Gamma_{j^{-1}s(X)}(X; \omega_X) \to \boldsymbol{R}\Gamma(X; \omega_X) \to \boldsymbol{C}. \tag{1.2.3}$$

Then the image of $\varphi \in \operatorname{Hom}(f^*\mathscr{F}, \mathscr{F})$ by their composition $\operatorname{Hom}(f^*\mathscr{F}, \mathscr{F}) \to \boldsymbol{C}$ coincides with $\operatorname{tr}(\varphi)$.

1.3. Assume moreover that, in the situation of Section 1.2, the fixed point set $j^{-1}s(X)$ is discrete. Then to each fixed point $x \in X$, we can associate the image of φ by the composition of $\mathrm{Hom}(f^*\mathscr{F}, \mathscr{F}) \to H^0_{j^{-1}s(X)}(X; \omega_X) \rightleftarrows \bigoplus_{y \in j^{-1}s(X)} H^0_{\{y\}}(X; \omega_X) \to H^0_{\{x\}}(X; \omega_X) \to \boldsymbol{C}$. We shall denote this by $\mathrm{tr}_x(\varphi)$. Then $\mathrm{tr}(\varphi)$ is expressed by the local contributions:

$$\mathrm{tr}(\varphi) = \sum_x \mathrm{tr}_x(\varphi), \tag{1.3.1}$$

where x ranges over the fixed point set.

1.4. We shall calculate explicitly $\mathrm{tr}_x(\varphi)$. Assume that

(1.4.1) *X is a real analytic manifold and the diagonal set and the graph o. f intersect transversally.*

Let x be a fixed point of f. Then f induces the homomorphism $f_*: T_xX \to T_xX$ of the tangent space and φ induces $\varphi: \nu_x(\mathscr{F}) \to \nu_x(\mathscr{F})$. Here ν denotes the normalization functor (See [KS]).

Let V_s be a vector subspace of T_xX invariant by f_* satisfying (1.4.2) and (1.4.3):

(1.4.2) No eigenvalue λ of $f_* \mid V_s$ satisfies $\lambda > 1$

(1.4.3) No eigenvalue λ of $f_* \mid T_xX/V_s$ satisfies $0 \leqq \lambda < 1$.

Then we can prove the following proposition.

Proposition 1.4.1. *Under the condition* (1.4.1), *we have*

$$\mathrm{tr}_x(\varphi) = \sum_i (-1)^i \, \mathrm{tr}(\varphi: H^i_{V_s}(T_xX; \nu_x(\mathscr{F}))).$$

Corollary 1.4.2. *If moreover X is a complex manifold and if $\mathscr{F}$ is $\boldsymbol{C}$-constructible then we have*

$$\mathrm{tr}_x(\varphi) = \sum_i (-1)^i \, \mathrm{tr}(\varphi; H^i_{\{x\}}(X; \mathscr{F}))$$
$$= \sum_i (-1)^i \, \mathrm{tr}(\varphi; \mathscr{H}^i(\mathscr{F})_x).$$

Example 1.4.3. Set $X = \boldsymbol{R} \cup \infty$, $f: x \mapsto ax$, $\mathscr{F} = \boldsymbol{C}_{\{x \geqq 0\}}$ and let $\varphi: f^*\mathscr{F} \to \mathscr{F}$ be the homomorphism such that $\varphi_0: \mathscr{F}_0 \to \mathscr{F}_0$ is the identity. Then, at $x=0$, $\mathrm{tr}_x(\varphi) = 0$ when $a > 1$ and $\mathrm{tr}_x(\varphi) = 1$ when $0 < a < 1$.

1.5. We shall generalize the situation in 1.4. Let X and Y be subanalytic spaces, $f, g: X \rightrightarrows Y$ two continuous subanalytic maps, and $F \in D_{\boldsymbol{R}-c}(Y)$. Let $\varphi: f^*\mathscr{F} \to g^*\mathscr{F}$ be a morphism. Let us denote by $s: X \to Y \times Y$ the map $x \mapsto (f(x), g(x))$ and let Δ_Y denote the diagonal set and

$Z=s^{-1}\Delta_Y=\{x\in X; f(x)=g(x)\}$. Then we have a chain of homomorphisms:

$$\begin{aligned}\boldsymbol{R}\operatorname{Hom}(\mathscr{F},\mathscr{F}) &\leftarrow\!\!\!\!\!\!\sim \boldsymbol{R}\Gamma_{\Delta_Y}(Y\times Y;\mathscr{F}\boxtimes\boldsymbol{D}\mathscr{F})\to\boldsymbol{R}\Gamma_Z(X;s^*(\mathscr{F}\boxtimes\boldsymbol{D}\mathscr{F}))\\ &\leftarrow\!\!\!\!\!\!\sim \boldsymbol{R}\Gamma_Z(X;f^*\mathscr{F}\otimes g^*\boldsymbol{D}\mathscr{F})\xrightarrow{\varphi}\boldsymbol{R}\Gamma_Z(X;g^*\mathscr{F}\otimes g^*\boldsymbol{D}\mathscr{F}))\\ &\leftarrow\!\!\!\!\!\!\sim \boldsymbol{R}\Gamma_Z(X;g^*(\mathscr{F}\otimes\boldsymbol{D}\mathscr{F}))\to\boldsymbol{R}\Gamma_Z(X;g^*\omega_Y).\end{aligned}$$

Hence $\mathrm{id}_{\mathscr{F}}\in\operatorname{Hom}(\mathscr{F},\mathscr{F})=H^0(\boldsymbol{R}\operatorname{Hom}(\mathscr{F},\mathscr{F}))$ gives an element $c(\mathscr{F},\varphi)\in H^0_Z(X;g^*\omega_Y)$. If $X=Y$ and $g=\mathrm{id}$, then this construction coincides with the former one.

1.6. We shall specialize the preceding construction to a group action case. Let X be a subanalytic space, G a Lie group operating on X. Let $\mathscr{F}$ be a G-equivariant $\boldsymbol{R}$-constructible complex of sheaves. In this note, we shall not investigate systematically the notion of G-equivariant $\boldsymbol{R}$-constructible complex of sheaves. However, this notion implies an isomorphism $\varphi\colon\mu^*\mathscr{F}\to\mathrm{pr}^*\mathscr{F}$, where $\mu\colon G\times X\to X$ is the composition map $(g,x)\mapsto gx$ and $\mathrm{pr}\colon G\times X\to X$ is the second projection. Thus we can apply the result of Section 1.5, and we obtain $c(\mathscr{F},\varphi)\in H^0_{\tilde{G}}(G\times X;\mathrm{pr}^*\omega_X)$. Here $\tilde{G}$ denotes the fixed point set $\{(g,x)\in G\times X; gx=x\}$. Note that $H^0_{\tilde{G}}(G\times X;\mathrm{pr}^*\omega_X)=H^0_{\tilde{G}}(G\times X;\omega_{G\times X}\otimes or_G[-\dim G])=H^{-\dim G}(\tilde{G};\omega_{\tilde{G}}\otimes or_G)=H^{\inf}_{\dim G}(\tilde{G};or_G)$. Here $H^{\inf}_n(\tilde{G};or_G)$ is the n-th homology group of or_G-valued locally finite chains. We shall denote $c(\mathscr{F},\varphi)$ by $\underline{\mathrm{ch}}(\mathscr{F})\in H^{\inf}_{\dim G}(\tilde{G};or_G)$ and call it the *character cycle* of $\mathscr{F}$.

1.7. When X has only finitely many G-orbits, the situation is simple. In fact, in such a case, we have $\tilde{G}=\bigcup\pi^{-1}(S)$, where π is the projection $\tilde{G}\to X$. Then $\pi^{-1}(S)$ being the fiber bundle over S with the isotropy subgroup as a fiber, $\pi^{-1}(S)$ is a $(\dim G)$-dimensional manifold. Hence we have $H^{\inf}_{\dim G}(\tilde{G};or_G)\subset A^N$, where N is the number of connected components of the regular locus of $\tilde{G}$.

1.8. In ($[\mathrm{K}_1]$), we defined the characteristic cycle for constructible sheaves. Let us reformulate this. Let X be a real analytic manifold and $\mathscr{F}\in\boldsymbol{D}_{\boldsymbol{R}-c}(X)$. Then, denoting by Δ_X the diagonal set and by μ the microlocalization functor (see [KS]), we have

$$\begin{aligned}\boldsymbol{R}\operatorname{Hom}(\mathscr{F},\mathscr{F})&\to\boldsymbol{R}\Gamma_{\Delta_X}(X\times X;\mathscr{F}\boxtimes\boldsymbol{D}\mathscr{F})\to\boldsymbol{R}\Gamma(T^*X;\mu_{\Delta_X}(\mathscr{F}\boxtimes\boldsymbol{D}\mathscr{F}))\\ &=\boldsymbol{R}\Gamma_{SS\mathscr{F}}(T^*X;\mu_{\Delta_X}(\mathscr{F}\boxtimes\boldsymbol{D}\mathscr{F}))\\ &\to\boldsymbol{R}\Gamma_{SS\mathscr{F}}(T^*X;\mu_{\Delta_X}(\boldsymbol{R}j_*j^*(\mathscr{F}\boxtimes\boldsymbol{D}\mathscr{F})))\to\boldsymbol{R}\Gamma_{SS\mathscr{F}}(T^*X;\mu_{\Delta_X}(j_*\omega_X))\\ &\simeq\boldsymbol{R}\Gamma_{SS\mathscr{F}}(T^*X;\pi^*\omega_X).\end{aligned}$$

Here $\pi: T^*X \to X$ is the cotangent bundle and $j: X \hookrightarrow X \times X$ is the diagonal embedding. We have furthermore

$$\boldsymbol{R}\Gamma_{SS\mathscr{F}}(T^*X;\, \pi^*\omega_X) = \boldsymbol{R}\Gamma_{SS\mathscr{F}}(T^*X;\, \omega_{T^*X} \otimes or_X[-\dim X]).$$

The image of $\mathrm{id}_{\mathscr{F}} \in \mathrm{Hom}(\mathscr{F}, \mathscr{F})$ by the homomorphism

$$\mathrm{Hom}(\mathscr{F}, \mathscr{F}) \to H^{-\dim X}_{SS\mathscr{F}}(T^*X;\, \omega_{T^*X} \otimes or_X) = H^{\inf}_{\dim X}(SS\mathscr{F};\, or_X)$$

is called the *characteristic cycle* of F, and denoted by $\underline{SS}(\mathscr{F})$. This definition coincides with the one given in [K_1].

1.9. Let X be a homogeneous space of a Lie group G and let H be a subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebra of G and H, respectively. Let $\mathscr{F}$ be an H-equivariant $\boldsymbol{R}$-constructible complex on X. Let us investigate the relation between the character cycle $\underline{\mathrm{ch}}(\mathscr{F})$ and the characteristic cycle $\underline{SS}(\mathscr{F})$ of F. Let $\rho: \tilde{G} \to G$ and $\pi: \tilde{G} \to X$ denote the projections. Let us consider the following chain of homomorphisms

$$\begin{aligned} \boldsymbol{R}\Gamma_{\rho^{-1}(H)}(H \times X;\, \boldsymbol{C}_H \boxtimes \omega_X) &= \boldsymbol{R}\Gamma(G \times X;\, \boldsymbol{R}\mathscr{H}om(\boldsymbol{C}_{\tilde{G}}, \boldsymbol{C}_H \boxtimes \omega_X)) \\ &\to \boldsymbol{R}\Gamma(T^*_{e \times X}(G \times X);\, \boldsymbol{R}\mathscr{H}om(\mu_{e \times X}(\boldsymbol{C}_{\tilde{G}}), \mu_{e \times X}(\boldsymbol{C}_H \boxtimes \omega_X))). \end{aligned} \tag{1.9.1}$$

Here $\mu_{e \times X}$ is the microlocalization functor along $\{e\} \times X$. Note that $T^*_{e \times X}(G \times X) = \mathfrak{g}^* \times X$ and

$$\mu_{e \times X}(\boldsymbol{C}_{\tilde{G}}) = \boldsymbol{C}_{T^*X} \otimes or_{\mathfrak{g}} \otimes or_X[\dim X - \dim G]$$

and

$$\mu_{e \times X}(\boldsymbol{C}_H \boxtimes \omega_X) = (\boldsymbol{C}_{\mathfrak{h}^\perp} \boxtimes \omega_X) \otimes or_{\mathfrak{h}}[\dim H] = \omega_{\mathfrak{h}^\perp \times X} \otimes or_{\mathfrak{g}}[-\dim G].$$

Here we identify T^*X with the subset of $\mathfrak{g}^* \times X$ by the moment map $\bar{\rho}: T^*X \to \mathfrak{g}^*$ and $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ denotes the orthogonal complement. Hence the last term of (1.9.1) coincides with

$$\begin{aligned} &\boldsymbol{R}\Gamma(\mathfrak{g}^* \times X;\, \boldsymbol{R}\Gamma_{T^*X}(\omega_{\mathfrak{h}^\perp \times X}) \otimes or_X[-\dim X]) \\ &\quad = \boldsymbol{R}\Gamma(\bar{\rho}^{-1}(\mathfrak{h}^\perp);\, \omega_{\bar{\rho}^{-1}(\mathfrak{h}^\perp)} \otimes or_X[-\dim X]). \end{aligned}$$

Thus we obtained

$$H^0_{\rho^{-1}(H)}(H \times X;\, \boldsymbol{C}_H \boxtimes \omega_X) \to H^{-\dim X}_{\bar{\rho}^{-1}(\mathfrak{h}^\perp)}(T^*X;\, \omega_{T^*X} \otimes or_X) \tag{1.9.2}$$

Since $\mathscr{F}$ is H-equivariant, $SS(\mathscr{F})$ is contained in $\bar{\rho}^{-1}(\mathfrak{h}^\perp)$ and we obtain

$$H^{-\dim X}_{SS(\mathscr{F})}(T^*X;\, \omega_{T^*X} \otimes or_X) \to H^{-\dim X}_{\bar{\rho}^{-1}(\mathfrak{h}^\perp)}(T^*X;\, \omega_{T^*X} \otimes or_X). \tag{1.9.3}$$

One can easily prove the following proposition.

Proposition 1.9.1. *The image of the character cycle* $\underline{\mathrm{ch}}(\mathscr{F})$ *by the homomorphism* (1.9.2) *coincides with the image of the characteristic cycle* $\underline{SS}(\mathscr{F})$ *by the homomorphism* (1.9.3).

If X has finitely many H-orbits, then the homomorphism (1.9.3) is injective because $\dim \bar{\rho}^{-1}(\mathfrak{h}^*) \leqq \dim X$. Therefore $\underline{SS}(\mathscr{F})$ is determined by $\underline{\mathrm{ch}}(\mathscr{F})$.

§ 2. Representation of semisimple groups

2.1. Let G be a connected complex semisimple group and let $G_{\boldsymbol{R}} \hookrightarrow G$ be a real form of G. Let $K_{\boldsymbol{R}}$ be a maximal compact subgroup of $G_{\boldsymbol{R}}$ and K the complexification of $K_{\boldsymbol{R}}$. Let X be the flag manifold of G. Set $\tilde{G} = \{(g, x) \in G \times X; gx = x\}$ and let $\rho: \tilde{G} \to G$ and $\pi; \tilde{G} \to X$ denote the projections. For $x \in X$, let $B(x)$ denote the Borel subgroup $\pi^{-1}(x)$ corresponding to x. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{g}_{\boldsymbol{R}}$, $\mathfrak{k}_{\boldsymbol{R}}$ and $\mathfrak{b}(x)$ denote the Lie algebra of G, K, $G_{\boldsymbol{R}}$, $K_{\boldsymbol{R}}$ and $B(x)$, respectively.

Let us denote by G_{reg} the set of regular semisimple elements of G.

2.2. Let $\mathscr{M}$ denote the $\mathscr{D}_G$-module for invariant eigendistributions. Hence $\mathscr{M}$ is a $\mathscr{D}_G$-module generated by a section u with the relation

$$(2.2.1) \qquad (\mathrm{Ad}\,\mathfrak{g}) \cdot u = 0, \qquad Pu = \chi(P)u \quad \text{for} \quad P \in \mathscr{Z}(\mathfrak{g}).$$

Here $\mathrm{Ad}\,\mathfrak{g}$ is the image of $\mathfrak{g} \to \Gamma(G; \Theta_G)$ derived by the adjoint action of G on G, and $\mathscr{Z}(\mathfrak{g})$ is the center of $U(\mathfrak{g})$ considered as the space of bi-invariant differential operators on G and χ is the trivial infinitesimal character $\mathscr{Z}(\mathfrak{g}) \to \mathscr{Z}(\mathfrak{g})/(\mathscr{Z}(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{g}) \to \boldsymbol{C}$. A similar argument to [HK] leads us the following proposition.

Proposition 2.2.1. (i) *$\mathscr{M}$ is a regular holonomic $\mathscr{D}_G$-module.*
(ii) $\boldsymbol{R}\mathscr{H}om(\mathscr{M}, \mathcal{O}_G) = \boldsymbol{R}\rho_* \boldsymbol{C}_{\tilde{G}}$.
(iii) *$\boldsymbol{R}\rho_* \boldsymbol{C}_{\tilde{G}}$ is the minimal extension of $\rho_* \boldsymbol{C}_{\tilde{G}}|_{G_{\mathrm{reg}}}$.*

2.3. Let us investigate the space of invariant eigendistributions. This is equal to $H^0(G_{\boldsymbol{R}}; \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_G}(\mathscr{M}, \mathscr{B}_{G_{\boldsymbol{R}}}))$. Here, $\mathscr{B}_{G_{\boldsymbol{R}}} = \mathscr{H}^{\dim G_{\boldsymbol{R}}}_{G_{\boldsymbol{R}}}(\mathcal{O}_X) \otimes or_G \otimes or_{G_{\boldsymbol{R}}}$ is the sheaf of hyperfunctions. We have

$$\begin{aligned}
&\boldsymbol{R}\Gamma(G_{\boldsymbol{R}}; \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_G}(\mathscr{M}, \mathscr{B}_{G_{\boldsymbol{R}}})) \\
&\quad = \boldsymbol{R}\Gamma(G; \boldsymbol{R}\Gamma_{G_{\boldsymbol{R}}}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_G}(\mathscr{M}; \mathcal{O}_X)) \otimes or_G \otimes or_{G_{\boldsymbol{R}}})[\dim G_{\boldsymbol{R}}] \\
&\quad = \boldsymbol{R}\Gamma(G; \boldsymbol{R}\Gamma_{G_{\boldsymbol{R}}}(\boldsymbol{R}\rho_* \boldsymbol{C}_{\tilde{G}}) \otimes or_G \otimes or_{G_{\boldsymbol{R}}})[\dim G_{\boldsymbol{R}}] \\
&\quad = \boldsymbol{R}\Gamma(\tilde{G}; \boldsymbol{R}\Gamma_{\rho^{-1}G_{\boldsymbol{R}}}(\boldsymbol{C}_{\tilde{G}}) \otimes or_G \otimes or_{G_{\boldsymbol{R}}})[\dim G_{\boldsymbol{R}}] \\
&\quad = \boldsymbol{R}\Gamma(\rho^{-1}G_{\boldsymbol{R}}; \omega_{\rho^{-1}G_{\boldsymbol{R}}} \otimes or_{G_{\boldsymbol{R}}})[-\dim G_{\boldsymbol{R}}].
\end{aligned}$$

Hence we have

Proposition 2.3.1. *The space of invariant eigendistribution on $G_{\boldsymbol{R}}$ coincides with* $H^{\inf}_{\dim G_{\boldsymbol{R}}}(\rho^{-1}G_{\boldsymbol{R}};\ {}_{o}\gamma_{G_{\boldsymbol{R}}})$.

We shall call this correspondence the *Hirai correspondence* by the reason explained in the next paragraph.

2.4. Let us write explicitly the correspondence in Proposition 2.3.1. Let $p=(g, x)$ be a point of $\tilde{G}$. The isotropy group $G_x=\pi^{-1}(x)$ acts on $T^*_xX=(\mathfrak{g}/\mathfrak{b}(x))^*$. Let us denote $\psi(p)=1/\det(1-g\colon T^*_xX)$. Then ψ is the meromorphic function with the pole in $\rho^{-1}G_{\text{reg}}$. If T is a Cartan subgroup containing g and contained in $B(x)$, we have

$$\psi=\frac{1}{\prod\limits_{\alpha>0}(1-e^{-\alpha(a)})}$$

for $g=e^a$, $a\in \operatorname{Lie}(T)$. For $\sigma\in H^{\inf}_{\dim G_{\boldsymbol{R}}}(\rho^{-1}(G_{\boldsymbol{R}});\ {}_{o}\gamma_{G\boldsymbol{R}})$, let us denote by f_σ the corresponding invariant eigendistribution. Then for a regular semi-simple element g of $G_{\boldsymbol{R}}$, we have

$$f_\sigma(g)=\sum_{p\in\rho^{-1}(g)}\sigma(p)\psi(p). \tag{2.4.1}$$

Here, $\sigma(p)$ is the intersection number of σ and $\rho^{-1}(g)$ at p.

If an invariant eigendistribution f on $G_{\boldsymbol{R},\text{reg}}$ is given, then it determines the ($\dim G$)-chain α in $\rho^{-1}G_{\boldsymbol{R}}$. Then f is extended to an invariant eigendistribution on $G_{\boldsymbol{R}}$ if and only if the boundary of α vanishes. If we write it down, we obtain Hirai's connection formula for invariant eigendistributions ([H]).

2.5. By Matsuki [M], there exists a correspondence between K-orbits and $G_{\boldsymbol{R}}$-orbits. This correspondence $S\leftrightarrow S^a$ is characterized by the following property:

$$S\cap S^a \text{ is compact and non empty.} \tag{2.5.1}$$

In such a case, $S\cap S^a$ is a homogeneous space over $K_{\boldsymbol{R}}$. Moreover, we have

$$K_{\boldsymbol{R}x}/K^\circ_x \xrightarrow{\sim} K/K^\circ_x \quad\text{and}\quad K_{\boldsymbol{R}x}/K^\circ_{\boldsymbol{R}x}\xrightarrow{\sim} G_{\boldsymbol{R}x}/G^\circ_{\boldsymbol{R}x} \quad \text{for } x\in S\cap S^a. \tag{2.5.2}$$

Here the subscript x signifies the isotropy subgroup at x and $\circ$ means the connected component containing the identity.

This shows immediately the following lemma.

Lemma 2.5.1. *The set of pairs* $(S, \mathscr{F})$ *of a* K*-orbit* S *and a* K*-equivariant local system* $\mathscr{F}$ *on* S *is isomorphic to the set of pairs* $(S^a, \mathscr{F}^a)$ *of a* $G_{\boldsymbol{R}}$*-orbit* S^a *and a* $G_{\boldsymbol{R}}$*-equivariant local system* $\mathscr{F}^a$ *on* S^a.

Note that $(S, \mathscr{F})$ and $(S^a, \mathscr{F}^a)$ correspond if we have (2.5.1) and

(2.5.3) $\quad \mathscr{F}|_{S\cap S^a} \cong \mathscr{F}^a|_{S\cap S^a}$ as $K_{\boldsymbol{R}}$-equivariant local systems.

We call this correspondence the *Matsuki correspondence.*

2.6. Let $\sigma=(S, \mathscr{F})$ be a pair of a K-orbit S and a K-equivariant local system $\mathscr{F}$ on S. Let $j: S \hookrightarrow X$ be the embedding. Let $\mathscr{M}$ be a regular holonomic $\mathscr{D}_X$-module such that $j_!\mathscr{F}[\operatorname{codim} S]=\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. Then $\mathscr{M}$ is a K-equivariant $\mathscr{D}_X$-module. Then $E=H^0(X: \mathscr{M})$ is a Harish-Chandra module. To E we can associate a representation of $G_{\boldsymbol{R}}$ such that the space of its $K_{\boldsymbol{R}}$-finite vectors is E. Let $\chi(E)$ be its character, which is an invariant eigendistribution on $G_{\boldsymbol{R}}$ (Harish-Chandra [HC]).

Let $(S^a, \mathscr{F}^a)$ be the associated pair to σ by the Matsuki correspondence. Denoting by $j_a: S^a \hookrightarrow X$ the imbedding, we set $\mathscr{F}'=\boldsymbol{R}j_{a*}(\mathscr{F}^a \otimes j_a^! C_X)$ $[-\operatorname{codim}_{\boldsymbol{C}} S]$ (See [K₂]). Since $\mathscr{F}'$ is a $G_{\boldsymbol{R}}$-equivariant sheaf, we can define its character cycle $\underline{\mathrm{ch}}(\mathscr{F}')$ as in Section 1.6. We have

$$\underline{\mathrm{ch}}(\mathscr{F}') \in H^{\inf}_{\dim G_{\boldsymbol{R}}}(\rho^{-1}G_{\boldsymbol{R}}, or_{\boldsymbol{R}}).$$

Conjecture. *The character* $\chi(E)$ *of* E *is equal to the invariant eigendistribution corresponding to* $\underline{\mathrm{ch}}(\mathscr{F}')$ *by the Hirai correspondence* (§ 2.3).

2.7. We shall prove the conjecture when the representation E is a discrete series. In such a case, S is a closed orbits, S^a is an open orbit, and $\mathscr{F}$ is a trivial local system. Let f be the invariant eigendistribution corresponding to $\underline{\mathrm{ch}}(\mathscr{F}')$. By the characterization of $\chi(E)$ due to Harish-Chandra ([HC]), it is enough to show

(2.7.1) $\quad f=\chi(E)$ on the regular part of a compact Cartan subgroup,

(2.7.2) $\quad |Df|$ is bounded on $G_{\boldsymbol{R},\mathrm{reg}}$.

Here D is the discriminant $|\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})|$.

2.8. In order to prove (2.7.1) and (2.7.2), we shall calculate $\underline{\mathrm{ch}}(\mathscr{F})$ for a $G_{\boldsymbol{R}}$-equivariant sheaf $\mathscr{F}$. Let g be a regular semisimple element of $G_{\boldsymbol{R}}$. Let $p=(g, x)$ be a point in $\tilde{G}$ above g.

Let H be a Cartan subgroup containing g and let $\mathfrak{g}=(\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)\oplus \mathrm{Lie}(H)$ be the root space decomposition with respect to $\mathrm{Lie}(H)$. Let $\mathfrak{h}=\bigoplus_{\alpha\in\Delta_-}\mathfrak{g}_\alpha\oplus\mathrm{Lie}(H)$ be the isotropy subalgebra at x and

(2.8.1) $$\mathfrak{n}(p)=\bigoplus_{\alpha\in\Delta_+,|(e^\alpha)(g)|<1}\mathfrak{g}_\alpha.$$

Here $\Delta=\Delta_+\cup\Delta_-$ is the corresponding positive and negative roots system. Then $\mathfrak{n}(p)$ is a nilpotent Lie algebra. Set $U(p)=\exp\mathfrak{n}(p)$. Then by Proposition 1.4.1, we can see easily

Proposition 2.8.1. $(\underline{\mathrm{ch}}(\mathscr{F})\cdot\rho^{-1}(g))_p=\mathrm{tr}(g\colon \boldsymbol{R}\Gamma_{U(p)\cdot x}(X;\mathscr{F}))$.

2.9. Coming back to the situation of Section 2.7, we shall prove (2.7.1). Let us take a compact Cartan subgroup H. Then any fixed point of H in X is contained in an open $G_{\boldsymbol{R}}$-orbit. Let us take a fixed point $x_0\in S^\alpha$ and choose a positive ordering $\Delta_+(x_0)$ of the root system of $(\mathfrak{g}, \mathrm{Lie}(H))$ such that $\Delta_+(x_0)=\Delta(T_{x_0}X)$.

Set $W_{\boldsymbol{R}}=N(H)/H$. Then we have

(2.9.1) $$L=\{x\in S^\alpha;\ Hx=x\}=W_{\boldsymbol{R}}\cdot x_0.$$

By Harish-Chandra [HC], we have

(2.9.2) $$(-1)^q\chi(E)=\frac{\sum_{w\in W_{\boldsymbol{R}}}(\mathrm{sgn}\ w)e^{w\rho(x_0)}}{\prod_{\alpha\in\Delta+(x_0)}(e^{\alpha/2}-e^{-\alpha/2})}=\sum_{x\in L}\frac{1}{\prod_{\alpha\in\Delta+(x)}(1-e^{-\alpha})}$$

where $q=\frac{1}{2}\dim(G_{\boldsymbol{R}}/K_{\boldsymbol{R}})$.

On the other hand, for $h\in H_{\mathrm{reg}}$ and $x\in L$

$$\mathrm{tr}(h\colon \boldsymbol{R}\Gamma_{\{x\}}(X;\mathscr{F}))=(-1)^{\mathrm{codim}\ S}$$

Hence by Proposition the value of f at h equals

$$\sum_{x\in L}\frac{1}{\prod_{\alpha\in\Delta+(x)}(1-e^{-\alpha})}(-1)^{\mathrm{codim}\ S}.$$

Hence (2.7.1) follows from $q=\mathrm{codim}\ S$.

2.10. Finally, we shall prove (2.7.2). Let us take $g\in G_{\boldsymbol{R}}$ and $p=(g, x)\in\tilde{G}$. As seen in Section 2.8, the contribution from p to $f(g)$ is

(2.10.1) $$c\cdot\frac{1}{\prod_{\alpha\in\Delta+(x)}(1-e^{-\alpha})}=\frac{ce^\rho}{\prod(e^{\alpha/2}-e^{-\alpha/2})}$$

Here $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha$, $c=\mathrm{tr}(g\colon \boldsymbol{R}\Gamma_{U(p)x}(\mathscr{F}'))$ and $e^\rho=\sqrt{\det(g\colon T_xX)}$

Hence in order to prove (2.7.2) it is enough to show

(2.10.2) If $c\neq 0$, then $|\det(g: T_xX)|\leqq 1$.

We have $R\Gamma_{U(p)x}(X; \mathscr{F}')=R\,\mathrm{Hom}((\mathscr{F}^a\otimes j_a^!C_X)_{U(p)x}; C_X)$. Hence it is zero if $U(p)x\cap S^a=\varnothing$. Therefore (2.10.2) is a consequence of

(2.10.3) If $U(p)x\cap S^a\neq\varnothing$, then $|\det(g: T_xX)|\leqq 1$.

This is an easy consequence of Lemma 7 of [OM] (p. 378). Thus we obtain

Proposition 2.10.1. *Conjecture is true if E is a discrete series.*

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