Character Degrees of Extensions of $\mathrm{PSL}_2(q)$ and $\mathrm{SL}_2(q)$

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Abstract

Denote by S the projective special linear group $\mathrm{PSL}_2(q)$ over the field of q elements. We determine, for all values of $q > 3$, the degrees of the irreducible complex characters of every group H such that $S \leqslant H \leqslant \mathrm{Aut}(S)$. We also determine the character degrees of certain extensions of the special linear group $\mathrm{SL}_2(q)$. Explicit knowledge of the character tables of $\mathrm{SL}_2(q)$, $\mathrm{GL}_2(q)$, $\mathrm{PSL}_2(q)$, and $\mathrm{PGL}_2(q)$ is used along with standard Clifford theory to obtain the degrees.

1 Introduction

In a series of recent articles, M. L. Lewis and the author have studied various properties of the set of degrees of irreducible complex characters of nonsolvable groups. This always requires detailed information on the character degrees of finite simple groups and, in order to extend results to general nonsolvable groups, often requires information on the degrees of almost simple groups; that is, groups H such that $S \leqslant H \leqslant \mathrm{Aut}(S)$ for some simple group S. The most interesting cases tend to be groups involving the smaller simple groups with few character degrees, in particular the 2-dimensional projective special linear groups $\mathrm{PSL}_2(q)$, which we will denote by $\mathrm{L}_2(q)$ in Atlas [2] notation.

Several of these studies required increasingly detailed information about the character degrees of groups H with $\mathrm{L}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{L}_2(q))$ (see [5, 6, 7, 8]). Of course, the character table of $\mathrm{L}_2(q)$ is well-known, as is the automorphism group, and so the character degrees of H are known in principle. Since $\mathrm{L}_2(q)$ is a normal subgroup of such a group H, each character degree of H will be $\chi(1) \cdot j$ for some irreducible character χ of $\mathrm{L}_2(q)$ and some divisor j of $|H : \mathrm{L}_2(q)|$. Determining the values of j for which $\chi(1) \cdot j$ is a character degree of H for a specific group H and character χ of $\mathrm{L}_2(q)$ is not theoretically difficult. However, given the more detailed information required in recent work and the number of different possibilities for H, χ, and q, it has become much more convenient to have a complete answer covering all cases than to derive the necessary information on a case-by-case basis. For more recent work, it has also become useful to have information about the degrees of extensions of the quasisimple groups $\mathrm{SL}_2(q)$ when q is odd. In Theorem A, we give the list of character degrees of H, where $\mathrm{L}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{L}_2(q))$, in all cases. In Theorem B and Corollary C, we give the character degrees of all extensions of $\mathrm{SL}_2(q)$ of a certain type.

Let $q = p^f$ for some prime p. The outer automorphism group of $\mathrm{L}_2(q)$ is generated by a field automorphism φ of order f and, if p is odd, a diagonal automorphism $\bar{\delta}$ of order 2. If $p = 2$, then $\bar{\delta}$ is an inner automorphism. The diagonal automorphism $\bar{\delta}$ is induced by conjugation on $\mathrm{SL}_2(q)$ by a diagonal matrix $\delta \in \mathrm{GL}_2(q)$ of order $q-1$. We have $\mathrm{L}_2(q)\langle\bar{\delta}\rangle = \mathrm{PGL}_2(q)$ and $\mathrm{SL}_2(q)\langle\delta\rangle = \mathrm{GL}_2(q)$. The character tables of $\mathrm{PGL}_2(q)$ and $\mathrm{GL}_2(q)$ are also known. In §3, we describe explicitly the actions of the automorphisms on the conjugacy classes of $\mathrm{SL}_2(q)$, $\mathrm{GL}_2(q)$, $\mathrm{L}_2(q)$, and $\mathrm{PGL}_2(q)$, and in §4, we describe the actions of the automorphisms on the irreducible characters.

If $q = p^f > 5$ is odd, the character degree set of $\mathrm{L}_2(q)$ is

$$\mathrm{cd}(\mathrm{L}_2(q)) = \{1, q, (q+\varepsilon)/2, q-1, q+1\},$$

where $\varepsilon = (-1)^{(q-1)/2}$, and the character degree set of $\mathrm{L}_2(q)$ for even q or $\mathrm{PGL}_2(q)$ for odd q is

$$\mathrm{cd}(\mathrm{L}_2(q)) = \{1, q, q-1, q+1\}.$$

The characters of degrees 1 and q are invariant in $\mathrm{Aut}(\mathrm{L}_2(q))$ and in fact extend to irreducible characters of H for any $\mathrm{L}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{L}_2(q))$. The two characters of degree $(q+\varepsilon)/2$ are invariant under φ and are interchanged by $\bar{\delta}$, so are easily handled. The characters of degrees $q-1$ and $q+1$ belong to parametrized families and are invariant under $\bar{\delta}$, but their stabilizers in $\langle\varphi\rangle$ depend on the parameters. In §5, we determine the subgroups of $\mathrm{Aut}(\mathrm{L}_2(q))$ that are stabilizers of characters of degree $q-1$ or $q+1$ of $\mathrm{L}_2(q)$ or $\mathrm{PGL}_2(q)$. We also determine the subgroups of $\langle\varphi\rangle$ that are stabilizers of characters of $\mathrm{SL}_2(q)$ of degree $q-1$ or $q+1$, and show that for each such subgroup there is a character of $\mathrm{SL}_2(q)$ and an extension of the character to $\mathrm{GL}_2(q)$ with the same stabilizer.

In §6, we show that for odd q, if H is any subgroup of $\mathrm{Aut}(\mathrm{L}_2(q))$ containing $\mathrm{L}_2(q)$ but not containing $\mathrm{PGL}_2(q)$, then $H/\mathrm{L}_2(q)$ is cyclic. Hence, in any case, if $\mathrm{L}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{L}_2(q))$, then either $H/\mathrm{L}_2(q)$ is cyclic or $H/\mathrm{PGL}_2(q)$ is cyclic. A character of $\mathrm{PGL}_2(q)$ or, if $\mathrm{PGL}_2(q) \not\leqslant H$, of $\mathrm{L}_2(q)$, will therefore extend to its stabilizer in H and then the extensions will induce irreducibly to H by Clifford's Theorem. We are then able to determine the character degrees of H using our knowledge of which subgroups of H appear as stabilizers.

In §7, we consider subgroups of $A = (\mathrm{SL}_2(q) \rtimes \langle\delta\rangle) \rtimes \langle\varphi\rangle = \mathrm{GL}_2(q) \rtimes \langle\varphi\rangle$ of the form $H = (\mathrm{SL}_2(q) \rtimes \langle\delta^\alpha\rangle) \rtimes \langle\varphi^\beta\rangle$. We determine the characters of $H_0 = \mathrm{SL}_2(q) \rtimes \langle\delta^\alpha\rangle$ lying over characters of $\mathrm{SL}_2(q)$ of each degree. We then find their stabilizers in $\langle\varphi^\beta\rangle$ in order to determine the degrees of the characters of H.

2 Notation and Main Theorems

If G is any finite group, $\mathrm{Irr}(G)$ will denote the set of irreducible complex characters of G. We denote by $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ the set of character degrees of G.

We set $q = p^f$, where p is prime and f is a positive integer. We will always assume $p^f > 3$ because the character degrees of the automorphism groups of the non-simple groups $\mathrm{L}_2(2) \cong S_3$ and $\mathrm{L}_2(3) \cong A_4$ are well-known and the general results we prove do not always apply in these cases.

It will be most convenient to work with the conjugacy classes and characters of $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$, which are described in [3] and [9], respectively. We will use the notation of those sources for the classes and characters.

The outer automorphism group of $\mathrm{L}_2(q)$ is of order $(q-1, 2) \cdot f$, and is generated by a field automorphism φ of order f and, if p is odd, a diagonal automorphism $\bar{\delta}$ of order 2 (see [1] or [2]). Observe that $\mathrm{PGL}_2(q) = \mathrm{L}_2(q)\langle\bar{\delta}\rangle$. If $p = 2$, then $\bar{\delta}$ is an inner automorphism and the center of $\mathrm{SL}_2(q)$ is trivial, so that $\mathrm{L}_2(q) \cong \mathrm{PGL}_2(q) \cong \mathrm{SL}_2(q)$.

We will denote by H a subgroup of $\mathrm{Aut}(\mathrm{L}_2(q))$ satisfying $\mathrm{L}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{L}_2(q))$. The following theorem describes the set of character degrees of H for any such subgroup of $\mathrm{Aut}(\mathrm{L}_2(q))$ for any q. Theorem A follows directly from Corollary 6.3 and Theorems 6.4, 6.5, 6.6, and 6.7.

Theorem A. *Let $S = \mathrm{L}_2(q)$, where $q = p^f > 3$ for a prime p, $A = \mathrm{Aut}(S)$, and let $S \leqslant H \leqslant A$. Set $G = \mathrm{PGL}_2(q)$ if $\bar{\delta} \in H$ and $G = S$ if $\bar{\delta} \notin H$, and let $|H : G| = d = 2^a m$, m odd. If p is odd, let $\varepsilon = (-1)^{(q-1)/2}$. The set of irreducible character degrees of H is*

$$\mathrm{cd}(H) = \{1, q, (q+\varepsilon)/2\} \cup \{(q-1)2^a i : i \mid m\} \cup \{(q+1)j : j \mid d\},$$

with the following exceptions:

i. If p is odd with $H \not\leq S\langle\varphi\rangle$ or if $p=2$, then $(q+\varepsilon)/2$ is not a degree of H.

ii. If f is odd, $p=3$, and $H=S\langle\varphi\rangle$, then $i \neq 1$.

iii. If f is odd, $p=3$, and $H=A$, then $j \neq 1$.

iv. If f is odd, $p=2$, 3, or 5, and $H=S\langle\varphi\rangle$, then $j \neq 1$.

v. If $f \equiv 2 \pmod 4$, $p=2$ or 3, and $H=S\langle\varphi\rangle$ or $H=S\langle\bar{\delta}\varphi\rangle$, then $j \neq 2$.

If q is odd, the diagonal automorphism $\bar{\delta}$ of $\mathrm{L}_2(q)$ is induced by conjugation on $\mathrm{SL}_2(q)$ by a diagonal matrix δ in $\mathrm{GL}_2(q)$ of order $q-1$, and $\mathrm{GL}_2(q)=\mathrm{SL}_2(q)\rtimes\langle\delta\rangle$. Conjugation of $\mathrm{SL}_2(q)$ by δ^2 is an inner automorphism. The field automorphism φ of $\mathrm{L}_2(q)$ of order f also acts as an outer automorphism of order f on $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$.

We determine the character degrees of certain subgroups of $A=(\mathrm{L}_2(q)\rtimes\langle\delta\rangle)\rtimes\langle\varphi\rangle=\mathrm{GL}_2(q)\rtimes\langle\varphi\rangle$ containing $\mathrm{SL}_2(q)$. However, in this case $A/\mathrm{SL}_2(q)\cong\mathbb{F}_q^*\rtimes\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, where $\mathbb{F}_q$ is the field of q elements and $\mathbb{F}_q^*$ its multiplicative group. The structure of this group is too complicated to allow us to determine the degrees for every subgroup, and so we restrict our attention to subgroups of the form $H=(S\rtimes\langle\delta^\alpha\rangle)\rtimes\langle\varphi^\beta\rangle$, where $\alpha \mid q-1$ and $\beta \mid f$. The next theorem is the general result for such a subgroup H. Theorem B follows directly from Theorems 7.3, 7.6, and 7.7. For coprime integers x and y, we denote by $O_y(x)$ the order of x modulo y.

Theorem B. *Let $A=(\mathrm{SL}_2(q)\rtimes\langle\delta\rangle)\rtimes\langle\varphi\rangle=\mathrm{GL}_2(q)\rtimes\langle\varphi\rangle$, where $q=p^f>3$ for an odd prime p, and let $H=(\mathrm{SL}_2(q)\rtimes\langle\delta^\alpha\rangle)\rtimes\langle\varphi^\beta\rangle$, where $\alpha \mid q-1$ and $\beta \mid f$. Set $f/\beta=d=2^a m$ with m odd, $\alpha'=\alpha/(2,\alpha)$, and $\ell=(q-1)/(2\alpha')$.*

The set of irreducible character degrees of H is

$$\begin{aligned}
\mathrm{cd}(H) \quad = \quad & \{k : k \mid d \text{ and } k=O_v(p^\beta) \text{ for some } v \mid (q-1)/\alpha\} \cup \\
& \{qk : k \mid d \text{ and } k=O_v(p^\beta) \text{ for some } v \mid (q-1)/\alpha\} \cup \\
& \left\{\frac{1}{(\alpha,2)}(q-1)k : k \mid d \text{ and } k=O_{2v}(p^\beta) \text{ for some } v \mid \ell \text{ with } \ell/v \text{ odd}\right\} \cup \\
& \left\{\frac{1}{(\alpha,2)}(q+1)k : k \mid d \text{ and } k=O_v(p^\beta) \text{ for some } v \mid \ell\right\} \cup \\
& \{(q-1)2^a i : i \mid m\} \cup \\
& \{(q+1)j : j \mid d\},
\end{aligned}$$

with the exception that if $p=3$, $\beta=1$, and f is odd, then $j \neq 1$.

Finally, we consider the special cases where $H=A$ or H is a particular subgroup of A of index 2. Corollary C follows from Theorem B. Observe that the characters of A of degree $q+1$ lie over the two characters of $\mathrm{SL}_2(q)$ of degree $(q+1)/2$ when $p=3$ and f is odd. There is no character of A of degree $q+1$ lying over a character of $\mathrm{SL}_2(q)$ of degree $q+1$ in this case.

Corollary C. *Let $A=(\mathrm{SL}_2(q)\rtimes\langle\delta\rangle)\rtimes\langle\varphi\rangle$, where $q=p^f>3$ for an odd prime p, and $A_0=(\mathrm{SL}_2(q)\rtimes\langle\delta^2\rangle)\rtimes\langle\varphi\rangle$. Let $f=2^a m$ with m odd.*

The set of irreducible character degrees of A is

$$\mathrm{cd}(A)=\{k, qk, (q-1)2^a i, (q+1)k : k \mid f,\ i \mid m\}.$$

The set of irreducible character degrees of A_0 is

$$\mathrm{cd}(A_0)=\{k, qk, \tfrac{1}{2}(q-1)2^a i, (q-1)2^a i, \tfrac{1}{2}(q+1)k, (q+1)k : k \mid f,\ i \mid m\},$$

with the exception that if $p=3$ and f is odd, there is no irreducible character of A_0 of degree $q+1$.

3 Conjugacy Classes and Automorphisms

Let $q = p^f$, where p is a prime. We are considering characters of extensions of $\mathrm{L}_2(q)$ and $\mathrm{PGL}_2(q)$, as well as of $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$. The conjugacy classes, characters, and automorphisms are more easily described for $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$. For $\mathrm{SL}_2(q)$, we will use the notation and character tables of [3, §38], and for $\mathrm{GL}_2(q)$ we will use the character table of [9].

Let ν be a generator of $\mathbb{F}_q^*$, the multiplicative group of the field $\mathbb{F}_q$ of q elements, τ a generator of $\mathbb{F}_{q^2}^*$, and $\gamma = \tau^{q-1}$. We denote

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ d = \begin{bmatrix} 1 & 0 \\ \nu & 1 \end{bmatrix},\ a = \begin{bmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{bmatrix},\ b = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{bmatrix}.$$

If q is odd, then every element of $\mathrm{SL}_2(q)$ is conjugate to one of 1, z, c, cz, d, dz, a^l for $1 \leqslant l \leqslant (q-3)/2$, or b^m for $1 \leqslant m \leqslant (q-1)/2$. If q is even, then every element of $\mathrm{SL}_2(q) = \mathrm{L}_2(q)$ is conjugate to one of 1, c, a^l for $1 \leqslant l \leqslant (q-2)/2$, or b^m for $1 \leqslant m \leqslant q/2$. Observe that in either case, $1 \leqslant l \leqslant [(q-2)/2]$ and $1 \leqslant m \leqslant [q/2]$, where $[x]$ denotes the greatest integer less than or equal to x.

We need to consider $\mathrm{GL}_2(q)$ and $\mathrm{PGL}_2(q)$ only when q is odd. We denote

$$A_1(l) = \begin{bmatrix} \nu^l & 0 \\ 0 & \nu^l \end{bmatrix},\ A_2(l) = \begin{bmatrix} \nu^l & 0 \\ 1 & \nu^l \end{bmatrix},\ A_3(l_1, l_2) = \begin{bmatrix} \nu^{l_1} & 0 \\ 0 & \nu^{l_2} \end{bmatrix},\ B_1(l) = \begin{bmatrix} \tau^l & 0 \\ 0 & \tau^{ql} \end{bmatrix}.$$

Every element of $\mathrm{GL}_2(q)$ is conjugate to one of $A_1(l)$ or $A_2(l)$ for $1 \leqslant l \leqslant q-1$, $A_3(l_1, l_2)$ for $1 \leqslant l_1 \leqslant q-1$, $1 \leqslant l_1 \leqslant q-1$, and $l_1 \neq l_2$, or $B_1(l)$ for $1 \leqslant l \leqslant q^2-1$ and $(q+1) \nmid l$.

The outer automorphism group of $\mathrm{L}_2(q)$, $q = p^f$, is of order df, where $d = (2, q-1)$. It is generated by a diagonal automorphism $\bar{\delta}$ and a field automorphism φ. The diagonal automorphism is induced by conjugation on $\mathrm{SL}_2(q)$ by the matrix

$$\delta = \begin{bmatrix} \nu & 0 \\ 0 & 1 \end{bmatrix}$$

and these automorphisms act on elements of $\mathrm{SL}_2(q)$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\delta = \begin{bmatrix} a & \nu^{-1}b \\ \nu c & d \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^\varphi = \begin{bmatrix} a^p & b^p \\ c^p & d^p \end{bmatrix}.$$

We have $\mathrm{GL}_2(q) = \mathrm{SL}_2(q) \rtimes \langle \delta \rangle$, and the field automorphism acts on $\mathrm{GL}_2(q)$ in the same way. Moreover, δ^2 acts as an inner automorphism on $\mathrm{SL}_2(q)$. Observe that $\langle \delta \rangle \cong \mathbb{F}_q^*$ and the action of φ on $\langle \delta \rangle$ is induced by the action of the generator of the Galois group of $\mathbb{F}_q$ over $\mathbb{F}_p$. Hence

$$(\mathrm{GL}_2(q) \rtimes \langle \varphi \rangle)/\,\mathrm{SL}_2(q) \cong \langle \delta \rangle \rtimes \langle \varphi \rangle \cong \mathbb{F}_q^* \rtimes \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p).$$

As the center Z of $\mathrm{SL}_2(q)$ is invariant under both δ and φ, these maps induce automorphisms $\bar{\delta}$ and $\bar{\varphi}$ on $\mathrm{L}_2(q) = \mathrm{SL}_2(q)/Z$ by $(gZ)^{\bar{\delta}} = g^\delta Z$ and $(gZ)^{\bar{\varphi}} = g^\varphi Z$, as usual. Denoting the induced automorphism $\bar{\varphi}$ on $\mathrm{L}_2(q)$ by φ as well, we have $\mathrm{L}_2(q)\langle \bar{\delta} \rangle \cong \mathrm{PGL}_2(q)$ and $\mathrm{Aut}(\mathrm{L}_2(q)) = \mathrm{L}_2(q)\langle \bar{\delta}, \varphi \rangle \cong \mathrm{PGL}_2(q)\langle \varphi \rangle$. Similarly, the center of $\mathrm{GL}_2(q)$ is invariant under φ and φ induces an automorphism $\bar{\varphi}$ on $\mathrm{PGL}_2(q)$, which we will also denote by φ.

If q is even, then $\bar{\delta}$ is an inner automorphism and $\mathrm{Aut}(\mathrm{L}_2(q)) = \mathrm{L}_2(q)\langle \varphi \rangle$, and if q is odd, then $\bar{\delta}$ is an outer automorphism but $\bar{\delta}^2$ is inner. Hence $\bar{\delta}$ is of order $d = (2, q-1)$ modulo inner automorphisms. Since entries in elements of $\mathrm{SL}_2(q)$ are from the field of $q = p^f$ elements, φ is of order f. Moreover, if q is odd, then $\bar{\delta}$ and φ commute modulo inner automorphisms, so that $\mathrm{Aut}(\mathrm{L}_2(q))/\,\mathrm{L}_2(q) \cong \langle \bar{\delta} \rangle \times \langle \varphi \rangle$, hence their actions on conjugacy classes or characters will commute.

We now describe the actions of δ and φ on conjugacy classes. These lemmas follow from straightforward calculations.

Lemma 3.1. *Let q be odd and assume notation as above. In $\mathrm{SL}_2(q)$, the diagonal automorphism δ interchanges the conjugacy classes of c and d, interchanges the conjugacy classes of cz and dz, and fixes all other conjugacy classes.*

Lemma 3.2. *Assume notation as above and let $1 \leqslant k < f$. In $\mathrm{SL}_2(q)$, the automorphism φ^k sends*

i. *the conjugacy class of a^l to the class of a^r, where $1 \leqslant r \leqslant [(q-2)/2]$ and $r \equiv \pm lp^k (\mathrm{mod}\ q-1)$,*

ii. *the conjugacy class of b^m to the class of b^s, where $1 \leqslant s \leqslant [q/2]$ and $s \equiv \pm mp^k (\mathrm{mod}\ q+1)$,*

and fixes all other conjugacy classes.

Lemma 3.3. *Assume notation as above with q odd and let $1 \leqslant k < f$. In $\mathrm{GL}_2(q)$, the automorphism φ^k sends*

i. *the conjugacy class of $A_1(l)$ to the class of $A_1(r)$, where $1 \leqslant r \leqslant q-1$ and $r \equiv lp^k (\mathrm{mod}\ q-1)$,*

ii. *the conjugacy class of $A_2(l)$ to the class of $A_2(r)$, where $1 \leqslant r \leqslant q-1$ and $r \equiv lp^k (\mathrm{mod}\ q-1)$,*

iii. *the conjugacy class of $A_3(l_1, l_2)$ to the class of $A_3(r_1, r_2)$, where $1 \leqslant r_1 \leqslant q-1$, $1 \leqslant r_2 \leqslant q-1$, $r_1 \equiv l_1 p^k (\mathrm{mod}\ q-1)$, and $r_2 \equiv l_2 p^k (\mathrm{mod}\ q-1)$, and*

iv. *the conjugacy class of $B_1(l)$ to the class of $B_1(t)$, where $1 \leqslant t \leqslant q^2-1$, $q+1 \nmid t$, and $t \equiv lp^k (\mathrm{mod}\ q^2-1)$.*

4 Characters and Automorphisms

In order to determine which subgroups of $\mathrm{Aut}(\mathrm{L}_2(q))$ and $(\mathrm{SL}_2(q) \rtimes \langle\delta\rangle) \rtimes \langle\varphi\rangle$ appear as stabilizers of irreducible characters of $\mathrm{SL}_2(q)$, $\mathrm{GL}_2(q)$, $\mathrm{L}_2(q)$, or $\mathrm{PGL}_2(q)$, we first determine conditions under which these characters are invariant under the action of powers of δ or φ.

It will be more convenient to work with characters and conjugacy class of $\mathrm{SL}_2(q)$ or $\mathrm{GL}_2(q)$. Let G be either $\mathrm{SL}_2(q)$ or $\mathrm{GL}_2(q)$ and let $Z = Z(G)$, so that G/Z is $\mathrm{L}_2(q)$ or $\mathrm{PGL}_2(q)$, respectively. An automorphism σ of G induces an automorphism $\bar{\sigma}$ on G/Z defined by $(gZ)^{\bar{\sigma}} = g^{\sigma}Z$. Similarly, the irreducible characters of G/Z are precisely those defined by $\bar{\chi}(gZ) = \chi(g)$, where $\chi \in \mathrm{Irr}(G)$ and $Z \leqslant \ker\chi$. It is straightforward to check that $\bar{\chi}^{\bar{\sigma}} = \bar{\chi}$ if and only if $\chi^{\sigma} = \chi$. Hence the irreducible characters of G/Z invariant under a particular automorphism σ are those characters of G invariant under σ with kernel containing Z.

We first determine the characters of $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$ whose kernels contain the center. Of course, if q is even, then the center of $\mathrm{SL}_2(q)$ is trivial and the diagonal automorphism is an inner automorphism, and so $\mathrm{SL}_2(q) \cong \mathrm{L}_2(q) \cong \mathrm{PGL}_2(q)$. In the notation of [3, §38], when q is even, $\mathrm{SL}_2(q)$ has irreducible characters

i. 1_G of degree 1,

ii. ψ of degree q (= St, the Steinberg character),

iii. χ_i, $1 \leqslant i \leqslant (q-2)/2$, of degree $q+1$, and

iv. θ_j, $1 \leqslant j \leqslant q/2$, of degree $q-1$.

We will assume q is odd in the following. For $\mathrm{SL}_2(q)$, we use the notation and character table in [3, §38] and for $\mathrm{GL}_2(q)$, we use [9, §2]. The following results are easily obtained from the respective character tables.

Lemma 4.1. *Let $G = \mathrm{SL}_2(q)$ with q odd. The irreducible characters of G with kernel containing $Z(G)$ are as follows:*

i. 1_G *of degree* 1;

ii. ψ *of degree* q (= St, *the Steinberg character);*

iii. χ_i, *for* $1 \leqslant i \leqslant (q-3)/2$ *and* i *even, of degree* $q+1$;

iv. θ_j *for* $1 \leqslant j \leqslant (q-1)/2$ *and* j *even, of degree* $q-1$;

v. ξ_1 *and* ξ_2 *of degree* $(q+1)/2$, *if* $q \equiv 1 (\mathrm{mod}\ 4)$;

vi. η_1 *and* η_2 *of degree* $(q-1)/2$, *if* $q \equiv -1 (\mathrm{mod}\ 4)$.

Lemma 4.2. *Let* $G = \mathrm{GL}_2(q)$ *with* q *odd. The irreducible characters of* G *with kernel containing* $Z(G)$ *are as follows:*

i. $\chi_1^{(n)}$ *and* $\chi_q^{(n)}$ *for* $n = (q-1)/2$ *and* $n = q-1$,

ii. $\chi_{q+1}^{(m,n)}$ *for* $1 \leqslant n \leqslant (q-3)/2$ *and* $m = (q-1) - n$,

iii. $\chi_{q-1}^{((q-1)n)}$ *for* $1 \leqslant n \leqslant (q-1)/2$.

Notation 4.3.
We will write $\chi_{q+1}^{(n)}$ *for the character* $\chi_{q+1}^{(m,n)}$ *with* $1 \leqslant n \leqslant (q-3)/2$ *and* $m = (q-1) - n$.
We will write $\theta_{q-1}^{(n)}$ *for the character* $\chi_{q-1}^{((q-1)n)}$ *with* $1 \leqslant n \leqslant (q-1)/2$.

We first consider the action of δ on the irreducible characters of $\mathrm{SL}_2(q)$ for odd q and relate the characters of $\mathrm{SL}_2(q)$ to those of $\mathrm{GL}_2(q)$ and the characters of $\mathrm{L}_2(q)$ to those of $\mathrm{PGL}_2(q)$.

Lemma 4.4. *Let* q *be odd. All characters of* $\mathrm{SL}_2(q)$ *of degrees* 1, q, $q+1$, *and* $q-1$ *are invariant under* δ *and each extends to* $q-1$ *distinct irreducible characters of* $\mathrm{GL}_2(q)$.

Each of $\{\xi_1, \xi_2\}$ *and* $\{\eta_1, \eta_2\}$ *is a single orbit under the action of* δ. *Each of these characters extends to* $(q-1)/2$ *irreducible characters of its stabilizer* $\mathrm{SL}_2(q) \rtimes \langle \delta^2 \rangle$ *in* $\mathrm{GL}_2(q)$ *and each extension then induces to an irreducible character of* $\mathrm{GL}_2(q)$.

Proof. Observe first that $\mathrm{GL}_2(q) = \mathrm{SL}_2(q) \rtimes \langle \delta \rangle$, so that $\mathrm{GL}_2(q)/\mathrm{SL}_2(q)$ is cyclic of order $q-1$. Hence by Gallagher's Theorem ([4, 6.17]), an invariant character of $\mathrm{SL}_2(q)$ extends to $q-1$ distinct irreducible characters of $\mathrm{GL}_2(q)$. The characters of degrees 1, q, $q+1$, and $q-1$ have the same value on the classes of c and d, and so also on the classes of cz and dz. Hence these characters are invariant under δ by Lemma 3.1.

The pairs of characters $\{\xi_1, \xi_2\}$ and $\{\eta_1, \eta_2\}$ are equal on all classes except the classes of c, d, cz, and dz. We have $\xi_1(c) = \xi_2(d)$, $\xi_1(d) = \xi_2(c)$, $\xi_1(cz) = \xi_2(dz)$, and $\xi_1(dz) = \xi_2(cz)$. Therefore, $\xi_1^\delta = \xi_2$ and $\xi_1^\delta = \xi_2$ by Lemma 3.1, and similarly for η_1 and η_2. Recall that δ^2 induces an inner automorphism on $\mathrm{SL}_2(q)$, and so the stabilizer in $\mathrm{GL}_2(q)$ of each of these characters is $\mathrm{SL}_2(q) \rtimes \langle \delta^2 \rangle$. The result then follows from Clifford's Theorem ([4, 6.11]) and Gallagher's Theorem. □

For the following lemma, let $\varepsilon = (-1)^{(q-1)/2}$ and denote by μ_1, μ_2 the irreducible characters of $\mathrm{SL}_2(q)$ of degree $(q+\varepsilon)/2$. Thus $\mu_i = \xi_i$ if $q \equiv 1 (\mathrm{mod}\ 4)$ and $\mu_i = \eta_i$ if $q \equiv -1 (\mathrm{mod}\ 4)$, so that $\mu_1, \mu_2 \in \mathrm{Irr}(\mathrm{L}_2(q))$ by Lemma 4.1.

Lemma 4.5. *Let* q *be odd. All characters of* $\mathrm{L}_2(q)$ *of degrees* 1, q, $q+1$, *and* $q-1$ *are invariant under* $\bar{\delta}$ *and each extends to two distinct irreducible characters of* $\mathrm{PGL}_2(q)$.

The characters μ_1 *and* μ_2 *of* $\mathrm{L}_2(q)$ *of degree* $(q+\varepsilon)/2$ *form a single orbit under the action of* $\bar{\delta}$ *and induce to a single irreducible character of* $\mathrm{PGL}_2(q)$ *of degree* $q+\varepsilon$.

Proof. This follows from Lemma 4.4 and the fact that $\mathrm{L}_2(q)$ is of index 2 in $\mathrm{PGL}_2(q) = \mathrm{L}_2(q)\langle \bar{\delta} \rangle$. □

We now consider the action of the field automorphism φ on the irreducible characters of $\mathrm{SL}_2(q)$, $\mathrm{L}_2(q)$, and $\mathrm{PGL}_2(q)$. The situation for $\mathrm{GL}_2(q)$ is more complicated and will be considered in §7. Unless stated otherwise, q may be either even or odd.

Lemma 4.6. *If χ is an irreducible character of $\mathrm{SL}_2(q)$, $\mathrm{L}_2(q)$, or $\mathrm{PGL}_2(q)$ of degree 1, q, $(q-1)/2$, or $(q+1)/2$, then χ is invariant under the action of φ.*

Proof. It is clear from the character tables that the characters of $\mathrm{SL}_2(q)$, and hence of $\mathrm{L}_2(q)$, of degree 1 and q are invariant under φ, as are the principal character and Steinberg character of $\mathrm{PGL}_2(q)$. For odd q, the remaining characters of degree 1, q of $\mathrm{PGL}_2(q)$ have value $(-1)^{k+l}$ on the class of $A_3(k,l)$, and either $(-1)^l$ or $-(-1)^l$ on the class of $B_1(l)$. By Lemma 3.3, φ sends $A_3(k,l)$ to $A_3(kp,lp)$ and $B_1(l)$ to $B_1(lp)$. Since p is odd in this case, $(-1)^{k+l}=(-1)^{kp+lp}$ and $(-1)^l=(-1)^{lp}$, and so these characters are also invariant under φ.

The characters of degree $(q\pm1)/2$ occur only for $\mathrm{SL}_2(q)$ and $\mathrm{L}_2(q)$ with q odd. By Lemma 3.2, the only conjugacy classes moved by φ are those of a^l and b^l. The values of the μ_i on these classes are 0, $(-1)^l$ or $-(-1)^l$. The class of a^l, b^l is sent to $a^{\pm lp}$, $b^{\pm lp}$, respectively. Again, since p is odd, $(-1)^l=(-1)^{\pm lp}$ and these characters are invariant under φ. □

We next determine conditions under which a given character of degree $q+1$ or $q-1$ is invariant under a power of φ. The following general result will be very useful.

Lemma 4.7. *If ϵ is a complex kth root of unity and i, j are integers, then $\epsilon^i+\epsilon^{-i}=\epsilon^j+\epsilon^{-j}$ if and only if $i\equiv\pm j \pmod k$.*

Proof. Observe that $\epsilon^i+\epsilon^{-i}=\epsilon^j+\epsilon^{-j}$ if and only if

$$\epsilon^i-\epsilon^j=\epsilon^{-j}-\epsilon^{-i}=\frac{\epsilon^i-\epsilon^j}{\epsilon^{i+j}}.$$

This holds if and only if either $\epsilon^i=\epsilon^j$, in which case $i\equiv j \pmod k$, or $\epsilon^{i+j}=1$, in which case $i\equiv -j \pmod k$. □

The next result also applies to characters of $\mathrm{L}_2(q)$, of course, as these are characters of $\mathrm{SL}_2(q)$.

Lemma 4.8. *Let $q=p^f$ for a prime p and $f\geqslant 2$, and let $1\leqslant k\leqslant f$.*

i. The character χ_n of $\mathrm{SL}_2(q)$ or $\chi_{q+1}^{(n)}$ of $\mathrm{PGL}_2(q)$ of degree $q+1$ is invariant under φ^k if and only if

$$p^f-1 \mid (p^k-1)n \text{ or } p^f-1 \mid (p^k+1)n.$$

ii. The character θ_n of $\mathrm{SL}_2(q)$ or $\theta_{q-1}^{(n)}$ of $\mathrm{PGL}_2(q)$ of degree $q-1$ is invariant under φ^k if and only if

$$p^f+1 \mid (p^k-1)n \text{ or } p^f+1 \mid (p^k+1)n.$$

Proof. Distinct characters of $\mathrm{SL}_2(q)$ or $\mathrm{PGL}_2(q)$ of degree $q+1$ differ only on the classes of a^l of $\mathrm{SL}_2(q)$ or $A_3(l_1,l_2)$ of $\mathrm{GL}_2(q)$. Similarly, distinct characters of degree $q-1$ differ only on the classes of b^m of $\mathrm{SL}_2(q)$ or $B_1(l)$ of $\mathrm{GL}_2(q)$.

Let ρ be a complex primitive $(q-1)$th root of unity, so that $\chi_n(a^l)=\rho^{nl}+\rho^{-nl}$ for the character χ_n of $\mathrm{SL}_2(q)$ of degree $q+1$. The character χ_n is then invariant under φ^k if and only if

$$\chi_n(a^l)=\chi_n((a^l)^{\varphi^k})=\chi_n(a^{lp^k})$$

for all l, $1\leqslant l\leqslant[(q-2)/2]$, by Lemma 3.2, hence if and only if

$$\rho^{nl}+\rho^{-nl}=\rho^{nlp^k}+\rho^{-nlp^k}$$

for all l. By Lemma 4.7, this holds if and only if $nlp^k \equiv \pm nl (\mathrm{mod}\ q-1)$ for all l. Observing that $q = p^f$ and that if the congruence holds for $l = 1$ then it holds for all l, we see that χ_n is invariant under φ^k if and only if $np^k \equiv \pm n (\mathrm{mod}\ p^f - 1)$ as claimed.

Similarly, for the character $\chi_{q+1}^{(n)}$ of $\mathrm{PGL}_2(q)$, we have

$$\chi_{q+1}^{(n)}(A_3(l_1, l_2)) = \rho^{n(l_2-l_1)} + \rho^{-n(l_2-l_1)}$$

and, by Lemma 3.3, $\chi_{q+1}^{(n)}$ is invariant under φ^k if and only if

$$\rho^{n(l_2-l_1)} + \rho^{-n(l_2-l_1)} = \rho^{n(l_2-l_1)p^k} + \rho^{-n(l_2-l_1)p^k}$$

for all $l_1 \neq l_2$, $1 \leqslant l_1 \leqslant q-1$, $1 \leqslant l_2 \leqslant q-1$. As before, this is equivalent to

$$n(l_2 - l_1)p^k \equiv \pm n(l_2 - l_1)(\mathrm{mod}\ q-1)$$

for all l_1, l_2. Since $q \geqslant 4$, this must hold in particular for $l_1 = 1$, $l_2 = 2$, and so holds for all l_1, l_2 if and only if $np^k \equiv \pm n(\mathrm{mod}\ p^f - 1)$ as claimed.

Let σ be a complex primitive $(q+1)$th root of unity, so that $\theta_n(b^m) = -(\sigma^{nm} + \sigma^{-nm})$ for the character θ_n of $\mathrm{SL}_2(q)$ of degree $q-1$. The character θ_n is then invariant under φ^k if and only if

$$\theta_n(b^m) = \theta_n((b^m)^{\varphi^k}) = \theta_n(b^{mp^k})$$

for all m, $1 \leqslant m \leqslant [q/2]$, by Lemma 3.2, hence if and only if

$$\sigma^{nm} + \sigma^{-nm} = \sigma^{nmp^k} + \sigma^{-nmp^k}$$

for all m. By Lemma 4.7, this holds if and only if $nmp^k \equiv \pm nm(\mathrm{mod}\ q+1)$ for all m. Again, $q = p^f$ and if the congruence holds for $m = 1$ then it holds for all m, hence θ_n is invariant under φ^k if and only if $np^k \equiv \pm n(\mathrm{mod}\ p^f + 1)$ as claimed.

For the character $\theta_{q-1}^{(n)}$ of $\mathrm{PGL}_2(q)$, we have

$$\theta_{q-1}^{(n)}(B_1(l)) = -(\sigma^{nl} + \sigma^{nlq}) = -(\sigma^{nl} + \sigma^{-nl}),$$

since $\sigma^q = \sigma^{-1}$. By Lemma 3.3, $\theta_{q-1}^{(n)}$ is invariant under φ^k if and only if

$$\theta^{nl} + \theta^{-nl} = \theta^{nlp^k} + \theta^{-nlp^k}$$

for all l, $1 \leqslant l \leqslant q^2 - 1$ and $q + 1 \nmid l$. As before, this is equivalent to

$$nlp^k \equiv \pm nl(\mathrm{mod}\ q+1),$$

which holds for all l if and only if $np^k \equiv \pm n(\mathrm{mod}\ p^f + 1)$. □

The following number-theoretic result will be helpful in applying Lemma 4.8.

Lemma 4.9. *If p is a prime and f, k are positive integers such that $k \mid f$, then*

i. $(p^f - 1, p^k - 1) = p^k - 1$,

ii. $(p^f - 1, p^k + 1) = \begin{cases} (p-1, 2) & \text{if } f/k \text{ is odd} \\ p^k + 1 & \text{if } f/k \text{ is even,} \end{cases}$

iii. $(p^f + 1, p^k - 1) = (p - 1, 2)$, *iv.* $(p^f+1, p^k+1) = \begin{cases} p^k+1 & \text{if } f/k \text{ is odd} \\ (p-1,2) & \text{if } f/k \text{ is even.} \end{cases}$

Proof. Observe first that $p^k \equiv 1 (\bmod\ p^k - 1)$ and $p^k \equiv -1 (\bmod\ p^k + 1)$, hence

$$p^f \equiv (p^k)^{f/k} \equiv (1)^{f/k} \equiv 1 (\bmod\ p^k - 1)$$

and

$$p^f \equiv (p^k)^{f/k} \equiv (-1)^{f/k} (\bmod\ p^k + 1).$$

It follows that $p^k - 1 \mid p^f - 1$, so that (i) holds. Similarly, if f/k is even, then $p^k + 1 \mid p^f - 1$ and $(p^f - 1, p^k + 1) = p^k + 1$, whereas if f/k is odd, then $p^k + 1 \mid p^f + 1$ and $(p^f + 1, p^k + 1) = p^k + 1$.

Since $p^k - 1 \mid p^f - 1$, we have that $(p^f + 1, p^k - 1)$ must divide 2. Similarly, if f/k is odd, then $p^k + 1 \mid p^f + 1$ and so $(p^f - 1, p^k + 1)$ divides 2, while if f/k is even, then $p^k + 1 \mid p^f - 1$ and $(p^f + 1, p^k + 1)$ divides 2. If $p = 2$, then all of $p^k \pm 1$ and $p^f \pm 1$ are odd, but if p is odd then all of these integers are even. Hence these greatest common divisors are 1 when $p = 2$ and 2 when p is odd, hence are equal to $(p - 1, 2)$ as claimed. □

5 Stabilizers of Characters of Degree $q - 1$ or $q + 1$

By Lemmas 4.4 and 4.5, the irreducible characters of $\mathrm{SL}_2(q)$ and $\mathrm{L}_2(q)$ of degree $q - 1$ or $q + 1$ are invariant in $\mathrm{GL}_2(q)$ and $\mathrm{PGL}_2(q)$, respectively. Also, recall that if q is even, then $\bar{\delta}$ is an inner automorphism and $\mathrm{SL}_2(q) \cong \mathrm{PGL}_2(q) = \mathrm{L}_2(q)$. We now determine the subgroups of $\langle\varphi\rangle$ that occur as stabilizers (in $\langle\varphi\rangle$) of characters of $\mathrm{SL}_2(q)$, $\mathrm{L}_2(q)$, $\mathrm{GL}_2(q)$, or $\mathrm{PGL}_2(q)$ of degree $q - 1$ or $q + 1$. Throughout this section, we denote $q = p^f$ for some prime p and integer $f \geqslant 2$, and K will denote a subgroup of $\langle\varphi\rangle$ of the form $K = \langle\varphi^k\rangle$ for some positive divisor k of f. We first determine when K is a stabilizer of a character of degree $q - 1$.

Lemma 5.1. *Let $q = p^f$, $k \mid f$ and $K = \langle\varphi^k\rangle$, and set $n = (p^f + 1)/(p^k + 1)$.*

i. If f/k is even, then K does not stabilize any irreducible character of $\mathrm{SL}_2(q)$ or $\mathrm{PGL}_2(q)$ of degree $q - 1$.

ii. If f/k is odd, then K is the stabilizer in $\langle\varphi\rangle$ of $\theta_n \in \mathrm{Irr}(\mathrm{SL}_2(q))$, $\theta_{q-1}^{(n)} \in \mathrm{Irr}(\mathrm{PGL}_2(q))$, and an extension $\hat{\theta}_n$ of θ_n to $\mathrm{GL}_2(q)$.

Proof. All characters of $\mathrm{SL}_2(q)$ of degree $q-1$ are θ_j for some $1 \leqslant j \leqslant [q/2]$ and for odd q, characters of $\mathrm{PGL}_2(q)$ of degree $q - 1$ are $\theta_{q-1}^{(j)}$ for $1 \leqslant j \leqslant (q - 1)/2$. In particular, note that

$$1 \leqslant j < (q + 1)/2 = (p^f + 1)/2$$

in any case. By Lemma 4.8, θ_j and $\theta_{q-1}^{(j)}$ are fixed by φ^k if and only if

$$p^f + 1 \mid (p^k - 1)j \text{ or } p^f + 1 \mid (p^k + 1)j.$$

If f/k is even, then

$$(p^f + 1, p^k - 1) = (p^f + 1, p^k + 1) = (2, p - 1)$$

by Lemma 4.9. Thus if φ^k stabilizes θ_j or $\theta_{q-1}^{(j)}$, then $(p^f + 1)/(2, p - 1)$ must divide j, which is impossible as $1 \leqslant j < (p^f + 1)/2$. Hence if f/k is even, φ^k does not stabilize any character of degree $q - 1$.

Assume now that f/k is odd, so by Lemma 4.9, $p^k + 1 \mid p^f + 1$. Set $n = (p^f + 1)/(p^k + 1)$. Since $p^k + 1 \geqslant 3$, we have

$$1 \leqslant n = \frac{p^f + 1}{p^k + 1} \leqslant \frac{p^f + 1}{3} < \frac{p^f + 1}{2} = \frac{q + 1}{2},$$

hence n is an integer and $n < (q+1)/2$. If q is odd, this implies $n \leqslant (q-1)/2$ and if q is even, this implies $n \leqslant q/2$. Hence θ_n is a character of $\mathrm{SL}_2(q)$ for any q and $\theta_{q-1}^{(n)}$ is a character of $\mathrm{PGL}_2(q)$ for odd q. Moreover, we have $p^f+1 \mid (p^k+1)n$, so that by Lemma 4.8, θ_n and $\theta_{q-1}^{(n)}$ are stabilized by K.

If the stabilizer, T, of θ_n or $\theta_{q-1}^{(n)}$ in $\langle\varphi\rangle$ properly contains K, then $T = \langle\varphi^t\rangle$ for some divisor t of k with $1 \leqslant t < k$. Since φ^t stabilizes θ_n or $\theta_{q-1}^{(n)}$, we have $p^f+1 \mid (p^t+1)n$ or $p^f+1 \mid (p^t-1)n$. Hence one of $(p^t+1)/(p^k+1)$ or $(p^t-1)/(p^k+1)$ is an integer. In any case, this implies $p^k+1 \leqslant p^t+1$, contradicting $1 \leqslant t < k$. Therefore, K is the stabilizer in $\langle\varphi\rangle$ of θ_n and $\theta_{q-1}^{(n)}$.

Finally, we show that K is also the stabilizer in $\langle\varphi\rangle$ of some extension $\hat{\theta}_n$ of θ_n to $\mathrm{GL}_2(q)$. We assume q is odd as the result is trivial if q is even. It is straightforward to check that for any j, one extension of θ_j to $\mathrm{GL}_2(q)$ is $\chi_{q-1}^{(j)}$. The irreducible characters of the cyclic group $\mathrm{GL}_2(q)/\mathrm{SL}_2(q)$ are precisely the characters $\chi_1^{(i)}$ of $\mathrm{GL}_2(q)$, for $1 \leqslant i \leqslant q-1$, and so by Gallagher's Theorem all of the extensions of θ_j are the characters $\chi_{q-1}^{(j)} \cdot \chi_1^{(i)}$. We claim that the extension $\hat{\theta}_n = \chi_{q-1}^{(n)} \cdot \chi_1^{(i)}$ where

$$n = \frac{p^f+1}{p^k+1} \text{ and } i = \frac{p^f-p^k}{p^{2k}-1} = \frac{p^k(p^{f-k}-1)}{p^{2k}-1},$$

has stabilizer K in $\langle\varphi\rangle$.

First, observe that since f/k is odd, $f/k-1$ is even, and so $2k$ divides $(f/k-1)k = f-k$. Hence $p^{2k}-1 \mid p^{f-k}-1$ and i is an integer with $1 \leqslant i \leqslant p^f-1 = q-1$. Thus $\hat{\theta}_n$ is an extension of θ_n, and so the stabilizer of $\hat{\theta}_n$ is contained in K. It remains to show that $\hat{\theta}_n$ is invariant under φ^k.

The values of the character $\hat{\theta}_n$ and its image under φ^{-k} are as follows:

	$\hat{\theta}_n$	$(\hat{\theta}_n)^{\varphi^{-k}}$
$A_1(l)$	$(q-1)\rho^{(n+2i)l}$	$(q-1)\rho^{(n+2i)p^k l}$
$A_2(l)$	$-\rho^{(n+2i)l}$	$-\rho^{(n+2i)p^k l}$
$A_3(l_1,l_2)$	0	0
$B_1(l)$	$-\epsilon^{[n+(q+1)i]l} - \epsilon^{[nq+(q+1)i]l}$	$-\epsilon^{[n+(q+1)i]p^k l} - \epsilon^{[nq+(q+1)i]p^k l}$

where ϵ is a complex primitive (q^2-1)th root of unity and $\rho = \epsilon^{q+1}$ is a complex primitive $(q-1)$th root of unity.

For n and i as defined above, we have

$$\begin{aligned} n+2i &= \frac{p^f-1}{p^k-1} \\ n+(q+1)i &= n \cdot \frac{p^f-1}{p^k-1} = \frac{q^2-1}{p^{2k}-1} \\ nq+(q+1)i &= np^k \cdot \frac{p^f-1}{p^k-1} = [n+(q+1)i]p^k. \end{aligned}$$

Thus it is clear that $\epsilon^{[nq+(q+1)i]l} = \epsilon^{[n+(q+1)i]p^k l}$ for all l. Also, $q^2-1 \mid [n+(q+1)i](p^{2k}-1)$, so that

$$[nq+(q+1)i]p^k l = [n+(q+1)i]p^{2k} l \equiv [n+(q+1)i]l \pmod{q^2-1},$$

and hence $\epsilon^{[n+(q+1)i]l} = \epsilon^{[nq+(q+1)i]p^k l}$. It follows that $\hat{\theta}_n$ and $(\hat{\theta}_n)^{\varphi^{-k}}$ have the same values on the classes $B_1(l)$. Finally, $q-1 \mid (n+2i)(p^k-1)$, so that $(n+2i)l \equiv (n+2i)p^k l \pmod{q-1}$ for all l. Therefore, $\hat{\theta}_n$ and $(\hat{\theta}_n)^{\varphi^{-k}}$ have the same values on all classes and $\hat{\theta}_n$ is invariant under φ^{-k}, and hence also under φ^k. □

Lemma 5.2. *Let $q = p^f$ be odd, $k \mid f$, $K = \langle \varphi^k \rangle$, and $n = 2(p^f + 1)/(p^k + 1)$.*

i. If f/k is even or $p^k = 3$, then K does not stabilize any irreducible character of $\mathrm{L}_2(q)$ of degree $q - 1$.

ii. If f/k is odd and $p^k \neq 3$, then K is the stabilizer in $\langle \varphi \rangle$ of $\theta_n \in \mathrm{Irr}(\mathrm{L}_2(q))$.

Proof. The characters of $\mathrm{L}_2(q)$ of degree $q-1$ are θ_j for some even integer j with $1 \leqslant j \leqslant (q-1)/2$. By Lemma 4.8, θ_j is fixed by φ^k if and only if

$$p^f + 1 \mid (p^k - 1)j \text{ or } p^f + 1 \mid (p^k + 1)j.$$

By the same argument as in Lemma 5.1, if f/k is even, then φ^k does not stabilize any character of degree $q - 1$.

Suppose now that $p = 3$ and $k = 1$, so that $K = \langle \varphi \rangle$, and $f \geqslant 3$ is odd. By Lemma 4.1, the characters of $\mathrm{L}_2(3^f)$ of degree $q - 1$ are the θ_j with $1 \leqslant j \leqslant (3^f - 1)/2$ and j even. By Lemma 4.8, θ_j is fixed by φ if and only if $3^f + 1 \mid 2j$ or $3^f + 1 \mid 4j$. Thus if $K = \langle \varphi \rangle$ stabilizes θ_j, then $3^f + 1 \mid 4j$. Since f is odd, $3^f + 1 \equiv 4 (\mathrm{mod}\ 8)$, and so $(3^f + 1)/4$ is odd and divides j. Since j is even, this implies $(3^f + 1)/2$ divides j, contradicting $1 \leqslant j \leqslant (3^f - 1)/2$. Therefore, if $p = 3$ and $k = 1$, then $K = \langle \varphi \rangle$ does not stabilize any character of $\mathrm{L}_2(q)$ of degree $q - 1$.

We now assume $p^k > 3$ and f/k is odd. Thus, by Lemma 4.9, $p^k + 1 \mid p^f + 1$, and so $n = 2(p^f + 1)/(p^k + 1)$ is an even integer. Since $p^k > 3$, we have

$$1 \leqslant n = 2 \cdot \frac{p^f + 1}{p^k + 1} < 2 \cdot \frac{p^f + 1}{4} = \frac{p^f + 1}{2} = \frac{q + 1}{2},$$

hence n is a positive integer and $n < (q + 1)/2$, which implies $n \leqslant (q - 1)/2$. Therefore, θ_n is a character of $\mathrm{L}_2(q)$ of degree $q - 1$, and since $p^f + 1 \mid (p^k + 1)n$, θ_n is stabilized by φ^k.

If the stabilizer of θ_n properly contains K, then there is a divisor t of k with $1 \leqslant t < k$ such that φ^t stabilizes θ_n. Hence $p^f + 1 \mid (p^t + 1)n$ or $p^f + 1 \mid (p^t - 1)n$, and so one of $2(p^t + 1)/(p^k + 1)$ or $2(p^t - 1)/(p^k + 1)$ is an integer. In particular, we must have $p^k + 1 \leqslant 2(p^t + 1)$; that is, $p^k \leqslant 2p^t + 1$. But $p \geqslant 3$ and $1 \leqslant t < k$, and so

$$p^k \geqslant p^{t+1} \geqslant 3p^t = 2p^t + p^t > 2p^t + 1,$$

a contradiction. Therefore, K is the stabilizer in $\langle \varphi \rangle$ of the character θ_n of $\mathrm{L}_2(q)$. $\square$

We next determine when $K = \langle \varphi^k \rangle$ is the stabilizer in $\langle \varphi \rangle$ of a character of degree $q + 1$.

Lemma 5.3. *Let $q = p^f$ be odd, $k \mid f$, and $K = \langle \varphi^k \rangle$.*

i. If $p^k \neq 3$, then K is the stabilizer in $\langle \varphi \rangle$ of $\chi_n \in \mathrm{Irr}(\mathrm{SL}_2(q))$, $\chi_{q+1}^{(n)} \in \mathrm{Irr}(\mathrm{PGL}_2(q))$, and an extension $\hat{\chi}_n$ of χ_n to $\mathrm{GL}_2(q)$, for $n = (p^f - 1)/(p^k - 1)$.

ii. If $p = 3$, $k = 1$, and f is even, then $K = \langle \varphi \rangle$ is the stabilizer in $\langle \varphi \rangle$ of $\chi_n \in \mathrm{Irr}(\mathrm{SL}_2(q))$, $\chi_{q+1}^{(n)} \in \mathrm{Irr}(\mathrm{PGL}_2(q))$, and an extension $\hat{\chi}_n$ of χ_n to $\mathrm{GL}_2(q)$, for $n = (3^f - 1)/4$.

iii. If $p = 3$, $k = 1$, and f is odd, then $K = \langle \varphi \rangle$ does not stabilize any irreducible character of $\mathrm{SL}_2(q)$ or $\mathrm{PGL}_2(q)$ of degree $q + 1$.

Proof. Characters of $\mathrm{SL}_2(q)$, $\mathrm{PGL}_2(q)$ of degree $q+1$ are χ_j, $\chi_{q+1}^{(j)}$ respectively, for $1 \leqslant j \leqslant (q-3)/2$. By Lemma 4.8, χ_j and $\chi_{q+1}^{(j)}$ are fixed by φ^k if and only if

$$p^f - 1 \mid (p^k - 1)j \text{ or } p^f - 1 \mid (p^k + 1)j.$$

Assume first that $p^k \geqslant 5$ and let $n = (p^f - 1)/(p^k - 1)$. We then have $p^k - 1 > 2$, so that $n < (p^f - 1)/2 = (q-1)/2$. Therefore,

$$n \leqslant \frac{q-1}{2} - 1 = \frac{q-3}{2},$$

and so $\chi_n \in \mathrm{Irr}(\mathrm{SL}_2(q))$ and $\chi_{q+1}^{(n)} \in \mathrm{Irr}(\mathrm{PGL}_2(q))$. Moreover, $(p^k - 1)n = p^f - 1$, hence $p^f - 1$ divides $(p^k - 1)n$ and χ_n and $\chi_{q+1}^{(n)}$ are invariant under $\langle \varphi^k \rangle$. Therefore, K is contained in the stabilizer, T, of χ_n and $\chi_{q+1}^{(n)}$ in $\langle \varphi \rangle$.

We have $K = \langle \varphi^k \rangle \leqslant \langle \varphi^t \rangle = T$ for some divisor t of k. Since χ_n and $\chi_{q+1}^{(n)}$ are invariant under φ^t, we have that $p^f - 1$ divides either $(p^t - 1)n$ or $(p^t + 1)n$, that is,

$$p^f - 1 \mid (p^t - 1) \cdot \frac{p^f - 1}{p^k - 1} \text{ or } p^f - 1 \mid (p^t + 1) \cdot \frac{p^f - 1}{p^k - 1}.$$

Hence either $p^k - 1 \mid p^t - 1$ or $p^k - 1 \mid p^t + 1$. Since $t \mid k$, we have $p^t - 1 \mid p^k - 1$, and so if $p^k - 1 \mid p^t + 1$, then $p^t - 1 \mid p^t + 1 = (p^t - 1) + 2$. Therefore, $p^t - 1 \mid 2$ and since p is odd, this means $p^t = 3$. In particular, $p^k - 1 = 3^k - 1$ divides $p^t + 1 = 4$, and hence $k = 1$ and $p^k = 3$, contradicting $p^k \geqslant 5$. Therefore, $p^k - 1 \mid p^t - 1$, so that $k \mid t$, and since $t \mid k$, we have $k = t$ and so $K = T$.

Now let $p^k = 3$ so that $p^k - 1 = 2$ and $p^k + 1 = 4$. It follows that if $\varphi^k = \varphi$ stabilizes χ_n or $\chi_{q+1}^{(n)}$ for some n, then $3^f - 1 \mid 2n$ or $3^f - 1 \mid 4n$. Hence $3^f - 1 \mid 4n$ in any case, and if f is odd, then $(3^f - 1)/2$ is odd and divides $2n$. Hence $(q-1)/2$ divides n, contradicting $1 \leqslant n \leqslant (q-3)/2$. Therefore, if φ stabilizes χ_n or $\chi_{q+1}^{(n)}$ for some n, then f is even.

Conversely, if f is even, then $4 \mid 3^f - 1$. Setting $n = (3^f - 1)/4 = (q-1)/4$, we have $n < (q-1)/2$, so that $n \leqslant (q-3)/2$ and $\chi_n \in \mathrm{Irr}(\mathrm{SL}_2(q))$ and $\chi_{q+1}^{(n)} \in \mathrm{Irr}(\mathrm{PGL}_2(q))$. Moreover, $3^f - 1 \mid 4n$ so that χ_n and $\chi_{q+1}^{(n)}$ are invariant under φ. Hence $K = \langle \varphi \rangle$ is the stabilizer of χ_n and $\chi_{q+1}^{(n)}$.

Finally, we show that unless f is odd and $p^k = 3$, K is also the stabilizer in $\langle \varphi \rangle$ of some extension $\hat{\chi}_n$ of χ_n to $\mathrm{GL}_2(q)$. It is straightforward to check that for any j, one extension of χ_j to $\mathrm{GL}_2(q)$ is $\chi_{q+1}^{(j,q-1)}$. The irreducible characters of the cyclic group $\mathrm{GL}_2(q)/\mathrm{SL}_2(q)$ are precisely the characters $\chi_1^{(i)}$ of $\mathrm{GL}_2(q)$, for $1 \leqslant i \leqslant q-1$, and so by Gallagher's Theorem all of the extensions of χ_j are the characters $\chi_{q+1}^{(j,q-1)} \cdot \chi_1^{(i)}$.

We claim that the extension $\hat{\chi}_n = \chi_{q+1}^{(n,q-1)} \cdot \chi_1^{(i)}$ has stabilizer K in $\langle \varphi \rangle$ for some choice of i. Since $\hat{\chi}_n$ is an extension of χ_n, the stabilizer of $\hat{\chi}_n$ is contained in K. We need to show that $\hat{\chi}_n$ is invariant under φ^k.

The values of the character $\hat{\chi}_n$ and its image under φ^{-k} are as follows:

	$\hat{\chi}_n$	$(\hat{\chi}_n)^{\varphi^{-k}}$
$A_1(l)$	$(q+1)\rho^{(n+2i)l}$	$(q+1)\rho^{(n+2i)p^k l}$
$A_2(l)$	$\rho^{(n+2i)l}$	$\rho^{(n+2i)p^k l}$
$A_3(l_1, l_2)$	$(\rho^{nl_1} + \rho^{nl_2})\rho^{i(l_1+l_2)}$	$(\rho^{nl_1 p^k} + \rho^{nl_2 p^k})\rho^{i(l_1+l_2)p^k}$
$B_1(l)$	0	0

where ρ is a complex primitive $(q-1)$th root of unity.

First assume $p^k \neq 3$ and let $i = q-1$, so that $\hat{\chi}_n = \chi_{q+1}^{(n,q-1)} \cdot \chi_1^{(q-1)} = \chi_{q+1}^{(n,q-1)}$ and $\rho^i = 1$. We have $q - 1 \mid n(p^k - 1)$, hence $np^k \equiv n \pmod{q-1}$ and $\rho^n = \rho^{np^k}$. It follows from the table of character values above that $\hat{\chi}_n$ is invariant under φ^{-k}, and so also under φ^k.

Now suppose $p = 3$, $k = 1$ (so $K = \langle \varphi \rangle$), and f is even. In this case, we have $n = (3^f - 1)/4$ and we take $i = (3^f - 1)/8$. Since f is even, i is an integer and $\hat{\chi}_n = \chi_{q+1}^{(n,q-1)} \cdot \chi_1^{(i)}$ is an extension of χ_n. We have $n+2i = (q-1)/2$ and $\rho^{n+2i} = -1$. As $p^k = 3$ is odd, it follows that $\hat{\chi}_n$ and $(\hat{\chi}_n)^{\varphi^{-1}}$ agree on the classes $A_1(l)$ and $A_2(l)$. We also have $n = 2i$ and $p^k = 3$, and so

$$\hat{\chi}_n(A_3(l_1, l_2)) = \rho^{(3l_1+l_2)i} + \rho^{(l_1+3l_2)i}$$

and

$$(\hat{\chi}_n)^{\varphi^{-1}}(A_3(l_1, l_2)) = \rho^{(3l_1+l_2)3i} + \rho^{(l_1+3l_2)3i}.$$

Finally, $9i = (q-1)+i$, hence $\rho^{9i} = \rho^i$, and so $\hat{\chi}_n$ and $(\hat{\chi}_n)^{\varphi^{-1}}$ agree on the classes $A_3(l_1, l_2)$. Therefore, $\hat{\chi}_n$ is invariant under φ. ☐

Lemma 5.4. *Let $q = p^f$, $k \mid f$, $K = \langle \varphi^k \rangle$, and $n = 2(p^f-1)/(p^k-1)$.*

i. If $p^k \notin \{2, 3, 2^2, 5, 3^2\}$, then K is the stabilizer in $\langle\varphi\rangle$ of $\chi_n \in \mathrm{Irr}(\mathrm{L}_2(q))$.

ii. For $p^k \in \{2, 3, 2^2, 5, 3^2\}$, K is the stabilizer in $\langle\varphi\rangle$ of an irreducible character of $\mathrm{L}_2(q)$ of degree $q+1$ if and only if f/k even.

iii. If $p^k = 2^2$ or 3^2 and $f/2$ is odd, then there is an irreducible character of $\mathrm{L}_2(q)$ of degree $q+1$ whose stabilizer is $\langle\varphi\rangle$ and hence is invariant under $K = \langle\varphi^2\rangle$.

Proof. Characters of $\mathrm{L}_2(q)$ of degree $q+1$ are χ_j for $1 \leqslant j \leqslant (q-2)/2$ if q is even and $1 \leqslant j \leqslant (q-3)/2$, j even, if q is odd. By Lemma 4.8, χ_j is fixed by φ^k if and only if

$$p^f - 1 \mid (p^k-1)j \text{ or } p^f - 1 \mid (p^k+1)j.$$

Assume first that K is the stabilizer of χ_j for some j, and suppose $p^k \in \{2, 3, 2^2, 5, 3^2\}$. We also assume f/k is odd and work for a contradiction.

If $p^k = 2$, 3, or 5, so $k = 1$, and K stabilizes χ_j, then $p^f-1 \mid (p-1)j$ or $p^f-1 \mid (p+1)j$. Since $f = f/k$ is odd, $(p^f-1, p+1) \mid 2$ by Lemma 4.9, and so if $p^f-1 \mid (p+1)j$, then $p^f-1 \mid 2j$, i.e., $q-1 \mid 2j$. This contradicts the fact that $j < (q-1)/2$ in all cases. Hence $p^f-1 \mid (p-1)j$, and so $(q-1)/(p-1) \mid j$. For $p = 2$ or 3, this again contradicts $j < (q-1)/2$. Finally, for $p = 5$, we have $(5^f-1)/4 \mid j$ and since f is odd, $(5^f-1)/4$ is odd. Recalling that j must be even, we then have $2 \cdot (5^f-1)/4 \mid j$, and so $(q-1)/2 \mid j$, again contradicting $j < (q-1)/2$.

Now suppose $p^k = 2^2$ or 3^2, so $k = 2$, and assume χ_j is invariant under K, hence under φ^2, for some j. In this case we have $p^f-1 \mid (p^2-1)j$ or $p^f-1 \mid (p^2+1)j$. Since $f/k = f/2$ is odd, we have $(p^f-1, p^2+1) \mid 2$ by Lemma 4.9, and as before, this implies that if $p^f-1 \mid (p^2+1)j$, then $q-1 \mid 2j$, a contradiction.

Hence we must have $p^f-1 \mid (p^2-1)j$. If $p = 2$, this means $2^f-1 \mid 3j$. Hence in fact $2^f-1 \mid (2^1+1)j$, and so χ_j is invariant under φ. If $p = 3$, we have $3^f-1 \mid 8j$. Since $f/2$ is odd, f is not divisible by 4, and it follows that $(3^f-1)/8$ is odd and divides j, which is even. Hence $2 \cdot (3^f-1)/8 \mid j$, and so $3^f-1 \mid 4j = (3^1+1)j$ and again this implies χ_j is invariant under φ. Therefore, in either case $K = \langle\varphi^2\rangle$ is not the full stabilizer of χ_j in $\langle\varphi\rangle$.

On the other hand, if $k = 2$ and $f/2$ is odd, then both $(2^f-1)/3$ and $(3^f-1)/4$ are integers, $(2^f-1)/3 \leqslant (2^f-2)/2$ and $(3^f-1)/4 \leqslant (3^f-3)/2$, and $(3^f-1)/4$ is even. Hence, setting $j = (2^f-1)/3$ when $p = 2$ and $j = (3^f-1)/4$ when $p = 3$, we have that $\chi_j \in \mathrm{Irr}(\mathrm{L}_2(q))$. Since $2^f-1 = 3j$ and $3^f-1 = 4j$, we obtain $2^f-1 \mid (2^1+1)j$ and $3^f-1 \mid (3^1+1)j$, and so χ_j is invariant under φ in either case.

We now have that if K is the stabilizer in $\langle\varphi\rangle$ of an irreducible character of $\mathrm{L}_2(q)$ of degree $q+1$ and $p^k \in \{2, 3, 2^2, 5, 3^2\}$, then f/k is even. We next consider the converse, and so let $K = \langle\varphi^k\rangle$ and if $p^k \in \{2, 3, 2^2, 5, 3^2\}$, we assume f/k is even. We will find a j in each case such that K is the stabilizer of χ_j.

We first suppose $p^k \in \{2, 3, 5\}$, so that $k = 1$ and f is even. If $p = 2$, this implies $3 \mid 2^f-1$ and we set $j = (2^f-1)/3$, so that $j < (2^f-1)/2$, so $j \leqslant (q-2)/2$ and $\chi_j \in \mathrm{Irr}(\mathrm{L}_2(q))$. If $p = 3$ or

5, then f even implies $8 \mid p^f - 1$ and we set $j = (p^f - 1)/4$. Thus j is even and strictly less than $(q-1)/2$, hence $j \leqslant (q-3)/2$, and so $\chi_j \in \mathrm{Irr}(\mathrm{L}_2(q))$. In all three cases, we have $j = (p^f-1)/(p\pm1)$ and so $p^f - 1 \mid (p \pm 1)j$, which implies that χ_j is invariant under φ. Hence the stabilizer of χ_j in $\langle\varphi\rangle$ is $K = \langle\varphi\rangle$.

Next, let $p^k = 2^2$ or 3^2, so that $k = 2$ and $f/k = f/2$ is even. By Lemma 4.9, $p^2+1 \mid p^f - 1$ and we set $j = (p^f-1)/(p^2+1)$. For $p = 2$, we have $j = (2^f-1)/5 < (2^f-1)/2$, hence $j \leqslant (q-2)/2$ and $\chi_j \in \mathrm{Irr}(\mathrm{L}_2(q))$. For $p = 3$, we have $j = (3^f - 1)/10 < (3^f - 1)/2$, hence $j \leqslant (q - 3)/2$. Also, since f is even, $8 \mid 3^f - 1$ and so j is even. Thus again $\chi_j \in \mathrm{Irr}(\mathrm{L}_2(q))$. In both cases, $p^f - 1 \mid (p^2 + 1)j$, and so χ_j is stabilized by $\varphi^2 = \varphi^k$. Also $p\pm1 < p^2+1$, so $(p\pm1)j < p^f-1$ and hence p^f-1 divides neither $(p+1)j$ nor $(p-1)j$. It follows that χ_j is not invariant under φ, and so the stabilizer of χ_j in $\langle\varphi\rangle$ is $K = \langle\varphi^2\rangle$, as claimed.

We now suppose $p^k \notin \{2, 3, 2^2, 5, 3^2\}$. Set $n = 2(p^f-1)/(p^k-1)$, which is even as $p^k-1 \mid p^f-1$. Since $p^k \geqslant 7$, we have

$$n = 2 \cdot \frac{q-1}{p^k-1} \leqslant 2 \cdot \frac{q-1}{6} < \frac{q-1}{2}.$$

Therefore, if q is even, then $n \leqslant (q-2)/2$ and if q is odd, then n is even and $n \leqslant (q-3)/2$, so that $\chi_n \in \mathrm{Irr}(\mathrm{L}_2(q))$ in any case. Moreover, $p^f - 1 \mid (p^k - 1)n$, and so χ_n is invariant under φ^k and K is contained in the stabilizer T of χ_n in $\langle\varphi\rangle$.

We know that T is of the form $T = \langle\varphi^t\rangle$ for some positive divisor t of k. In particular, observe that $t \mid k$ implies $p^t - 1 \mid p^k - 1$. Also, since φ^t stabilizes χ_n, we have $p^f - 1 \mid (p^t \pm 1)n$; that is

$$p^f - 1 \mid (p^t \pm 1) \cdot 2 \cdot \frac{p^f-1}{p^k-1},$$

which implies that either $p^k - 1 \mid 2(p^t - 1)$ or $p^k - 1 \mid 2(p^t + 1)$.

First, we show that $p^k - 1 \mid 2(p^t + 1)$ cannot occur if $p^k \notin \{2, 3, 2^2, 5, 3^2\}$. If $p^k - 1 \mid 2(p^t + 1)$, then $t \mid k$ implies $p^t - 1 \mid 2(p^t + 1) = 2(p^t - 1) + 4$. Hence $p^t - 1 \mid 4$ and $p^t = 2$, 3, or 5, so $t = 1$ and we have $p^k - 1 \mid 2(p + 1)$ with $p = 2$, 3, or 5. If $p = 2$, then $2^k - 1 \mid 6$, hence $k = 1$ or 2 and $p^k = 2$ or 2^2. If $p = 3$, then $3^k - 1 \mid 8$, hence $k = 1$ or 2 and $p^k = 3$ or 3^2. If $p = 5$, then $5^k - 1 \mid 12$, hence $k = 1$ and $p^k = 5$. Therefore, if $p^k \notin \{2, 3, 2^2, 5, 3^2\}$, then $p^k - 1 \nmid 2(p^t + 1)$.

Finally, suppose $p^k - 1 \mid 2(p^t - 1)$. If $p = 2$, this implies $p^k - 1 \mid (p^t - 1)$, hence $k \mid t$ and so $t = k$ and $T = K$. If p is odd, this implies $(p^k - 1)/(p^t - 1)$ divides 2. Suppose $k > t$, so that $(p^k - 1)/(p^t - 1) = 2$ and $p^k - 1 = 2(p^t - 1)$. We have $p^k \geqslant p \cdot p^t$, hence

$$2(p^t - 1) = p^k - 1 \geqslant p \cdot p^t - 1 = p(p^t - 1) + (p - 1) > 2(p^t - 1),$$

a contradiction. Hence $t = k$ and $T = K$ in this case as well. Therefore, if $p^k \notin \{2, 3, 2^2, 5, 3^2\}$, then K is the stabilizer in $\langle\varphi\rangle$ of χ_n. □

6 Subgroups of $\mathrm{Aut}(\mathrm{L}_2(q))$ and Their Degrees

Our goal in this section is to determine the character degrees of every group H with $\mathrm{L}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{L}_2(q))$. Recall that if q is even, then $\bar{\delta}$ is an inner automorphism, $\mathrm{PGL}_2(q) = \mathrm{L}_2(q)$, and $\mathrm{Aut}(\mathrm{L}_2(q)) = \mathrm{L}_2(q)\langle\varphi\rangle$. Hence, if $\mathrm{L}_2(q) < H \leqslant \mathrm{Aut}(\mathrm{L}_2(q))$ with q even, then $H = \mathrm{L}_2(q)\langle\varphi^k\rangle$ for some $k \mid f$ with $1 \leqslant k < f$, and $H/\mathrm{L}_2(q)$ is cyclic.

If q is odd, then $\mathrm{Aut}(\mathrm{L}_2(q)) = \mathrm{L}_2(q)\langle\bar{\delta}, \varphi\rangle$ and the outer automorphism group of $\mathrm{L}_2(q)$ is $\mathrm{Aut}(\mathrm{L}_2(q))/\mathrm{L}_2(q) \cong \langle\bar{\delta}\rangle \times \langle\bar{\varphi}\rangle$, where $\bar{\delta}$ is of order 2 and $\bar{\varphi}$ is of order f. The following elementary lemma will be useful in describing the subgroups of $\mathrm{Aut}(\mathrm{L}_2(q))$ in this case.

Lemma 6.1. *If $A \cong \langle x\rangle \times \langle y\rangle$, where $|\langle x\rangle| = 2$ and $|\langle y\rangle| = f$, then any subgroup of A that does not contain x is cyclic.*

Proof. If f is odd, then A is cyclic, so we may assume f is even. In this case, A contains exactly three elements of order 2, as does any noncyclic subgroup of A. Thus a subgroup of A not containing the element x of order 2 cannot contain three elements of order 2, hence is cyclic. $\square$

Corollary 6.2. *If* $\mathrm{L}_2(q) < H \leqslant \mathrm{Aut}(\mathrm{L}_2(q))$ *with* $q = p^f$, p *an odd prime, then one of the following occurs:*

i. $\bar{\delta} \in H$ *so that* $\mathrm{PGL}_2(q) \leqslant H$ *and* $H = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$ *for some* $k \mid f$ *with* $1 \leqslant k \leqslant f$*;*

ii. $H = \mathrm{L}_2(q)\langle\varphi^k\rangle$ *for some* $k \mid f$ *with* $1 \leqslant k < f$*;*

iii. $H = \mathrm{L}_2(q)\langle\bar{\delta}\varphi^k\rangle$ *for some* $k \mid f$ *with* $1 \leqslant k < f$ *and* f/k *even.*

Proof. If $\bar{\delta} \in H$, then H contains $\mathrm{L}_2(q)\langle\bar{\delta}\rangle \cong \mathrm{PGL}_2(q)$. If $\bar{\delta}$ is not in H, then $H/\mathrm{L}_2(q)$ is cyclic by the lemma, hence $H = \mathrm{L}_2(q)\langle\sigma\rangle$ for some outer automorphism σ. If H is not a subgroup of $\mathrm{L}_2(q)\langle\varphi\rangle$, then $\sigma = \bar{\delta}\varphi^k$ for some $k \mid f$. Finally, f/k is the order of φ^k, and so if f/k is odd, then $\bar{\delta}$ is in H. $\square$

Corollary 6.3. *Let* $\mathrm{L}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{L}_2(q))$ *and set* $G = \mathrm{PGL}_2(q)$ *if* $\bar{\delta} \in H$ *and* $G = \mathrm{L}_2(q)$ *if* $\bar{\delta} \notin H$*. If* $\hat{\chi} \in \mathrm{Irr}(H)$*, then* $\hat{\chi}(1) = \chi(1)|H : I_H(\chi)|$*, where* χ *is a constituent of* $\hat{\chi}_G$ *and* $I_H(\chi)$ *is the stabilizer of* χ *in* H*.*

Proof. We have that $G \trianglelefteq H$ and H/G is cyclic. Hence a character $\chi \in \mathrm{Irr}(G)$ extends to its stabilizer $I_H(\chi)$ in H and each extension induces irreducibly to H by Theorem 6.11 of [4]. Every character of H lying over χ will therefore have degree $\chi(1)|H : I_H(\chi)|$. $\square$

We first determine the stabilizers of characters of $\mathrm{L}_2(q)$ of degrees 1, q, and $(q+\varepsilon)/2$, and the degrees of the characters of H lying over these.

Theorem 6.4. *Let* $S = \mathrm{L}_2(q)$ *and let* $S \leqslant H \leqslant \mathrm{Aut}(S)$*. If* $\chi \in \mathrm{Irr}(S)$ *has degree* 1 *or* q*, then every irreducible character of* H *lying over* χ *has degree* $\chi(1)$*.*

Proof. By Lemmas 4.5 and 4.6, χ is invariant in H. If H does not contain $\bar{\delta}$, then by Corollary 6.2, H/S is cyclic and χ extends to H. The result then follows from Gallagher's Theorem (see [4, 6.17]).

If $\bar{\delta} \in H$, then by Lemma 4.5, χ extends to two irreducible characters of $\mathrm{PGL}_2(q)$. By Lemma 4.6, both of these are invariant in H. Since $H/\mathrm{PGL}_2(q)$ is cyclic, these characters extend to H and the result again follows from Gallagher's Theorem. $\square$

Theorem 6.5. *Let* $S = \mathrm{L}_2(q)$ *with* q *odd and let* $S \leqslant H \leqslant \mathrm{Aut}(S)$*. Let* $\mu \in \mathrm{Irr}(S)$ *with* $\mu(1) = (q+\varepsilon)/2$*.*

i. If $H \leqslant S\langle\varphi\rangle$*, then* μ *is invariant in* H *and every irreducible character of* H *lying over* μ *is of degree* $(q+\varepsilon)/2$*.*

ii. If $H \nleqslant S\langle\varphi\rangle$*, then the stabilizer of* μ *in* H *is* $I_H(\mu) = H \cap (S\langle\varphi\rangle)$ *and* $|H : I_H(\mu)| = 2$*. Every irreducible character of* H *lying over* μ *is of degree* $q+\varepsilon$*.*

Proof. If $H \leqslant S\langle\varphi\rangle$, then μ is invariant in H by Lemma 4.6. Since H/S is cyclic, μ extends to H and (*i*) follows from Gallagher's Theorem ([4, 6.17]).

Assume now that $H \nleqslant S\langle\varphi\rangle$, so $\bar{\delta}\varphi^k \in H$ for some integer k. By Lemma 4.5, μ is not fixed by this automorphism, and so $I_H(\mu) < H$. By Lemma 4.6, μ is invariant in $H \cap (S\langle\varphi\rangle)$ and we have $H \cap (S\langle\varphi\rangle) \leqslant I_H(\mu) < H$. Since $\bar{\delta}^2$ is an inner automorphism, $|H : H \cap (S\langle\varphi\rangle)| = 2$, and hence $I_H(\mu) = H \cap (S\langle\varphi\rangle)$.

We have that $I_H(\mu) \leqslant S\langle\varphi\rangle$, and so $I_H(\mu)/S$ is cyclic. Thus μ extends to $I_H(\mu)$ and, by Gallagher's Theorem, each extension has degree $(q+\varepsilon)/2$. Finally, by Clifford's Theorem, each extension induces to an irreducible character of H of degree $|H : I_H(\mu)|(q+\varepsilon)/2 = q+\varepsilon$. $\square$

Theorem 6.6. *Let $S = \mathrm{L}_2(q)$, where $q = p^f > 3$ for a prime p, $A = \mathrm{Aut}(S)$, and let $S \leqslant H \leqslant A$. Let $G = \mathrm{PGL}_2(q)$ if $\bar{\delta} \in H$ and $G = S$ if $\bar{\delta} \notin H$, and set $|H : G| = d = 2^a m$, m odd.*

The degrees of the irreducible characters of H lying over characters of G of degree $q-1$ are precisely $(q-1)2^a\ell$, where ℓ is a positive divisor of m, with the exception that if $p = 3$, f is odd, and $H = \mathrm{L}_2(q)\langle\varphi\rangle$, then $\ell \neq 1$.

Proof. First suppose $\bar{\delta} \in H$, so that $H = \mathrm{PGL}_2(q)\langle\varphi^{f/d}\rangle$. (We include here the case where q is even.) By Corollary 6.3, for $j \mid d$, there is a character of H of degree $(q-1)j$ lying over a character θ of $\mathrm{PGL}_2(q)$ of degree $q-1$ if and only if $|H : I_H(\theta)| = j$, hence if and only if $I_H(\theta) = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$, where $k = (f/d)j$. By Lemma 5.1, such a character θ of $\mathrm{PGL}_2(q)$ exists if and only if $f/k = d/j$ is odd, that is, $j = 2^a\ell$ for some $\ell \mid m$.

Suppose now that q is odd and $\bar{\delta} \notin H$. By Corollary 6.2, $H = \mathrm{L}_2(q)\langle\bar{\delta}^c\varphi^{f/d}\rangle$, where $c \in \{0, 1\}$ and if d is odd, then $c = 0$ since $\bar{\delta} \notin H$. Again by Corollary 6.3, there is a character of H of degree $(q-1)j$ lying over a character θ of $\mathrm{L}_2(q)$ of degree $q-1$ if and only if $I_H(\theta) = \mathrm{L}_2(q)\langle\bar{\delta}^{cj}\varphi^k\rangle$, where $k = (f/d)j$. By Lemma 5.2, this implies $f/k = d/j$ must be odd, that is, $j = 2^a\ell$ for some $\ell \mid m$. Conversely, if d/j is odd, then such a character θ of $\mathrm{L}_2(q)$ exists except when $p = 3$ and $k = 1$. Since $d \mid f$, $k = (f/d)j = 1$ if and only if $d = f$ (so $f/d = 1$) and $j = 1$. Observe that since $d = f$ is odd in this case, $c = 0$ and $H = \mathrm{L}_2(q)\langle\varphi\rangle$. □

Theorem 6.7. *Let $S = \mathrm{L}_2(q)$, where $q = p^f > 3$ for a prime p, $A = \mathrm{Aut}(S)$, and let $S \leqslant H \leqslant A$. Let $G = \mathrm{PGL}_2(q)$ if $\bar{\delta} \in H$ and $G = S$ if $\bar{\delta} \notin H$, and set $|H : G| = d$.*

The degrees of the irreducible characters of H lying over characters of G of degree $q+1$ are precisely $(q+1)j$, where j is a positive divisor of d, with the following exceptions:

i. if f is odd, $p = 3$, and $H = A$, then $j \neq 1$;

ii. if f is odd, $p = 2$, 3, or 5, and $H = S\langle\varphi\rangle$, then $j \neq 1$;

iii. if $f \equiv 2 \pmod 4$, $p = 2$ or 3, and $H = S\langle\varphi\rangle$ or $H = S\langle\bar{\delta}\varphi\rangle$, then $j \neq 2$.

Proof. First, suppose q is odd and $\bar{\delta} \in H$, so that $H = \mathrm{PGL}_2(q)\langle\varphi^{f/d}\rangle$. By Corollary 6.3, for $j \mid d$, there is a character of H of degree $(q+1)j$ lying over a character χ of $\mathrm{PGL}_2(q)$ of degree $q+1$ if and only if $|H : I_H(\chi)| = j$, hence if and only if $I_H(\chi) = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$, where $k = (f/d)j$. By Lemma 5.3, such a character χ of $\mathrm{L}_2(q)$ exists except when f is odd, $p = 3$, and $k = 1$. Since $d \mid f$, $k = (f/d)j = 1$ if and only if $f = d$ (hence $K = A$) and $j = 1$, which is exception (i) in the statement of the theorem.

Suppose now that either q is odd and $\bar{\delta} \notin H$ or q is even. By Corollary 6.2, $H = \mathrm{L}_2(q)\langle\bar{\delta}^c\varphi^{f/d}\rangle$, where $c \in \{0, 1\}$ and if d is odd, then $c = 0$ since $\bar{\delta} \notin H$. Again, Corollary 6.3 implies that if $j \mid d$, there is a character of H of degree $(q+1)j$ lying over a character χ of $\mathrm{L}_2(q)$ of degree $q+1$ if and only if $I_H(\chi) = \mathrm{L}_2(q)\langle\bar{\delta}^{cj}\varphi^k\rangle$, where $k = (f/d)j$. Lemma 5.4 implies that if $p^k \notin \{2, 3, 2^2, 5, 3^2\}$ or if $p^k \in \{2, 3, 2^2, 5, 3^2\}$ and f/k is even, then such a character χ of $\mathrm{L}_2(q)$ exists and H has a character of degree $(q+1)j$. It therefore remains to consider the cases where $p^k \in \{2, 3, 2^2, 5, 3^2\}$ and f/k is odd.

Suppose $p = 2$, 3, or 5, $k = (f/d)j = 1$, and f is odd. Since $d \mid f$, $k = 1$ implies that $j = 1$ and $d = f$, which is odd, so that $H = \mathrm{L}_2(q)\langle\varphi\rangle$. In this case, Lemma 5.4 implies there is no character of $\mathrm{L}_2(q)$ of degree $q+1$ stabilized by H, hence H has no character of degree $(q+1)j$ for $j = 1$, which is exception (ii) in the statement of the theorem.

Finally, suppose $p = 2$ or 3, $k = (f/d)j = 2$, and $f/2$ is odd, that is, $f \equiv 2 \pmod 4$. In this case, $k = (f/d)j = 2$ implies that either $j = 1$ and $d = f/2$, or $j = 2$ and $d = f$.

If $j = 1$ and $d = f/2$, we have $H = \mathrm{L}_2(q)\langle\varphi^2\rangle$. By Lemma 5.4, there is a character χ of $\mathrm{L}_2(q)$ of degree $q+1$ invariant under φ, hence under H, and so H has a character of degree $(q+1)j$ with $j = 1$.

If $j = 2$ and $d = f$, then either $H = \mathrm{L}_2(q)\langle\varphi\rangle$ or $H = \mathrm{L}_2(q)\langle\bar{\delta}\varphi\rangle$, and the subgroup of H of index $j = 2$ is $K = \mathrm{L}_2(q)\langle\varphi^2\rangle$. However, by Lemma 5.4, K is not the stabilizer in H of any character of $\mathrm{L}_2(q)$ of degree $q+1$ (any such character invariant under φ^2 is also invariant under φ). Hence H does not have a character of degree $(q+1)j$ for $j = 2$, which is exception (*iii*) in the statement of the theorem. □

Finally, we observe that Theorem A now follows from Corollary 6.3 and Theorems 6.4, 6.5, 6.6, and 6.7.

7 Subgroups of $(\mathrm{SL}_2(q) \rtimes \langle\delta\rangle) \rtimes \langle\varphi\rangle$ and Their Degrees

Throughout this section we will denote $A = (\mathrm{SL}_2(q) \rtimes \langle\delta\rangle) \rtimes \langle\varphi\rangle = \mathrm{GL}_2(q) \rtimes \langle\varphi\rangle$, where $q = p^f$ is odd. Our goal is to determine the character degrees of all subgroups of A of the form $H = (\mathrm{SL}_2(q) \rtimes \langle\delta^\alpha\rangle) \rtimes \langle\varphi^\beta\rangle$, where $\alpha \mid q-1$ and $\beta \mid f$.

We will first determine the characters of $H_0 = \mathrm{SL}_2(q) \rtimes \langle\delta^\alpha\rangle$ lying over the irreducible characters of $\mathrm{SL}_2(q)$ of degree 1, q, $(q-1)/2$, and $(q+1)/2$ and then determine the stabilizers of these characters in H. As before, since H/H_0 is cyclic, each character of H_0 will extend to its stabilizer in H and then induce irreducibly to H. For the characters of $\mathrm{SL}_2(q)$ of degrees $q-1$ and $q+1$, we have shown that each of the characters of degree $q-1$ or $q+1$ from Lemmas 5.1 and 5.3 has an extension to $\mathrm{GL}_2(q)$, hence also to H_0, with the same stabilizer in $\langle\varphi\rangle$.

The following lemma will be necessary for computing restrictions of characters of $\mathrm{GL}_2(q)$ to H_0. It is straightforward to verify the lemma using the list of conjugacy class representatives in [9].

Lemma 7.1. *Let $\alpha \mid q-1$, $H_0 = \mathrm{SL}_2(q) \rtimes \langle\delta^\alpha\rangle$, and $\alpha' = \alpha/(2,\alpha)$. An element X of $\mathrm{GL}_2(q)$ is in H_0 if and only if the determinant of X is a power of ν^α. Representatives of the the conjugacy classes of $\mathrm{GL}_2(q)$ that are contained in H_0 are as follows:*

i. $A_1(j\alpha')$, *where* $1 \leqslant j \leqslant (q-1)/\alpha'$;

ii. $A_2(j\alpha')$, *where* $1 \leqslant j \leqslant (q-1)/\alpha'$;

iii. $A_3(l_1, l_2)$, *where* $1 \leqslant l_i \leqslant q-1$, $l_1 \neq l_2$, *and* $l_1 + l_2 = j\alpha$ *for some integer* j;

iv. $B_1(j\alpha)$, *where* $1 \leqslant j \leqslant (q^2-1)/\alpha$, $q+1 \nmid j\alpha$.

Observe that if α is odd, each conjugacy class of $\mathrm{GL}_2(q)$ contained in H_0 is a single conjugacy class of H_0, but if α is even, then the class of $A_2(l)$ in $\mathrm{GL}_2(q)$ splits into two conjugacy classes $A_2'(l)$ and $A_2''(l)$ of H_0.

7.1 Characters of H Lying Over 1_{SL} and ψ

The irreducible characters of $\mathrm{SL}_2(q)$ of degree 1 and q are the principal character 1_{SL} and the Steinberg character ψ, respectively. These extend to the principal and Steinberg character of $\mathrm{GL}_2(q)$, denoted in [9] by $1_{\mathrm{GL}} = \chi_1^{(q-1)}$ and $\mathrm{St} = \chi_q^{(q-1)}$, respectively. By Gallagher's Theorem, the remaining irreducible characters of $\mathrm{GL}_2(q)$ lying over 1_{SL} and ψ are the products of these characters with the $q-1$ irreducible characters of $\mathrm{GL}_2(q)/\mathrm{SL}_2(q)$, which is cyclic of order $q-1$. Of course, the characters of this quotient are precisely the extensions of 1_{SL}, which are denoted $\chi_1^{(n)}$ for $n = 1, 2, \ldots, q-1$ in [9], and the extensions of the Steinberg character are $\chi_q^{(n)} = \mathrm{St} \cdot \chi_1^{(n)} = \chi_q^{(q-1)} \cdot \chi_1^{(n)}$.

The irreducible characters of H_0 lying over 1_{SL} and ψ are therefore the restrictions of $\chi_1^{(n)}$ and $\chi_q^{(n)} = \mathrm{St} \cdot \chi_1^{(n)}$ to H_0. As the Steinberg character St is invariant under φ, the stabilizer in $\langle\varphi\rangle$ of the restriction of $\chi_q^{(n)}$ to H_0 is equal to the stabilizer of the restriction of $\chi_1^{(n)}$.

Lemma 7.2. *Let $q = p^f$ be odd, $H_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^\alpha \rangle$ for $\alpha \mid q-1$, and $\ell = (q-1)/\alpha = |H_0 : \mathrm{SL}_2(q)|$. Let $\beta \mid f$. For $n = 1, 2, \ldots, q-1$, the stabilizer in $\langle \varphi^\beta \rangle$ of $(\chi_1^{(n)})_{H_0}$ and $(\chi_q^{(n)})_{H_0}$ is $\langle \varphi^{\beta k} \rangle$, where k is the order of p^β modulo $\ell/(\ell, n)$.*

Proof. As noted above, we only need to consider the restriction of $\chi_1^{(n)}$ to H_0. By Lemma 7.1, Lemma 3.3, and the values of $\chi_1^{(n)}$ given in [9], the values of $(\chi_1^{(n)})_{H_0}$ and its image under φ^{-r} for any $r \mid f$ are as follows:

	$(\chi_1^{(n)})_{H_0}$	$(\chi_1^{(n)})_{H_0}^{\varphi^{-r}}$
$A_1(j\alpha')$	$\rho^{2nj\alpha'}$	$\rho^{2nj\alpha' p^r}$
$A_2(j\alpha')$	$\rho^{2nj\alpha'}$	$\rho^{2nj\alpha' p^r}$
$A_3(l_1, l_2)$	$\rho^{nj\alpha}$	$\rho^{nj\alpha p^r}$
$B_1(j\alpha)$	$\rho^{nj\alpha}$	$\rho^{nj\alpha p^r}$

where ρ is a complex primitive $(q-1)$th root of unity. Note that since $\alpha \mid q-1$, we know that $q+1 \nmid \alpha$ and so $B_1(\alpha)$ is a conjugacy class of H_0. Therefore, if $(\chi_1^{(n)})_{H_0}$ is invariant under φ^r (and hence under φ^{-r}) we must have $\rho^{n\alpha} = \rho^{n\alpha p^r}$, and thus $n\alpha \equiv n\alpha p^r \pmod{q-1}$. Conversely, if this congruence holds, then, since $2\alpha'$ is either α or 2α, it is clear that $(\chi_1^{(n)})_{H_0}$ is invariant under φ^r. We therefore have that $(\chi_1^{(n)})_{H_0}$ is invariant under φ^r if and only if $p^f - 1 \mid n\alpha(p^r - 1)$.

Suppose now that φ^r is in $\langle \varphi^\beta \rangle$, and let $r = \beta k$ for a positive integer k. Replacing α with $(q-1)/\ell$, we have that $(\chi_1^{(n)})_{H_0}$ is invariant under $\varphi^{\beta k}$ if and only if $p^f - 1 \mid n((p^f-1)/\ell)(p^{\beta k} - 1)$, hence if and only if $\ell \mid n(p^{\beta k} - 1)$, which is equivalent to $\ell/(\ell, n) \mid (p^{\beta k} - 1)$. Therefore, if k is the order of p^β modulo $\ell/(\ell, n)$, then by definition $\varphi^{\beta k}$ stabilizes $(\chi_1^{(n)})_{H_0}$, but no smaller positive power of φ^β does, and so the stabilizer of $(\chi_1^{(n)})_{H_0}$ in $\langle \varphi^\beta \rangle$ is $\langle \varphi^{\beta k} \rangle$. □

Theorem 7.3. *Let $q = p^f$ be odd and set $H = (\mathrm{SL}_2(q) \rtimes \langle \delta^\alpha \rangle) \rtimes \langle \varphi^\beta \rangle$ and $H_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^\alpha \rangle$, where $\alpha \mid q-1$ and $\beta \mid f$. Denote $d = f/\beta = |H : H_0|$ and $\ell = (q-1)/\alpha = |H_0 : \mathrm{SL}_2(q)|$.*

The set of degrees of irreducible characters of H lying over the principal character of $\mathrm{SL}_2(q)$ is

$$\{k : k \mid d \text{ and } k = O_v(p^\beta) \text{ for some } v \mid \ell\}$$

and the set of degrees of irreducible characters of H lying over the Steinberg character of $\mathrm{SL}_2(q)$ of degree q is

$$\{qk : k \mid d \text{ and } k = O_v(p^\beta) \text{ for some } v \mid \ell\}.$$

Proof. By the remarks above, the irreducible characters of H lying over 1_{SL} and ψ are precisely those lying over the restrictions of $\chi_1^{(n)}$ and $\chi_q^{(n)}$ to H_0. As $H/H_0 \cong \langle \varphi^\beta \rangle$ is cyclic, each irreducible character χ of H_0 will extend to its stabilizer I in H and then induce to H. Hence the degree of each character of H lying over χ is $|H : I|\chi(1)$. Therefore, k is the degree of an irreducible character of H lying over the principal character of $\mathrm{SL}_2(q)$ if and only if $I_k = H_0 \rtimes \langle \varphi^{\beta k} \rangle$ is the stabilizer in H of $(\chi_1^{(n)})_{H_0}$ for some $1 \leqslant n \leqslant q-1$.

By Lemma 7.2, if I_k is the stabilizer in H of $(\chi_1^{(n)})_{H_0}$, then k is the order of p^β modulo the divisor $v = \ell/(\ell, n)$ of ℓ. Conversely, suppose $v \mid \ell$ and k is the order of p^β modulo v. Setting $n = \ell/v$, we have $1 \leqslant n \leqslant \ell \leqslant q-1$ and $(\ell, n) = n = \ell/v$. Hence k is the order of p^β modulo $v = \ell/(\ell, n)$, and I_k is the stabilizer in H of $(\chi_1^{(n)})_{H_0}$.

Therefore, the degrees of characters of H lying over the principal character of $\mathrm{SL}_2(q)$ are as claimed. As the stabilizers of $(\chi_1^{(n)})_{H_0}$ and $(\chi_q^{(n)})_{H_0}$ in H are the same, the second conclusion in the lemma follows from the first. □

7.2 Characters of H Lying Over ξ_1, ξ_2, η_1, and η_2

We next consider characters of H lying over the irreducible characters ξ_1, ξ_2, η_1, and η_2 of $\mathrm{SL}_2(q)$. These characters are invariant under φ and under δ^2, while $\xi_i^\delta = \xi_j$ and $\eta_i^\delta = \eta_j$ for $i \neq j$. Each character ξ_i, η_i extends to $(q-1)/2$ irreducible characters of its stabilizer $G_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^2 \rangle$ in $\mathrm{GL}_2(q)$ and then induces irreducibly to $\mathrm{GL}_2(q)$. It is not difficult to determine from the character tables of $\mathrm{GL}_2(q)$ and $\mathrm{SL}_2(q)$ that the characters of $\mathrm{GL}_2(q)$ lying over ξ_1 and ξ_2 are $\chi_{q+1}^{(n,n+\frac{q-1}{2})}$ for $1 \leqslant n \leqslant (q-1)/2$, and the characters of $\mathrm{GL}_2(q)$ lying over η_1 and η_2 are $\chi_{q-1}^{(n\cdot\frac{q+1}{2})}$ for $1 \leqslant n \leqslant q-1$ and n odd. For $1 \leqslant n \leqslant (q-1)/2$, denote by $\xi_i^{(n)}$ the extension of ξ_i to G_0 such that $\xi_1^{(n)}$ and $\xi_2^{(n)}$ induce to $\chi_{q+1}^{(n,n+\frac{q-1}{2})}$, and for $1 \leqslant n \leqslant q-1$ and n odd, denote by $\eta_i^{(n)}$ the extension of η_i to G_0 such that $\eta_1^{(n)}$ and $\eta_2^{(n)}$ induce to $\chi_{q-1}^{(n\cdot\frac{q+1}{2})}$. As before, set $H = (\mathrm{SL}_2(q) \rtimes \langle \delta^\alpha \rangle) \rtimes \langle \varphi^\beta \rangle$ and $H_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^\alpha \rangle$, where $\alpha \mid q-1$ and $\beta \mid f$.

Let $\alpha' = \alpha/(2,\alpha)$, so that $H_0 \cap G_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^{2\alpha'} \rangle$. The characters of $H_0 \cap G_0$ lying over ξ_1, ξ_2, η_1, and η_2 are just the restrictions of $\xi_i^{(n)}$ and $\eta_i^{(n)}$ to $H_0 \cap G_0$. Setting $Z = Z(\mathrm{GL}_2(q))$, we have that G_0 is a central product of $\mathrm{SL}_2(q)$ and Z, with $\mathrm{SL}_2(q) \cap Z = \langle z \rangle$ of order 2. Thus $G_0 \cong (\mathrm{SL}_2(q) \times Z)/\langle (z,z) \rangle$. Viewing Z as the cyclic group of order $q-1$, it is straightforward to calculate the values of the characters of G_0 lying over ξ_1, ξ_2, η_1, and η_2. Their restrictions to H_0 are given in the following lemma.

Lemma 7.4. *Let $q = p^f$ be odd, $G_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^2 \rangle$, and $H_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^\alpha \rangle$, where $\alpha \mid q-1$. Let $\alpha' = \alpha/(2,\alpha)$, so that $H_0 \cap G_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^{2\alpha'} \rangle$. The extensions of ξ_i and η_i to $H_0 \cap G_0$ are $\hat{\xi}_i^{(n)} = (\xi_i^{(n)})_{H_0 \cap G_0}$ and $\hat{\eta}_i^{(n)} = (\eta_i^{(n)})_{H_0 \cap G_0}$, and the values of these characters are as follows:*

	$\hat{\xi}_1^{(n)}$	$\hat{\xi}_2^{(n)}$	$\hat{\eta}_1^{(n)}$	$\hat{\eta}_2^{(n)}$
$A_1(j\alpha')$	$\frac{1}{2}(q+1)(-1)^{j\alpha'}\rho^{2nj\alpha'}$	$\frac{1}{2}(q+1)(-1)^{j\alpha'}\rho^{2nj\alpha'}$	$\frac{1}{2}(q-1)(-1)^{j\alpha'}\rho^{nj\alpha'}$	$\frac{1}{2}(q-1)(-1)^{j\alpha'}\rho^{nj\alpha'}$
$A_2'(j\alpha')$	$\frac{1+\sqrt{\varepsilon q}}{2}(-1)^{j\alpha'}\rho^{2nj\alpha'}$	$\frac{1-\sqrt{\varepsilon q}}{2}(-1)^{j\alpha'}\rho^{2nj\alpha'}$	$\frac{-1+\sqrt{\varepsilon q}}{2}(-1)^{j\alpha'}\rho^{nj\alpha'}$	$\frac{-1-\sqrt{\varepsilon q}}{2}(-1)^{j\alpha'}\rho^{nj\alpha'}$
$A_2''(j\alpha')$	$\frac{1-\sqrt{\varepsilon q}}{2}(-1)^{j\alpha'}\rho^{2nj\alpha'}$	$\frac{1+\sqrt{\varepsilon q}}{2}(-1)^{j\alpha'}\rho^{2nj\alpha'}$	$\frac{-1-\sqrt{\varepsilon q}}{2}(-1)^{j\alpha'}\rho^{nj\alpha'}$	$\frac{-1+\sqrt{\varepsilon q}}{2}(-1)^{j\alpha'}\rho^{nj\alpha'}$
$A_3(l_1,l_2)$	$(-1)^{l_1}\rho^{2nj\alpha'}$	$(-1)^{l_1}\rho^{2nj\alpha'}$	0	0
$B_1(2j\alpha')$	0	0	$-\rho^{nj\alpha'}$	$-\rho^{nj\alpha'}$

*where ρ is a complex primitive $(q-1)$*th *root of unity, $\varepsilon = (-1)^{(q-1)/2}$, and $l_1 + l_2 = 2j\alpha'$.*

If α is even, then $H_0 = H_0 \cap G_0$.

If α is odd, then for each $1 \leqslant n \leqslant (q-1)/2$, $\hat{\xi}_1^{(n)}$ and $\hat{\xi}_2^{(n)}$ induce irreducibly to a character of H_0 of degree $q+1$ and for each odd n, $1 \leqslant n \leqslant q-1$, $\hat{\eta}_1^{(n)}$ and $\hat{\eta}_2^{(n)}$ induce irreducibly to a character of H_0 of degree $q-1$.

We next determine the stabilizer in $\langle \varphi^\beta \rangle$ of the irreducible characters of H_0 lying over ξ_1, ξ_2, η_1, and η_2.

Lemma 7.5. *Let $q = p^f$ be odd and set $G_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^2 \rangle$ and $H_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^\alpha \rangle$ for $\alpha \mid q-1$. Set $\alpha' = \alpha/(2,\alpha)$ and $\ell = (q-1)/(2\alpha') = |H_0 \cap G_0 : \mathrm{SL}_2(q)|$. Let $\beta \mid f$.*

i. The stabilizer in $\langle \varphi^\beta \rangle$ of the characters $\hat{\xi}_1^{(n)}$ and $\hat{\xi}_2^{(n)}$ of $H_0 \cap G_0$, for $1 \leqslant n \leqslant (q-1)/2$, is $\langle \varphi^{\beta k} \rangle$, where k is the order of p^β modulo $\ell/(\ell,n)$.

ii. The stabilizer in $\langle \varphi^\beta \rangle$ of the characters $\hat{\eta}_1^{(n)}$ and $\hat{\eta}_2^{(n)}$ of $H_0 \cap G_0$, for $1 \leqslant n \leqslant q-1$ and n odd, is $\langle \varphi^{\beta k} \rangle$, where k is the order of p^β modulo $2\ell/(\ell,n)$.

iii. Moreover, if α is odd, the stabilizer of the induced character $(\hat{\xi}_i^{(n)})^{H_0}$ or $(\hat{\eta}_i^{(n)})^{H_0}$ is equal to the stabilizer of $\hat{\xi}_i^{(n)}$ or $\hat{\eta}_i^{(n)}$ respectively.

Proof. For $r \mid f$, the effect of φ^{-r} acting on any of $\hat{\xi}_i^{(n)}$ or $\hat{\eta}_i^{(n)}$ is to replace $j\alpha'$ in each character value with $j\alpha' p^r$ and l_1 with $l_1 p^r$. As p^r is odd, this has no effect on the value of either $(-1)^{j\alpha'}$ or $(-1)^{l_1}$. Also, if α is odd, $(\hat{\xi}_i^{(n)})^{H_0} = \hat{\xi}_1^{(n)} + \hat{\xi}_2^{(n)}$ and $(\hat{\eta}_i^{(n)})^{H_0} = \hat{\eta}_1^{(n)} + \hat{\eta}_2^{(n)}$ on $H_0 \cap G_0$ and both vanish on $H_0 - (H_0 \cap G_0)$, and so *(iii)* holds.

If φ^r (and hence φ^{-r}) stabilizes $\hat{\xi}_i^{(n)}$, then because $A_1(\alpha')$ is a conjugacy class of $H_0 \cap G_0$, we must have $2n\alpha' \equiv 2n\alpha' p^r (\mathrm{mod}\ q-1)$. Conversely, if this congruence holds, then $2nj\alpha' \equiv 2nj\alpha' p^r (\mathrm{mod}\ q-1)$ for every integer j, and so φ^r stabilizes $\hat{\xi}_i^{(n)}$. Therefore, φ^r stabilizes $\hat{\xi}_i^{(n)}$ if and only if $p^f - 1 \mid 2n\alpha'(p^r - 1)$.

Suppose now that φ^r is in $\langle \varphi^\beta \rangle$, and let $r = \beta k$ for a positive integer k. Replacing $2\alpha'$ with $(q-1)/\ell$, we have that $\hat{\xi}_i^{(n)}$ is invariant under $\varphi^{\beta k}$ if and only if $p^f - 1 \mid n((p^f-1)/\ell)(p^{\beta k} - 1)$, hence if and only if $\ell \mid n(p^{\beta k} - 1)$, which is equivalent to $\ell/(\ell, n) \mid (p^{\beta k} - 1)$. Therefore, if k is the order of p^β modulo $\ell/(\ell, n)$, then by definition $\varphi^{\beta k}$ stabilizes $\hat{\xi}_i^{(n)}$, but no smaller positive power of φ^β does, and so the stabilizer of $\hat{\xi}_i^{(n)}$ in $\langle \varphi^\beta \rangle$ is $\langle \varphi^{\beta k} \rangle$.

Similarly, φ^r stabilizes $\hat{\eta}_i^{(n)}$ if and only if $p^f - 1 \mid n\alpha'(p^r - 1)$. If $r = \beta k$ for a positive integer k, replacing α' with $(q-1)/(2\ell)$ yields $\hat{\eta}_i^{(n)}$ is invariant under $\varphi^{\beta k}$ if and only if $2\ell \mid n(p^{\beta k} - 1)$, which is equivalent to $2\ell/(\ell, n) \mid (p^{\beta k} - 1)$, since n is odd. The conclusion *(ii)* now follows as before. □

Theorem 7.6. *Let $q = p^f$ be odd and set $G_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^2 \rangle$, $H = (\mathrm{SL}_2(q) \rtimes \langle \delta^\alpha \rangle) \rtimes \langle \varphi^\beta \rangle$, and $H_0 = \mathrm{SL}_2(q) \rtimes \langle \delta^\alpha \rangle$, where $\alpha \mid q-1$ and $\beta \mid f$. Denote $d = f/\beta = |H : H_0|$, $\alpha' = \alpha/(2, \alpha)$, and $\ell = (q-1)/(2\alpha') = |H_0 \cap G_0 : \mathrm{SL}_2(q)|$.*

i. *The set of degrees of irreducible characters of H lying over ξ_1 and ξ_2 is*

$$\left\{ \frac{1}{(\alpha, 2)}(q+1)k : k \mid d \text{ and } k = O_v(p^\beta) \text{ for some } v \mid \ell \right\}.$$

ii. *The set of degrees of irreducible characters of H lying over η_1 and η_2 is*

$$\left\{ \frac{1}{(\alpha, 2)}(q-1)k : k \mid d \text{ and } k = O_{2v}(p^\beta) \text{ for some } v \mid \ell \text{ with } \ell/v \text{ odd} \right\}.$$

Proof. The irreducible characters of H lying over ξ_i and η_i are precisely those lying over the extensions $\hat{\xi}_i^{(n)}$ and $\hat{\eta}_i^{(n)}$ to $H_0 \cap G_0$. Let $\chi = \xi_i$ or $\chi = \eta_i$ and let $\hat{\chi}$ be an extension of χ to $H_0 \cap G_0$. Denote by $I = I_{\langle \varphi^\beta \rangle}(\hat{\chi})$ the stabilizer of $\hat{\chi}$ in $\langle \varphi^\beta \rangle$.

If α is even, then $H_0 \cap G_0 = H_0$ and $H/H_0 \cong \langle \varphi^\beta \rangle$ is cyclic, thus $\hat{\chi}$ extends to its stabilizer $H_0 \rtimes I$ in H and then induces irreducibly to H. Hence the degree of each character of H lying over χ is $|H : H_0 \rtimes I|\chi(1) = |\langle \varphi^\beta \rangle : I|\chi(1)$.

If α is odd, then $\hat{\chi}$ induces irreducibly to $(\hat{\chi})^{H_0} \in \mathrm{Irr}(H_0)$. By Lemma 7.5, the stabilizer of $(\hat{\chi})^{H_0}$ in $\langle \varphi^\beta \rangle$ is the same as that of $\hat{\chi}$. Hence $(\hat{\chi})^{H_0}$ extends to its stabilizer $H_0 \rtimes I$ in H and then induces irreducibly to H as before. We have $|H_0 : H_0 \cap G_0| = 2$, and so $\hat{\chi}^{H_0}(1) = 2\chi(1)$. Thus if α is odd, the degree of each character of H lying over χ is $2|H : H_0 \rtimes I|\chi(1) = 2|\langle \varphi^\beta \rangle : I|\chi(1)$. Hence, for any α, the degree is $\frac{2}{(2,\alpha)}|\langle \varphi^\beta \rangle : I|\chi(1)$.

Therefore, $\frac{1}{(\alpha,2)}(q+1)k$ is the degree of an irreducible character of H lying over ξ_1 or ξ_2 if and only if $I_k = \langle \varphi^{\beta k} \rangle$ is the stabilizer in $\langle \varphi^\beta \rangle$ of $\hat{\xi}_i^{(n)}$ for some $1 \leqslant n \leqslant (q-1)/2$. By Lemma 7.5, if I_k is the stabilizer in $\langle \varphi^\beta \rangle$ of $\hat{\xi}_i^{(n)}$, then k is the order of p^β modulo the divisor $v = \ell/(\ell, n)$ of ℓ. Conversely, suppose $v \mid \ell$ and k is the order of p^β modulo v. Setting $n = \ell/v$, we have $1 \leqslant n \leqslant \ell \leqslant (q-1)/2$ and $(\ell, n) = n = \ell/v$. Hence k is the order of p^β modulo $v = \ell/(\ell, n)$ and I_k is the stabilizer in $\langle \varphi^\beta \rangle$ of $\hat{\xi}_i^{(n)}$.

Similarly, $\frac{1}{(\alpha,2)}(q-1)k$ is the degree of an irreducible character of H lying over η_1 or η_2 if and only if $I_k = \langle \varphi^{\beta k} \rangle$ is the stabilizer in $\langle \varphi^\beta \rangle$ of $\hat{\eta}_i^{(n)}$ for some $1 \leqslant n \leqslant q-1$ with n odd. By Lemma 7.5, if I_k is the stabilizer in $\langle\varphi^\beta\rangle$ of $\hat{\eta}_i^{(n)}$, then k is the order of p^β modulo $2v$ for the divisor $v = \ell/(\ell, n)$ of ℓ, and $\ell/v = n$ is odd. Conversely, suppose $v \mid \ell$, with ℓ/v odd, and k is the order of p^β modulo $2v$. Setting $n = \ell/v$, we have $1 \leqslant n \leqslant \ell \leqslant q-1$ and $(\ell, n) = n = \ell/v$ is odd. Hence k is the order of p^β modulo $2v = 2\ell/(\ell, n)$ and I_k is the stabilizer in $\langle\varphi^\beta\rangle$ of $\hat{\eta}_i^{(n)}$.

The degrees of characters of H lying over ξ_1, ξ_2, η_1, and η_2 are therefore as claimed. $\square$

7.3 Characters of H Lying Over θ_j and χ_i

We next consider characters of H lying over the irreducible characters θ_j and χ_i of $\mathrm{SL}_2(q)$. These characters are invariant under δ, and in Lemmas 5.1 and 5.3 we determined the subgroups $\langle\varphi^k\rangle$ of $\langle\varphi\rangle$ that occur as stabilizers of some θ_j or χ_i. Moreover, we showed that for each of these subgroups, there is an character of $\mathrm{SL}_2(q)$ and an extension of that character to $\mathrm{GL}_2(q)$ with stabilizer $\langle\varphi^k\rangle$ in $\langle\varphi\rangle$.

Theorem 7.7. *Let $q = p^f$ be odd and set $H = (\mathrm{SL}_2(q) \rtimes \langle\delta^\alpha\rangle) \rtimes \langle\varphi^\beta\rangle$ and $H_0 = \mathrm{SL}_2(q) \rtimes \langle\delta^\alpha\rangle$, where $\alpha \mid q-1$ and $\beta \mid f$. Denote $d = 2^a m = f/\beta = |H : H_0|$, where m is odd.*

i. The set of degrees of irreducible characters of H lying over θ_n for some n is $\{(q-1)2^a i : i \mid m\}$.

ii. The set of degrees of irreducible characters of H lying over χ_n for some n is $\{(q+1)j : j \mid d\}$, with the exception that if $p = 3$, $\beta = 1$, and f is odd, then $j \neq 1$.

Proof. If $\chi = \theta_n$ or $\chi = \chi_n$ is a character of $\mathrm{SL}_2(q)$ of degree $q-1$ or $q+1$, then χ is invariant under δ and so extends to a character $\hat{\chi}$ of H_0. As $H/H_0 \cong \langle\varphi^\beta\rangle$ is cyclic, $\hat{\chi}$ extends to its stabilizer $I_H(\hat{\chi})$ in H and then induces irreducibly to a character of H of degree $|H : I_H(\hat{\chi})|\chi(1)$. Since χ is invariant under δ, $I_H(\hat{\chi}) = H_0 \rtimes \langle\varphi^{\beta j}\rangle$ for some $j \mid d$, and $|H : I_H(\hat{\chi})| = j$. Hence, for $j \mid d$, there is an irreducible character of H of degree $\chi(1)j$ lying over χ if and only if there is an extension $\hat{\chi}$ of χ to H_0 such that $I_{\langle\varphi^\beta\rangle}(\hat{\chi}) = \langle\varphi^{\beta j}\rangle$. Observe also that $I_{\langle\varphi^\beta\rangle}(\hat{\chi}) \leqslant I_{\langle\varphi^\beta\rangle}(\chi) \leqslant I_{\langle\varphi\rangle}(\chi)$.

We first consider degrees of characters of H lying over some character of $\mathrm{SL}_2(q)$ of degree $q-1$. Let $j \mid d$. If $d/j = f/(\beta j)$ is even, then by Lemma 5.1, no irreducible character of $\mathrm{SL}_2(q)$ is invariant under $\varphi^{\beta j}$. Hence there is no character of H of degree $(q-1)j$ lying over a character of $\mathrm{SL}_2(q)$ of degree $q-1$. If d/j is odd, that is, $j = 2^a i$ for some $i \mid m$, then by Lemma 5.1 there is $\theta_n \in \mathrm{Irr}(\mathrm{SL}_2(q))$ and an extension of θ_n to $\mathrm{GL}_2(q)$ that both have stabilizer $\langle\varphi^{\beta j}\rangle$ in $\langle\varphi\rangle$. Hence there is an extension of θ_n to H_0 with this stabilizer as well, and so there is a character of H of degree $(q-1)j = (q-1)2^a i$ lying over θ_n.

Finally, we consider degrees of characters of H lying over some character of $\mathrm{SL}_2(q)$ of degree $q+1$. Let $j \mid d$. By Lemma 5.3, unless $p = 3$, f is odd, and $\beta = j = 1$, there is $\chi_n \in \mathrm{Irr}(\mathrm{SL}_2(q))$ and an extension of χ_n to $\mathrm{GL}_2(q)$ that both have stabilizer $\langle\varphi^{\beta j}\rangle$ in $\langle\varphi\rangle$. Hence there is also an extension of χ_n to H_0 with this stabilizer, and so there is a character of H of degree $(q+1)j$ lying over χ_n. If $p = 3$, f is odd, and $\beta = j = 1$ (so that $\varphi^{\beta j} = \varphi$), then there is no character of $\mathrm{SL}_2(q)$ of degree $q+1$ that is invariant under φ, and hence no character of H_0 lying over any $\chi_n \in \mathrm{Irr}(\mathrm{SL}_2(q))$ is invariant under φ. Thus there is no character of H of degree $q+1$ lying over any $\chi_n \in \mathrm{Irr}(\mathrm{SL}_2(q))$. $\square$

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