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Character Formula of $C<1$ Unitary Representation of $N=2$ Superconformal Algebra

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The character formula of $c<1$ unitary representation of $N=2$ superconformal algebra is obtained. As a corollary of our formula, we can easily express Waterson's bosonic construction of the algebra in character form.

In the study of two dimensional conformal field theory, the importance of the character of the representation of the Virasoro algebra and current algebra has recently been recognized. In particular, the recent breakthrough in the determination of field content of the conformal field theory was made possible via the character formula of the Virasoro algebra.[1] (Cardy[2]) Prior to this development, it was not possible to determine which combination of primary fields appear in concrete models, for example, the critical 3-state Potts model, although it was known[3] that the minimal degenerate conformal field theories consist of only a finite number of primary fields. The critical observation of Cardy is that if we consider the theory on a torus, instead of complex plane, we may introduce a new guiding principle, i.e., "modular invariance of the partition function" which is familiar in closed string theory,[4] and we can determine the field content of the theory. What is fortunate is that the character of the Virasoro algebra transforms in a well-defined manner under modular transformation.

It is natural to extend this program to other infinite dimensional Lie algebra. Extension to the system with the Kac-Moody symmetry,[5] $N=1$ superconformal algebra[6),7)] was made.

In this paper we calculate the character formula of $N=2$ superconformal algebra. In the representation theory of $N=2$ superconformal algebra, it is known[8] that besides the case $\tilde{c}<1$ where $\tilde{c}$ is central charge of the $N=2$ superconformal algebra, there are also degenerate representations for $\tilde{c}>1$, and it might be related to the compactification of heterotic string theory. From the viewpoint of modular properties, however, we may have to combine infinite primary fields to obtain modular invariant partition function[2] when $\tilde{c}>1$ because we have infinitely many primary fields. So in this paper, we limit our consideration to the $\tilde{c}<1$ minimal unitary theory.[8),9)]

$N=2$ superconformal algebra is defined by (anti-) commutation relations of four generators,[10] i.e., the stress-energy tensor $T(z)=\sum_n L_n/z^{n+2}$, two supersymmetry generators $G^i(z)=\sum_n G_n{}^i/z^{n+3/2}$ and $U(1)$ Kac-Moody generator $J(z)=\sum_n J_n/z^{n+1}$. Their operator production expansions are

$$T(z)T(w)\sim 3\tilde{c}/2(z-w)^4+2T(w)/(z-w)^2+T'(w)/(z-w)\,,$$

$$T(z)G^i(w)\sim 3G^i/2(z-w)^2+G^{i\prime}(w)/(z-w)\,,$$
$$T(z)J(w)\sim J(w)/(z-w)^2+J'(w)/(z-w)\,,$$
$$G^i(z)G^j(w)\sim \tilde{c}\delta^{ij}/(z-w)^3+\delta^{ij}T(w)/(z-w)$$
$$\qquad -i\epsilon^{ij}[J(w)/(z-w)^2+J'(w)/(z-w)]\,,$$
$$J(z)G^i(w)\sim -i\epsilon^{ij}G^j(w)/(z-w)\,,$$
$$J(z)J(w)\sim \tilde{c}/(z-w)^2\,. \tag{1}$$

Representation space is split into three sectors according to the boundary condition of supersymmetry generators,[8)]

$$\begin{aligned} &A \text{ sector}: && G^i(ze^{2\pi i})=-G^i(z)\,, && (i=1,2)\\ &P \text{ sector}: && G^i(ze^{2\pi i})=G^i(z)\,, && (i=1,2)\\ &T \text{ sector}: && G^1(ze^{2\pi i})=-G^1(z)\,, &&\\ & && G^2(ze^{2\pi i})=G^2(z)\,. && \end{aligned}$$

Furthermore P sector splits into two subsectors, $P^\pm$ sectors[8)] by the specification of the vacuum. For A sector and $P^\pm$ sector we have two bosonic global charges, i.e., conformal dimension L_0 and global $U(1)$ charge J_0. Using these charges, we can define the character $\chi^{A,P\pm}(x,y)$ as follows:

$$\chi^{A,P\pm}(x,y)=\mathrm{Tr}(x^{L_0}y^{J_0})\,, \tag{2}$$

where trace is taken over all states of an irreducible representation. For the T sector we have only one glocal charge L_0. So the character is defined as follows:

$$\chi^T(x)=\mathrm{Tr}(x^{L_0})\,. \tag{3}$$

When the central charge ($\tilde{c}$) is smaller than unity, $\tilde{c}$ must take the following discrete values in order to define a unitary theory:[8)]

$$\tilde{c}=1-2/m\,. \qquad (m=2,3,4,\cdots) \tag{4}$$

For each integer m, the conformal dimension h of the highest weight vectors and their global $U(1)$ charge q must take the following values:[8)]

$$\begin{aligned} &A \text{ sector}: && h^A_{jk}=(jk-1/4)/m\,, \qquad q^A_{jk}=(j-k)/m\,,\\ & && j,k\in Z+1/2\,, \qquad 0<j,k,j+k\leq m-1\,, \end{aligned} \tag{5a}$$

$$\begin{aligned} &P^\pm \text{ sector}: && h^{P\pm}_{jk}=jk/m+\tilde{c}/8\,, \qquad q^{P\pm}_{jk}=\pm(j-k)/m\,,\\ & && j,k\in Z\,, \qquad 0\leq j-1,k,j+k\leq m-1\,, \end{aligned} \tag{5b}$$

$$\begin{aligned} &T \text{ sector}: && h_r{}^T=(m-2r)^2/16m+\tilde{c}/8\,,\\ & && r\in Z\,, \qquad 1\leq r\leq m/2\,. \end{aligned} \tag{5c}$$

When we calculate the characters, we should carefully treat the null states of the

Verma module. In order to consistently subtract their contributions it is convenient to write down the embedding relations of Verma modules of singular vectors.[1] This embedding relations are easily calculated using the Kac formula. For detail see Ref. 1). For the $N=2$ superconformal algebra, the Kac formula was obtained in Ref. 8) and proved in Ref. 9). Using this we can obtain the character formula of $N=2$ superconformal algebra. Here we only describe the result.

For the A and $P^{\pm}$ sectors, the embedding pattern is more complicated than that of conformal algebra.[1] The effective part which is relevant to the character formula is shown in Fig. 1. We can obtain the character formula easily from this diagram. The result is

$$A \text{ sector}; \quad \chi_{jk}^{(m)A}(x,y)=\varphi_A(x,y)x^{h_{jk}{}^A}y^{q_{jk}{}^A}\Gamma_{jk}^{(m)}(x,y)\,, \tag{6a}$$

$$P^{\pm} \text{ sector}; \quad \chi_{jk}^{(m)P^{\pm}}(x,y)=\varphi_P(x,y)x^{h_{jk}{}^{P^{\pm}}}y^{q_{jk}{}^{P^{\pm}}}\Gamma_{jk}^{(m)}(x,y^{\pm1})\,, \tag{6b}$$

where

$$\varphi_A(x,y)=\prod_{n=1}^{\infty}\frac{(1+x^{n-1/2}y)(1+x^{n-1/2}y^{-1})}{(1-x^n)^2}\,, \tag{7a}$$

$$\varphi_P(x,y)=(y^{1/2}+y^{-1/2})\prod_{n=1}^{\infty}\frac{(1+x^ny)(1+x^ny^{-1})}{(1-x^n)^2}\,, \tag{7b}$$

$$\begin{aligned}\Gamma_{jk}^{(m)}(x,y)=&\sum_{n=0}^{\infty}x^{mn^2+(j+k)n}\left(1-\frac{x^{nm+j}y^{-1}}{1+x^{nm+j}y^{-1}}-\frac{x^{nm+k}y}{1+x^{nm+k}y}\right)\\&-\sum_{n=1}^{\infty}x^{mn^2-(j+k)n}\left(1-\frac{x^{nm-j}y}{1+x^{nm-j}y}-\frac{x^{nm-k}y^{-1}}{1+x^{nm-k}y^{-1}}\right),\end{aligned} \tag{8}$$

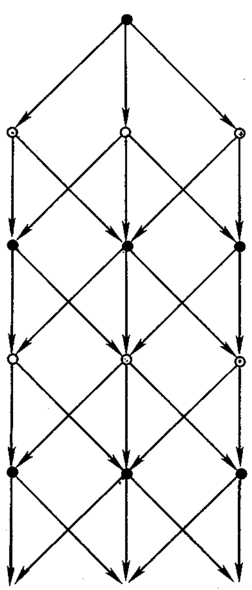

Fig. 1. Embedding diagram of Verma modules: A and $P^{\pm}$ sectors.

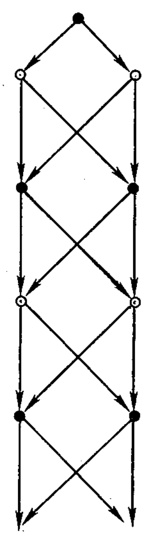

Fig. 2. Embedding diagram of Verma modules: T sector.

$$=\prod_{n=1}^{\infty}\frac{(1-x^{mn+j+k-m})(1-x^{nm-j-k})(1-x^{mn})^2}{(1+x^{nm-j}y)(1+x^{nm-m+j}y^{-1})(1+x^{nm-k}y^{-1})(1+x^{nm+k-m}y)}\,. \tag{9}$$

The denominator factors in (8) come from fermionic nature of singular vectors. The transformation from infinite sum form (8) to infinite product form (9) can be proved by comparing their singularities and their residues.

For the T sector, the embedding pattern is the same as that of conformal algebra.[1] (Fig. 2) From this we can get the following character formula:

$$\chi_r{}^{(m)T}(x)=\varphi_T(x)x^{h_r{}^T}\tilde{\Gamma}_r{}^{(m)}(x)\,, \tag{10}$$

where

$$\varphi_T(x)=\prod_{n=1}^{\infty}\frac{1+x^{n/2}}{1-x^{n/2}}\,, \tag{11}$$

$$\tilde{\Gamma}_r{}^{(m)}(x)=\sum_{n=-\infty}^{\infty}(-1)^n x^{(mn^2+n(m-2r))/4}\,, \tag{12}$$

$$=\prod_{n=1}^{\infty}(1-x^{mn/2})(1-x^{(mn-r)/2})(1-x^{(mn-m+r)/2})\,. \tag{13}$$

The main feature of the form of $N=2$ superconformal algebra is that they can be written in one infinite product forms (9) and (13) independent of m which is not the case for conformal or $N=1$ superconformal algebra. This fact should be related to the fact that the unitary representation of $N=2$ superconformal algebra can be written by (nonlocal) parafermion current[10] and one free boson.[11]

As the consistency check of the character formula, it will be interesting to calculate the characters explicitly for small m.

Let us begin the calculation for $m=2$. In this case the central charge $\tilde{c}$ is zero. Accordingly there are no physical states other than vacuum. In fact we can show easily the following identity using the product forms (9) and (13) which meets our expectations:

$$\chi^{(2)A}_{1/2\ 1/2}(x,y)=\chi^{(2)P\pm}_{10}(x,y)=\chi_1{}^{(2)T}(x)=1\,. \tag{14}$$

Next let us consider the case $m=3$. In this case the central charge is $\tilde{c}=1/3$ ($c=1$). We can show the following desirable results:

$$\chi^{(3)A,P\pm}_{jk}(x)=\frac{\sum_{n=-\infty}^{\infty}x^{3/2(n+a_{jk}{}^{A,P\pm})^2}y^{(n+a_{jk}{}^{A,P\pm})}}{\prod_{n=1}^{\infty}(1-x^n)}\,, \tag{15a}$$

$$\left\{\begin{array}{ll} a^{A}_{1/2\ 1/2}=0\,, & a^{A}_{3/2\ 1/2}=-a^{A}_{1/2\ 3/2}=\dfrac{1}{3}\,,\\ a^{P\pm}_{10}=-a^{P\pm}_{20}=\mp\dfrac{1}{6}\,, & a_{11}{}^{P\pm}=\dfrac{1}{2}\,, \end{array}\right. \tag{16}$$

$$\chi_1{}^{(3)T}(x)=x^{1/16}\prod_{n=1}^{\infty}(1-x^{n-1/2})^{-1}=x^{1/16}\prod_{n=1}^{\infty}(1+x^{n-1/2})(1+x^n)\,. \tag{15b}$$

The partition function (15a) has precisely the expected form, because as Waterson has shown[12] the $\tilde{c}=1/3$ $N=2$ superconformal algebra can be written by only one boson compactified on a circle such that its momentums are multiples of $\sqrt{3}$ with appropriate momentum shift. For the twist sector, the character is the partition function of antiperiodic boson[13),14] or one Ramond and one Neveu-Schwarz Majorana fermions.

We have obtained the character formula of $N=2$ superconformal algebra using the technique of Rocha-Caridi.[1] It correctly reproduces the expected form for $m=2$, 3 cases. The natural next step is to calculate the modular transformation of characters and find the invariant partition functions.

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Note added: After we finished this work, we received the corrected version of Ref. 15) where he also obtained the infinite sum form of character formulas (8) and (12).