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CHARACTER SHEAVES ON DISCONNECTED GROUPS, IX

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ABSTRACT. We associate a two-sided cell to any (parabolic) character sheaf. We study the interaction between the duality operator for character sheaves and the operation of "twisted induction".

INTRODUCTION

Throughout this paper, G denotes a fixed, not necessarily connected, reductive algebraic group over an algebraically closed field $\mathbf{k}$. This paper is a part of a series [L9] which attempts to develop a theory of character sheaves on G.

One of the main constructions in [L3] (going back to [L14]) was a procedure which to any character sheaf on G^0 associates a certain two-sided cell in an (extended) Coxeter group. A variant of this construction (restricted to "unipotent" character sheaves) was later given by Grojnowski [Gr]. Here we give a construction which generalizes that in [L3] (and takes into account the approach in [Gr]) which to any (parabolic) character sheaf on $Z_{J,D}$ associates a certain type of two-sided cell.

The paper is organized as follows. In Section 40 we study certain equivariant sheaves on $G^0/U^* \times G^0/U^*$ (where U^* is the unipotent radical of a Borel in G^0) under the convolution operation. Some results in this section are implicit in [L14, Ch.1]. In Section 41 we study the character sheaves on $Z_{\emptyset,D}$ (where D is a connected component of G) by connecting them with sheaves on $G^0/U^* \times G^0/U^*$. We use this study to attach a two-sided cell to any character sheaf on $Z_{J,D}$. (See 41.4.) In Section 42 we study the interaction between the duality operation $\mathbf{d}$ (see 38.10, 38.11) and the functor $\mathfrak{f}_{\emptyset,\mathbf{I}}$ (see 36.4). The main result in this section is Proposition 42.9 which contains [L3, III, Cor. 15.8(b)] as a special case (with $G = G^0, v = 1$).

Notation. We fix a 1-dimensional $\bar{\mathbf{Q}}_l$-vector space V with a given isomorphism $V^{\otimes 2} \xrightarrow{\sim} \bar{\mathbf{Q}}_l(1)$ (Tate twist of $\bar{\mathbf{Q}}_l$). For $n \in \mathbf{N}$ we set $\bar{\mathbf{Q}}_l(n/2) = V^{\otimes n}$. For $n \in \mathbf{Z}, n < 0$ let $\bar{\mathbf{Q}}_l(n/2)$ be the dual space of $\bar{\mathbf{Q}}_l(-n/2)$. If X is an algebraic variety and $A \in \mathcal{D}(X), n \in \mathbf{Z}$, we write $A(n/2)$ instead of $A \otimes \bar{\mathbf{Q}}_l(n/2)$ and $A[[n/2]]$ instead of $A[n](n/2)$. (When n is even, this agrees with the notation in [L9, II, p. 73].)

CONTENTS

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40. Sheaves on $G^0/U^* \times G^0/U^*$

40.1. Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$. Let $\hat{H}$ (resp. H) be the $\mathcal{A}$-module consisting of all formal (resp. finite) linear combinations $\sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}} a_{w,\lambda}\tilde{T}_w 1_\lambda$ with $a_{w,\lambda} \in \mathcal{A}$. Note that H is naturally an $\mathcal{A}$-submodule of $\hat{H}$ with $\mathcal{A}$-basis $\{\tilde{T}_w 1_\lambda; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}\}$. For any $n \in \mathbf{N}^*_{\mathbf{k}}$, the $\mathcal{A}$-submodule of H spanned by $\{\tilde{T}_w 1_\lambda; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n\}$ may be naturally identified with H_n (see 31.2(a)). There is a unique $\mathcal{A}$-algebra structure on $\hat{H}$ in which the product of two elements

$$h = \sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}} a_{w,\lambda}\tilde{T}_w 1_\lambda,\ h' = \sum_{w'\in\mathbf{W},\lambda'\in\underline{\mathfrak{s}}} a'_{w',\lambda'}\tilde{T}_{w'} 1_{\lambda'}$$

as above is $hh' = \sum_{y\in\mathbf{W},\nu\in\underline{\mathfrak{s}}} b_{y',\nu}\tilde{T}_y 1_\nu$ where for any $\nu \in \underline{\mathfrak{s}}$,

$$\sum_{w,w'\in\mathbf{W}} a_{w,w'^{-1}\nu} a'_{w',\nu}\tilde{T}_w\tilde{T}_{w'} 1_\nu = \sum_{y\in\mathbf{W}} b_{y,\nu}\tilde{T}_y 1_\nu$$

is computed in the algebra structure of H_n for any n such that $\nu \in \underline{\mathfrak{s}}_n$. Thus $\hat{H}$ becomes an associative algebra with 1; H is a subalgebra (without 1) and, for $n \in \mathbf{N}^*_{\mathbf{k}}$, H_n is a subalgebra (with a different 1) with the algebra structure as in 31.2.

Now in the definition of $\hat{H}$ given above, although $\tilde{T}_w 1_\lambda$ is defined, the elements $\tilde{T}_w, 1_\lambda$ are not defined separately (as was the case in H_n). To remedy this we set $\tilde{T}_w = \sum_{\lambda\in\underline{\mathfrak{s}}} \tilde{T}_w 1_\lambda \in \hat{H}$ (for $w \in \mathbf{W}$) and $1_\lambda = \tilde{T}_1 1_\lambda \in H$ (for $\lambda \in \underline{\mathfrak{s}}$). Then $\tilde{T}_w 1_\lambda$ is the product of $\tilde{T}_w, 1_\lambda$ in the algebra $\hat{H}$. Note that $\tilde{T}_1$ is the unit element of $\hat{H}$ and the following equalities hold in $\hat{H}$:

$1_\lambda 1_\lambda = 1_\lambda$ for $\lambda \in \underline{\mathfrak{s}}, 1_\lambda 1_{\lambda'} = 0$ for $\lambda \neq \lambda'$ in $\underline{\mathfrak{s}}$;
$\tilde{T}_w\tilde{T}_{w'} = \tilde{T}_{ww'}$ for $w, w' \in \mathbf{W}$ such that $l(ww') = l(w) + l(w')$;
$\tilde{T}_w 1_\lambda = 1_{w\lambda}\tilde{T}_w$ for $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}$;
$\tilde{T}_s^2 = \tilde{T}_1 + (v - v^{-1})\sum_{\lambda\in\underline{\mathfrak{s}}; s\in\mathbf{W}_\lambda} \tilde{T}_s 1_\lambda$ for $s \in \mathbf{I}$.

By a standard argument we see that

(a) H is exactly the $\mathcal{A}$-algebra defined by the generators $\tilde{T}_w 1_l$ ($w \in \mathbf{W}$, $\lambda \in \underline{\mathfrak{s}}$) and the relations:

$(\tilde{T}_w 1_\lambda)(\tilde{T}_{w'} 1_{\lambda'}) = 0$ if $w, w' \in \mathbf{W}, \lambda, \lambda' \in \underline{\mathfrak{s}}, w'\lambda' \neq \lambda$,
$(\tilde{T}_w 1_{w'\lambda'}(\tilde{T}_{w'} 1_{\lambda'}) = \tilde{T}_{ww'} 1_{\lambda'}$ if $w, w' \in \mathbf{W}, \lambda, \lambda' \in \underline{\mathfrak{s}}$, $l(ww') = l(w) + l(w')$,
$(\tilde{T}_s 1_{s\lambda'})(\tilde{T}_s 1_{\lambda'}) = \tilde{T}_1 1_{\lambda'} + (v - v^{-1})c\tilde{T}_s 1_{\lambda'}$ if $s \in \mathbf{I}$, $\lambda' \in \underline{\mathfrak{s}}$ where $c = 1$ for $s \in \mathbf{W}_{\lambda'}$ and $c = 0$ for $s \notin \mathbf{W}_{\lambda'}$.

40.2. Let R, R^+ be as in 28.3. The following result is well known:

(a) *If $w \in \mathbf{W}, \alpha \in R^+$ and s_α is as in 28.3, then we have $l(ws_\alpha) > l(w)$ if and only if $w(\alpha) \in R^+$.*

Let $\lambda \in \underline{\mathfrak{s}}$. Let $R_\lambda, R^+_\lambda, \mathbf{W}_\lambda, H_\lambda$ be as in 34.2. We write $\vee_\lambda$ instead of $\vee^D_\lambda$ (as in 34.4 with $D = G^0$). We show that

(b) *if $w \in \mathbf{W}$, then $w\mathbf{W}_\lambda$ contains a unique element w_1 of minimal length; it is characterized by the property $w_1(R^+_\lambda) \subset R^+$.*

Let w_1 be an element of minimal length in $w\mathbf{W}_\lambda$. Let $\alpha \in R^+_\lambda$. Then $l(w_1 s_\alpha) \geq l(w_1)$. Since $l(w_1 s_\alpha) = l(w_1) + 1 \mod 2$ we see that $l(w_1 s_\alpha) > l(w_1)$. By (a) we have $w_1(\alpha) \in R^+$. Thus, $w_1(R^+_\lambda) \subset R^+$. Now let $u \in \mathbf{W}_\lambda - \{1\}$. We pick $\alpha \in R^+_\lambda$ such that $u(\alpha)^{-1} \in R^+_\lambda$; then $w_1 u(\alpha)^{-1} \in R^+$. If $w_1 u$ has minimal length in $w\mathbf{W}_\lambda$, then by an earlier part of the argument applied to w_1u instead of w_1 we have $w_1u(\alpha) \in R^+$, a contradiction. We see that w_1 is the unique element of minimal length in $w\mathbf{W}_\lambda$. It remains to show that if $u \in \mathbf{W}_\lambda$ satisfies $w_1u(R_\lambda^+) \subset R^+$, then $u = 1$. If $u \neq 1$, then by an earlier part of the argument we have $w_1u(\alpha)^{-1} \in R^+$ for some $\alpha \in R_\lambda^+$, a contradiction. This proves (b).

We show that

(c) *if* $s \in \mathbf{I}$ *and* $w \in \mathbf{W}$ *has minimal length in* $w\mathbf{W}_\lambda$*, then either (i)* sw *has minimal length in* $sw\mathbf{W}_\lambda$ *or (ii)* $w^{-1}sw \in \mathbf{W}_\lambda$*.*

There is a unique $\beta \in R^+$ such that $s(\beta)^{-1} \in R^+$. Assume that (i) does not hold. By (b) there exists $\alpha \in R_\lambda^+$ such that $sw(\alpha)^{-1} \in R^+$; moreover, $w(\alpha) \in R^+$. Hence $w(\alpha) = \beta$. We have $w^{-1}(\beta) = \alpha \in R_\lambda$, hence $w^{-1}sw \in \mathbf{W}_\lambda$ and (ii) holds. This proves (c).

For $z \in \mathbf{W}_\lambda$ let $\tilde{T}_z^\lambda, c_z^\lambda \in H_\lambda$ be as in 34.2 . Then $c_z^\lambda = \sum_{z' \in \mathbf{W}_\lambda} p_{z',z}^\lambda \tilde{T}_{z'}^\lambda$ where $p_{z',z}^\lambda \in \mathbf{Z}[v^{-1}]$ are uniquely defined.

For any $w \in \mathbf{W}$, $\lambda \in \underline{\mathfrak{s}}$ there is a unique element of H which is equal to $c_{w,\lambda} \in H_n$ (see 34.4) for any n such that $\lambda \in \underline{\mathfrak{s}}_n$; we denote this element again by $c_{w,\lambda}$. We have $c_{w,\lambda} = \sum_{w' \in \mathbf{W}} \pi_{w',w,\lambda} \tilde{T}_{w'} 1_\lambda$ where $\pi_{w',w,\lambda} \in \mathbf{Z}[v^{-1}]$ are uniquely defined. Note that $\{c_{w,\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}\}$ *is an* $\mathcal{A}$*-basis of* H*.*

Now:

(d) *Let* $w, w' \in \mathbf{W}$*. We write* $w = w_1z, w' = w_1'z'$ *where* w_1 *has minimal length in* $w\mathbf{W}_\lambda$*,* w_1' *has minimal length in* $w'\mathbf{W}_\lambda$ *and* $z, z' \in \mathbf{W}_\lambda$*. If* $w_1 \neq w_1'$*, then* $\pi_{w',w,\lambda} = 0$*. If* $w_1 = w_1'$*, then* $\pi_{w',w,\lambda} = p_{z',z}^\lambda$*.*

From the definitions we see that if $w\lambda \neq w'\lambda$, then $\pi_{w',w,\lambda} = 0$. Thus we may assume that $w\lambda = w'\lambda$. We choose a sequence $s_1, s_2, \dots, s_r$ in $\mathbf{I}$ such that $w\lambda = w'\lambda = s_r s_{r-1} \dots s_1\lambda \neq s_{r-1} \dots s_1\lambda \neq \dots \neq s_1\lambda \neq \lambda$.

We show that for $k \in [0, r]$, $s_k s_{k-1} \dots s_1$ has minimal length in $s_k s_{k-1} \dots s_1\mathbf{W}_\lambda$. We argue by induction. For $k = 0$ the result is obvious. Assume now that $k \in [1, r]$. Since $s_{k-1} \dots s_1$ has minimal length in $s_{k-1} \dots s_1\mathbf{W}_\lambda$ and $s_k s_{k-1} \dots s_1\lambda \neq s_{k-1} \dots s_1\lambda$, we see from (c) that $s_k s_{k-1} \dots s_1$ has minimal length in $s_k s_{k-1} \dots s_1\mathbf{W}_\lambda$ as required.

In particular, $s_r s_{r-1} \dots s_1$ has minimal length in $s_r s_{r-1} \dots s_1\mathbf{W}_\lambda$. Since $w\lambda = s_r s_{r-1} \dots s_1\lambda$ we have $w = s_r s_{r-1} \dots s_1 h_1 h_2$ where $h_1 \in \vee_\lambda$, $h_2 \in \mathbf{W}_\lambda$. Then both w_1 and $s_r s_{r-1} \dots s_1 h_1$ have minimal length in $s_r s_{r-1} \dots s_1 h_1\mathbf{W}_\lambda = w\mathbf{W}_\lambda = w_1\mathbf{W}_\lambda$; using (b) we deduce that $s_r s_{r-1} \dots s_1 h_1 = w_1$. Hence $s_1 \dots s_r w = s_1 \dots s_r w_1 z = h_1 z$. Similarly, $s_1 \dots s_r w' = h_1' z'$ where $h_1' \in \vee_\lambda$.

From the results in 34.7–34.10 we see that $\pi_{w',w,\lambda} = p_{s_1 \dots s_r w', s_1 \dots s_r w}^\lambda = p_{h_1'z', h_1z}^\lambda$. Using $h_1, h_1' \in \vee_\lambda$ and the definitions (34.2) we see that $p_{h_1'z', h_1z}^\lambda = 0$ if $h_1 \neq h_1'$ and $p_{h_1'z', h_1z}^\lambda = p_{z',z}^\lambda$ if $h_1 = h_1'$.

It remains to show that we have $w_1 = w_1'$ if and only if $h_1 = h_1'$. We have $s_r s_{r-1} \dots s_1 = h_1^{-1} w_1$ and similarly $s_r s_{r-1} \dots s_1 = (h_1')^{-1} w_1'$. Hence $h_1^{-1} w_1 = (h_1')^{-1} w_1'$. We see that $w_1 = w_1'$ if and only if $h_1 = h_1'$. This proves (d).

For $w' \leq w$ in $\mathbf{W}$, $\lambda \in \underline{\mathfrak{s}}$ and $i \in \mathbf{Z}$ we define $N_{i,w',w,\lambda} \in \mathbf{Z}$ by

(e) $\pi_{w',w,\lambda} = v^{l(w')-l(w)} \sum_{i \in \mathbf{Z}} N_{i,w',w,\lambda} v^i$, that is,

$$p_{z',z}^\lambda = v^{l(w')-l(w)} \sum_{i \in \mathbf{Z}} N_{i,w',w,\lambda} v^i$$

if $w'\mathbf{W}_\lambda = w\mathbf{W}_\lambda$ and z, z' are as in (d), then

$$N_{i,w',w,\lambda} = 0 \text{ if } w'\mathbf{W}_\lambda \neq w\mathbf{W}_\lambda.$$

Note that $N_{i,w',w,\lambda}$ is 0 unless i is even.

40.3. Let $B^* \in \mathcal{B}$. Let $U^* = U_{B^*}$ and let T be a maximal torus of B^*. Let $\mathbf{r} = \dim \mathbf{T}$. Let $W_T = N_{G^0}T/T$. We identify $T = \mathbf{T}, W_T = \mathbf{W}$ as in 28.5. For any $w \in \mathbf{W}$ we denote by $\dot{w}$ a representative of w in $N_{G^0}T$.

Let $C = G^0/U^* \times G^0/U^*$. We have a partition $C = \bigcup_{w\in\mathbf{W}} C_w$ where

$$C_w = \{(hU^*, h'U^*) \in C; h^{-1}h' \in B^*\dot{w}B^*\}.$$

For $w \in \mathbf{W}$ let $d_w = \dim C_w$ and let

$$\bar{C}_w = \{(hU^*, h'U^*) \in C; h^{-1}h' \in \overline{B^*\dot{w}B^*}\}$$

(closure in G^0). Now $\bar{C}_w$ is an irreducible variety and we have a partition $\bar{C}_w = \bigcup_{w';w'\le w} C_{w'}$ with C_w smooth, open dense in $\bar{C}_w$.

Define $\gamma_{\dot{w}} : B^*\dot{w}B^* \to T$ by $\gamma_{\dot{w}}(g) = t$ where $g \in U^*\dot{w}tU^*$ with $t \in T$. Define $\psi : C_w \to T$ by $\psi(hU^*, h'U^*) = \gamma_{\dot{w}}(h^{-1}h')$.

For $\mathcal{L} \in \mathfrak{s}$ we set $\mathcal{L}_w = \psi^*\mathcal{L}$, a local system on C_w. (Using 28.1(c) we see that the isomorphism class of $\psi^*\mathcal{L}$ is independent of the choice of $\dot{w}$.) Let $\mathcal{L}_w^\sharp = IC(\bar{C}_w, \mathcal{L}_w) \in \mathcal{D}(\bar{C}_w)$.

40.4. For $w \in \mathbf{W}, \mathcal{L} \in \mathfrak{s}$ let $\underline{\mathcal{L}}_w = j_{w!}\mathcal{L}_w$, $\underline{\mathcal{L}}_w^\sharp = \bar{j}_{w!}\mathcal{L}_w^\sharp$ where $j_w : C_w \to C$, $\bar{j}_w : \bar{C}_w \to C$ are the inclusions. Let $\hat{C}$ be the full subcategory of $\mathcal{D}(C)$ whose objects are the simple perverse sheaves on C which are equivariant for the $G^0\times T\times T$ action

(a) $(x, t, t') : (hU^*, h'U^*) \mapsto (xht^nU^*, xh't'^nU^*)$

on C (for some $n \in \mathbf{N}_{\mathbf{k}}^*$) or equivalently, are isomorphic to $\underline{\mathcal{L}}_w^\sharp[d_w]$ for some $\mathcal{L} \in \mathfrak{s}$ and some $w \in \mathbf{W}$. Let $\mathcal{D}^{cs}(C)$ be the subcategory of $\mathcal{D}(C)$ whose objects are those $K \in \mathcal{D}(C)$ such that for any j, any simple subquotient of ${}^pH^jK$ is in $\hat{C}$.

If $w, \mathcal{L}$ are as above, then $\underline{\mathcal{L}}_w \in \mathcal{D}^{cs}(C)$. Indeed this constructible sheaf is equivariant for the action (a) (for some n), hence so is each ${}^pH^j(\underline{\mathcal{L}}_w)$.

We have a diagram $C \times C \xleftarrow{r} (G^0/U^*)^3 \xrightarrow{s} C$ where

$$r(h_1U^*, h_2U^*, h_3U^*) = ((h_1U^*, h_2U^*), (h_2U^*, h_3U^*)),$$
$$s(h_1U^*, h_2U^*, h_3U^*) = (h_1U^*, h_3U^*).$$

We define a bi-functor $\mathcal{D}(C) \times \mathcal{D}(C) \to \mathcal{D}(C)$ by $A, A' \mapsto A * A' = s_!r^*(A \boxtimes A')$. The "product" $A * A'$ is associative in an obvious sense. We show that

(b) $A, A' \mapsto A * A'$ restricts to a bi-functor $\mathcal{D}^{cs}(C) \times \mathcal{D}^{cs}(C) \to \mathcal{D}^{cs}(C)$.

Let $A, A' \in \mathcal{D}^{cs}(C)$. To show that $A * A' \in \mathcal{D}^{cs}(C)$ we may assume that $A, A' \in \hat{C}$. Then each ${}^pH^j(A * A')$ is equivariant for the action (a) (for some n). This proves (b).

40.5. For $w' \le w$ in $\mathbf{W}$, $\lambda \in \underline{\mathfrak{s}}$, $\mathcal{L} \in \lambda$ and $i \in \mathbf{Z}$ we show that

(a) $\mathcal{H}^i(\mathcal{L}_w^\sharp)|_{C_{w'}} \cong (\mathcal{L}_{w'}(-i/2))^{\oplus N_{i,w',w,\lambda}}$.

(Both sides are 0 unless i is even.)

Let

$$\tilde{C}_w = \{(h,h') \in G^0 \times G^0; h^{-1}h' \in B^*\dot{w}B^*\} \times \mathbf{k}^*,$$
$$\bar{\tilde{C}}_w = \{(h,h') \in G^0 \times G^0; h^{-1}h' \in \overline{B^*\dot{w}B^*}\} \times \mathbf{k}^*.$$

Now $\bar{\tilde{C}}_w$ is an irreducible variety and we have a partition $\bar{\tilde{C}}_w = \bigcup_{w';w'\le w} \tilde{C}_{w'}$ with $\tilde{C}_w$ smooth, open dense in $\bar{\tilde{C}}_w$. Define $\bar{d} : \bar{\tilde{C}}_w \to \bar{C}_w$, $d : \tilde{C}_w \to \bar{C}_w$ by $(h,h',z) \mapsto (hU^*, h'U^*)$. Let $\tilde{\mathcal{L}}_w = d^*\mathcal{L}_w$, a local system on $\tilde{C}_w$. Let $\tilde{\mathcal{L}}^\sharp_w = IC(\bar{\tilde{C}}_w, \tilde{\mathcal{L}}_w)$. Since $d, \bar{d}$ are principal $U^* \times \mathbf{k}^*$-bundles, it is enough to show that

(b) $\mathcal{H}^i(\tilde{\mathcal{L}}^\sharp_w)|_{\tilde{C}_{w'}} \cong (\tilde{\mathcal{L}}_{w'}(-i/2))^{\oplus N_{i,w',w,\lambda}}$.

(Both sides are 0 unless i is even.)

We choose $\kappa \in \mathrm{Hom}(T, \mathbf{k}^*), \mathcal{E} \in \mathfrak{s}(\mathbf{k}^*)$ such that $\mathcal{L} \cong \kappa^*\mathcal{E}$; see 28.1(c).

Now B^* acts on $(B^*\dot{w}B^*) \times \mathbf{k}^*$ and on $(\overline{B^*\dot{w}B^*}) \times \mathbf{k}^*$ by

$$t_1u : (g,z) \mapsto (g(t_1u)^{-1}, \kappa(t_1)z)$$

where $t_1 \in T$, $u \in U^*$. Let $\bar{\mathbf{P}}^\kappa_w = ((\overline{B^*\dot{w}B^*}) \times \mathbf{k}^*)/B^*$, $PP^\kappa_w = ((B^*\dot{w}B^*) \times \mathbf{k}^*)/B^*$. Now $\mathbf{P}^\kappa_w$ is a smooth open dense subvariety of the irreducible variety $\bar{\mathbf{P}}^k_w$ and $\bar{\mathbf{P}}^\kappa_w = \bigcup_{w';w'\le w} \mathbf{P}^\kappa_{w'}$ is a partition. The morphism $(B^*\dot{w}B^*) \times \mathbf{k}^* \to \mathbf{k}^*$ given by $(g,z) \mapsto \kappa(\gamma_{\dot{w}}(g))z$ factors through a morphism $\phi : \mathbf{P}^\kappa_w \to \mathbf{k}^*$. Let $\mathcal{E}^\kappa_w = \phi^*\mathcal{E}$, a local system of rank 1 on $\mathbf{P}^\kappa_w$. Let $\mathcal{E}^{\kappa\sharp}_w = IC(\bar{\mathbf{P}}^\kappa_w, \mathcal{E}^\kappa_w) \in \mathcal{D}(\bar{\mathbf{P}}^\kappa_w)$. From [L14, 1.24] we see that

(c) $\mathcal{H}^i(\mathcal{E}^{\kappa\sharp}_w)|_{\mathbf{P}^\kappa_{w'}} \cong (\mathcal{E}^\kappa_{w'}(-i/2))^{\oplus N_{i,w',w,\lambda}}$.

(Both sides are 0 unless i is even.)

We can find $n \in \mathbf{N}^*_{\mathbf{k}}$ such that $\mathcal{E} \in \mathfrak{s}_n(\mathbf{k}^*)$. Define $\bar{c} : \bar{\tilde{C}}_w \to \bar{\mathbf{P}}_w$, $c : \bar{\tilde{C}}_w \to \bar{\mathbf{P}}_w$ by $(h,h',z) \mapsto B^*-$orbit of $(h^{-1}h', z^n)$. Now $\bar{c}, c$ are locally trivial fibrations with smooth fibres of pure dimension. Hence (b) follows from (c) provided that we can show that $c^*\mathcal{E}^\kappa_{w'} = \tilde{\mathcal{L}}_{w'}$ for $w' \le w$. We may assume that $w = w'$. We have a commutative diagram

$$\begin{array}{ccccc} \mathbf{P}^\kappa_w & \xleftarrow{c} & \tilde{C}_w \times \mathbf{k}^* & \xrightarrow{d} & C_w \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi'}\downarrow & & {\scriptstyle\kappa\psi}\downarrow \\ \mathbf{k}^* & \xleftarrow{c'} & \mathbf{k}^* \times \mathbf{k}^* & \xrightarrow{d'} & \mathbf{k}^* \end{array}$$

with ϕ, ψ, c, d as above, $\phi'(h,h',z) = (\kappa(\gamma_{\dot{w}}(h^{-1}h')), z)$, $c'(z',z) = z'z^n$, $d'(z',z) = z'$. Using this and the definitions we have $\tilde{\mathcal{L}}_w = \phi'^*d'^*\mathcal{E}$, $c^*\mathcal{E}_w = \phi'^*c'^*\mathcal{E}$. It remains to show that $d'^*\mathcal{E} = c'^*\mathcal{E}$. This expresses the fact that $\mathcal{E}$ is equivariant for the $\mathbf{k}^*$-action $z_1 : z \mapsto z_1^n z$ on $\mathbf{k}^*$ which follows from $\mathcal{E} \in \mathfrak{s}_n(\mathbf{k}^*)$. This proves (b), hence (a).

40.6. Let $w, w' \in \mathbf{W}$, $\mathcal{L}, \mathcal{L}' \in \mathfrak{s}$. We set $L = \underline{\mathcal{L}}_w * \underline{\mathcal{L}}'_{w'} \in \mathcal{D}^{cs}(C)$. Let

$$X = \{(h_1U^*, h_2U^*, h_3U^*) \in (G^0/U^*)^3; h_1^{-1}h_2 \in B^*\dot{w}B^*, h_2^{-1}h_3 \in B^*\dot{w}'B^*\},$$
$$\bar{X} = \{(h_1U^*, h_2B^*, h_3U^*) \in G^0/U^* \times G^0/B^* \times G^0/U^*;$$
$$h_1^{-1}h_2 \in B^*\dot{w}B^*, h_2^{-1}h_3 \in B^*\dot{w}'B^*\}.$$

We have a commutative diagram with a cartesian square

$$\begin{array}{ccccc} X & \xrightarrow{f} & \bar{X} & \xrightarrow{\bar{\sigma}} & C \\ {\scriptstyle\tau}\downarrow & & {\scriptstyle\bar{\tau}}\downarrow & & \\ T\times T & \xrightarrow{f'} & T & & \end{array}$$

where f is given by $(h_1U^*, h_2U^*, h_3U^*) \mapsto (h_1U^*, h_2B^*, h_3U^*)$,
f' is $(t,t') \mapsto \mathrm{Ad}(\dot{w}')^{-1}(t)t'$,
τ is $(h_1U^*, h_2U^*, h_3U^*) \mapsto (t,t')$ with $h_1^{-1}h_2 \in U^*\dot{w}tU^*, h_2^{-1}h_3 \in U^*\dot{w}'t'U^*$,
$\bar{\tau}$ is $(h_1U^*, h_2B^*, h_3U^*) \mapsto \mathrm{Ad}(\dot{w}')^{-1}(t)t'$ with t,t' as in the definition of τ,
$\bar{\sigma}$ is $(h_1U^*, h_2B^*, h_3U^*) \mapsto (h_1U^*, h_3U^*)$.
From the definitions we have $L = \bar{\sigma}_! f_! \tau^*(\mathcal{L} \boxtimes \mathcal{L}')$. Using the diagram above, we have $L = \bar{\sigma}_! \bar{\tau}^* f'_!(\mathcal{L} \boxtimes \mathcal{L}')$. From the definitions we see that either (i) or (ii) below holds:

(i) $\mathcal{L} \not\cong (\mathrm{Ad}(\dot{w}')^{-1})^*\mathcal{L}'$ and $f'_!(\mathcal{L} \boxtimes \mathcal{L}') = 0$;
(ii) $\mathcal{L} \cong (\mathrm{Ad}(\dot{w}')^{-1})^*\mathcal{L}'$ and $\mathcal{L} \boxtimes \mathcal{L}' = f'^*\mathcal{L}'$.

If (i) holds, then $K = 0$. If (ii) holds, then, as in 32.16, we have

$$f'_!(\mathcal{L} \boxtimes \mathcal{L}') = f'_! f'^* \mathcal{L}' = \mathcal{L}' \otimes f'_! \bar{\mathbf{Q}}_l \approx \{\mathcal{L}' \otimes \mathcal{H}^e(f'_!\bar{\mathbf{Q}}_l)[-e], e \in \mathbf{Z}\},$$

$$\mathcal{L}' \otimes \mathcal{H}^e(f'_!\bar{\mathbf{Q}}_l)[-e] \approx \{\mathcal{L}'(\mathbf{r}-e), \dots, \mathcal{L}'(\mathbf{r}-e), (\binom{\mathbf{r}}{2\mathbf{r}-e} \text{ copies})\}.$$

Setting $\bar{L} = \bar{\sigma}_!\bar{\tau}^*(\mathcal{L}')$, it follows that

$$L \approx \{\bar{L}(\mathbf{r}-e)[-e], \dots, \bar{L}(\mathbf{r}-e)[-e], (\binom{\mathbf{r}}{2\mathbf{r}-e} \text{ copies}), e \in \mathbf{Z}\}.$$

We now consider $\bar{L}$ for certain choices of w, w'.

If w, w' satisfy $l(ww') = l(w) + l(w')$, then $\bar{\sigma}$ restricts to an isomorphism $\bar{X} \to C_{ww'}$ and $\bar{L} = \underline{\mathcal{L}}'_{ww'}$.

Now assume that $\alpha, \check{\alpha}, s_\alpha$ are as in 28.3 and that $w = w' = s_\alpha \in \mathbf{I}$. We have

$$\bar{L} \approx \{j_{u!}\bar{L}_u; u \in \mathbf{W}\}$$

where $j_u : C_u \to C$ is the inclusion and $\bar{L}_u = j_u^*\bar{L}$. Let $\bar{X}_u = \bar{\sigma}^{-1}(C_u)$. Then $\bar{L}_u = \bar{\sigma}_{u!}\bar{\tau}_u^*(\mathcal{L}')$ where $\bar{\sigma}_u : \bar{X}_u \to C_u$, $\bar{\tau}_u : \bar{X}_u \to T$ are the restrictions of $\bar{\sigma}, \bar{\tau}$.

If $u \notin \{1, s_\alpha\}$, then $\bar{X}_u = \emptyset$ and $\bar{L}_u = 0$. If $u = 1$, then $\bar{\sigma}_u : \bar{X}_u \to C_u$ is an affine line bundle and $\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_u^*\mathcal{L}'_u$; hence $\bar{\sigma}_{u!}\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_{u!}\bar{\sigma}_u^*\mathcal{L}'_u = \mathcal{L}'_u[[-1]]$. If $u = s_\alpha$, then $\bar{\sigma}_u : \bar{X}_u \to C_u$ is a principal $\mathbf{k}^*$-bundle and either (iii) or (iv) below holds:

(iii) $\check{\alpha}^*\mathcal{L}' \not\cong \bar{\mathbf{Q}}_l$ and $\bar{\sigma}_{u!}\bar{\tau}_u^*(\mathcal{L}') = 0$,
(iv) $\check{\alpha}^*\mathcal{L}' \cong \bar{\mathbf{Q}}_l$ and $\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_u^*\mathcal{L}'_u$.

If (iv) holds, then, as in case (ii) above, we have

$$\bar{\sigma}_{u!}\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_{u!}\bar{\sigma}_u^*\mathcal{L}'_u = \mathcal{L}'_u \otimes \bar{\sigma}_{u!}\bar{\mathbf{Q}}_l \approx \{\mathcal{L}'_u \otimes \mathcal{H}^e(\bar{\sigma}_{u!}\bar{\mathbf{Q}}_l)[-e], e \in \mathbf{Z}\},$$

$$\mathcal{L}'_u \otimes \mathcal{H}^e(\bar{\sigma}_{u!}\bar{\mathbf{Q}}_l)[-e] \approx \{\mathcal{L}'_u(1-e), \dots, \mathcal{L}'_u(1-e), (\binom{1}{2-e} \text{ copies})\}.$$

40.7. In this subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field. Now the $\mathcal{A}$-module $\mathfrak{K}(C)$ is defined as in 36.8 (the character sheaves on C are taken to be the objects in $\hat{C}$).

For $(w,\lambda) \in \mathbf{W} \times \underline{\mathfrak{s}}$, let $[w;\lambda]$ be the basis element of $\mathfrak{K}(C)$ given by $\underline{\mathcal{L}}_w^\sharp[[d_w/2]]$; we choose $\mathcal{L} \in \lambda$ and we regard $\underline{\mathcal{L}}_w, \underline{\mathcal{L}}_w^\sharp$ as mixed complexes on C whose restriction to C_w is pure of weight 0; then $gr(\underline{\mathcal{L}}_w), gr(\underline{\mathcal{L}}_w^\sharp)$ are defined in $\mathfrak{K}(C)$ as in 36.8. We denote these elements of $\mathfrak{K}(C)$ by $[w;\lambda]', [w;\lambda]'^\sharp$ respectively. From 40.5(a) we see that

(a) $(-v)^{d_w}[w;\lambda] = [w;\lambda]'^\sharp = \sum_{w' \in \mathbf{W}} \sum_{i \in 2\mathbf{Z}} N_{i,w',w,\lambda} v^i [w';\lambda]'$ in $\mathfrak{K}(C)$

where $N_{i,w',w,\lambda}$ is as in 40.2(e).

Let r, s be as in 40.4. By 40.4(b), $s_! r^* : \mathcal{D}(C \times C) \to \mathcal{D}(C)$ restricts to a functor $\mathcal{D}^{cs}(C \times C) \to \mathcal{D}^{cs}(C)$ where the character sheaves on $C \times C$ are by definition complexes of the form $A \boxtimes A'$ with $A \in \hat{C}, A' \in \hat{C}$. Hence the $\mathcal{A}$-linear map $gr(s_! r^*) : \mathfrak{K}(C \times C) \to \mathfrak{K}(C)$ or equivalently $\mathfrak{K}(C) \otimes_{\mathcal{A}} \mathfrak{K}(C) \to \mathfrak{K}(C)$ is well defined. (We have canonically $\mathfrak{K}(C \times C) = \mathfrak{K}(C) \otimes_{\mathcal{A}} \mathfrak{K}(C)$.) We write $\xi * \xi'$ instead of $gr(s_! r^*)(\xi \boxtimes \xi')$ where $\xi, \xi' \in \mathfrak{K}(C)$. Note that $\xi, \xi' \mapsto \xi * \xi'$ defines an associative $\mathcal{A}$-algebra structure on $\mathfrak{K}(C)$.

Let $w, w' \in \mathbf{W}$, $\lambda, \lambda' \in \underline{\mathfrak{s}}$. From 40.6 we see that

if $w'\lambda' \ne \lambda$, then $[w;\lambda]' * [w';\lambda']' = 0$ in $\mathfrak{K}(C)$;

if $w'\lambda' = \lambda$ and $l(ww') = l(w) + l(w')$, then $[w;\lambda]' * [w',\lambda']' = (v^2-1)^{\mathbf{r}}[ww';\lambda']'$ in $\mathfrak{K}(C)$;

if $s \in \mathbf{I}$ and $s\lambda' = \lambda$, then $[s;\lambda]' * [s,\lambda']' = (v^2-1)^{\mathbf{r}}(v^2[1;\lambda']' + (v^2-1)c[s;\lambda']')$ where $c = 1$ for $s \in \mathbf{W}_{\lambda'}$ and $c = 0$ for $s \notin \mathbf{W}_{\lambda'}$.

Using this and (a), 40.1(a), 40.2(e), we see that

(b) *the unique $\mathcal{A}$-linear isomorphism $\omega : \mathfrak{K}(C) \to H$ (H as in 40.1) given by $[w,\lambda]' \mapsto v^{l(w)}\tilde{T}_w 1_\lambda$ for $w \in \mathbf{W}$, $\lambda \in \underline{\mathfrak{s}}$, satisfies $\omega([w,\lambda]) = (-v)^{-d_w} v^{l(w)} c_{w,\lambda}$ for $w \in \mathbf{W}$, $\lambda\ in \underline{\mathfrak{s}}$ and $\omega(x * x') = (v^2-1)^{\mathbf{r}}\omega(x)\omega(x')$ for any $x, x' \in \mathfrak{K}(C)$.*

40.8. For $w, w' \in \mathbf{W}$ and $\lambda, \lambda' \in \underline{\mathfrak{s}}$ we have $c_{w,\lambda} c_{w',\lambda'} = \sum_{y \in \mathbf{W}, \nu \in \underline{\mathfrak{s}}} \gamma_{y,\nu}^{w,\lambda;w',\lambda'} c_{y,\lambda}$ in the algebra H. Here $\gamma_{y,\nu}^{w,\lambda;w',\lambda'} \in \mathcal{A}$. We have:

(a) $\gamma_{y,\nu}^{w,\lambda;w',\lambda'} \in \mathbf{N}[v, v^{-1}]$.

By the arguments in 34.4–34.10 (with $D = G^0$) this is reduced to the analogous (well-known) statement for the structure constants of the algebra H_λ^D with its basis (c_w^λ) (see 34.2).

40.9. For any $J \subset \mathbf{I}$ let H_J be the $\mathcal{A}$-submodule of H spanned by $\{c_{w,\lambda}; w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}\}$ or equivalently by $\{\tilde{T}_w 1_\lambda; w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}\}$. From the definitions we see that H_J is a subalgebra of H. For any $J \subset \mathbf{I}, J' \subset \mathbf{I}$ we define a relation $\preceq_{J,J'}$ on $\mathbf{W} \times \underline{\mathfrak{s}}$ as follows. We say that $(y,\nu) \preceq_{J,J'} (w,\lambda)$ if there exist $w_1 \in \mathbf{W}_J, w_2 \in \mathbf{W}_{J'}$, $\lambda_1, \lambda_2 \in \underline{\mathfrak{s}}$ such that in the expansion (in the algebra H)

$$c_{w_1,\lambda_1} c_{w,\lambda} c_{w_2,\lambda_2} = \sum_{y' \in \mathbf{W}, \nu' \in \underline{\mathfrak{s}}} a_{y',\nu'} c_{y',\nu'}$$

(with $a_{y',\nu'} \in \mathcal{A}$) we have $a_{y,\nu} \ne 0$.

Using the associativity of the product in H, the fact that $H_J, H_{J'}$ are subalgebras of H and 40.8(a), we see that $\preceq_{J,J'}$ is transitive. Using the formula $c_{1,w\lambda} c_{w,\lambda} c_{1,\lambda} = c_{w,\lambda}$ we see that it is reflexive. Thus, it is a preorder. Let $\sim_{J,J'}$ be the equivalence relation attached to $\preceq_{J,J'}$; thus, $(y,\nu) \sim_{J,J'} (w,\lambda)$ if

$(y,\nu) \preceq_{J,J'} (w,\lambda)$ and $(w,\lambda) \preceq_{J,J'} (y,\nu)$. The equivalence classes for $\sim_{J,J'}$ are called (J,J')-two-sided cells. The $(\mathbf{I},\mathbf{I})$-two-sided cells in $\mathbf{W}\times\underline{\mathfrak{s}}$ are also called two-sided cells.

40.10. Let $w,w',w''\in\mathbf{W}$, $\mathcal{L},\mathcal{L}',\mathcal{L}''\in\mathfrak{s}$. We set $K=\underline{\mathcal{L}}_w * \underline{\mathcal{L}}'^{\sharp}_{w'} * \underline{\mathcal{L}}''_{w''}\in\mathcal{D}^{cs}(C)$. Let

$$X=\{(h_1U^*,h_2U^*,h_3U^*,h_4U^*)\in(G^0/U^*)^4;$$
$$h_1^{-1}h_2\in B^*\dot wB^*, h_2^{-1}h_3\in\overline{B^*\dot w'B^*}, h_3^{-1}h_4\in B^*\dot w''B^*\},$$

an irreducible variety. Let X_0 be the smooth open dense subset of X defined by the condition $h_2^{-1}h_3\in B^*\dot w'B^*$. Define $\sigma: X\to C$ by

$$(h_1U^*,h_2U^*,h_3U^*,h_4U^*)\mapsto(h_1U^*,h_4U^*).$$

Define $\tau: X_0\to T\times T\times T$ by

$$(h_1U^*,h_2U^*,h_3U^*,h_4U^*)\mapsto(t,t',t'')$$

with $h_1^{-1}h_2\in U^*\dot wtU^*, h_2^{-1}h_3\in U^*\dot w't'U^*, h_3^{-1}h_4\in U^*\dot w''t''U^*$.

Let $\mathcal{F}=\tau^*(\mathcal{L}\boxtimes\mathcal{L}'\boxtimes\mathcal{L}'')$, a local system on X_0. Then $\mathcal{F}^\sharp:=IC(X,\mathcal{F})\in\mathcal{D}(X)$ is defined and we have $K=\sigma_!\mathcal{F}^\sharp$.

Let $\bar X$ (resp. $\bar X_0$) be the the variety of all $(h_1U^*,h_2B^*,h_3B^*,h_4U^*)\in G^0/U^*\times G^0/B^*\times G^0/B^*\times G^0/U^*$ that satisfy the same equations as those defining X (resp. X_0). Note that $\bar X$ is irreducible and $\bar X_0$ is an open dense smooth subset of $\bar X$. We have a cartesian diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & \bar X & \xrightarrow{\bar\sigma} & C \\ \uparrow & & \uparrow & & \\ X_0 & \xrightarrow{f_0} & \bar X_0 & & \\ \tau\downarrow & & \bar\tau\downarrow & & \\ T\times T\times T & \xrightarrow{f'} & T & & \end{array}$$

where $X_0\to X, \bar X_0\to\bar X$ are the obvious imbeddings,

f,f_0 are given by $(h_1U^*,h_2U^*,h_3U^*,h_4U^*)\mapsto(h_1U^*,h_2B^*,h_3B^*,h_4U^*)$,

f' is $(t,t',t'')\mapsto\mathrm{Ad}(\dot w'\dot w'')^{-1}(t)\mathrm{Ad}(\dot w'')^{-1}(t')t''$,

$\bar\tau$ is $(h_1U^*,h_2B^*,h_3B^*,h_4U^*)\mapsto\mathrm{Ad}(\dot w'\dot w'')^{-1}(t)\mathrm{Ad}(\dot w'')^{-1}(t')t''$ with t,t',t'' as in the definition of τ, $\bar\sigma$ is $(h_1U^*,h_2B^*,h_3B^*,h_4U^*)\mapsto(h_1U^*,h_4U^*)$. Assume that $\mathcal{L}\cong(\mathrm{Ad}(\dot w')^{-1})^*\mathcal{L}'$ and $\mathcal{L}'\cong(\mathrm{Ad}(\dot w'')^{-1})^*\mathcal{L}''$. Then $\mathcal{L}\boxtimes\mathcal{L}'\boxtimes\mathcal{L}''=f'^*\mathcal{L}''$. We have $\mathcal{F}=\tau^*f'^*\mathcal{L}''=f_0^*\bar\tau^*\mathcal{L}''$. Since f is a principal $T\times T$-bundle and $X_0=f^{-1}(\bar X_0)$ it follows that $\mathcal{F}^\sharp=f^*IC(\bar X,\bar\tau^*\mathcal{L}'')$. Note that $f_!\bar{\mathbf{Q}}_l\Bumpeq\{\mathcal{H}^e(f_!\bar{\mathbf{Q}}_l)[-e],2\mathbf{r}\le e\le4\mathbf{r}\}$,

$$\mathcal{H}^e(f_!\bar{\mathbf{Q}}_l)\Bumpeq\{\bar{\mathbf{Q}}_l(2\mathbf{r}-e),\dots,\bar{\mathbf{Q}}_l(2\mathbf{r}-e),(\binom{2\mathbf{r}}{4\mathbf{r}-e}\text{ copies})\}.$$

Hence setting $\bar K=\bar\sigma_!(IC(\bar X,\bar\tau^*\mathcal{L}''))$ we have

$$K=\sigma_!f^*IC(\bar X,\bar\tau^*\mathcal{L}'')=\bar\sigma_!f_!f^*IC(\bar X,\bar\tau^*\mathcal{L}'')=\bar\sigma_!(IC(\bar X,\bar\tau^*\mathcal{L}'')\otimes f_!\bar{\mathbf{Q}}_l),$$

(a) $$K\Bumpeq\{\bar K(2\mathbf{r}-e)[-e],\dots,\bar K(2\mathbf{r}-e)[-e],(\binom{2\mathbf{r}}{4\mathbf{r}-e}\text{ copies}),2\mathbf{r}\le e\le4\mathbf{r}\}.$$

We now show that

(b) *if $A\in\hat C$ is such that $A\dashv\bar K$, then $A\dashv K$.*

We may regard $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ as mixed local systems (with respect to a rational structure over a sufficiently large finite subfield of $\mathbf{k}$) which are pure of weight 0. Then $K, \bar{K}$ are naturally mixed complexes and (a) is compatible with the mixed structures. For any mixed perverse sheaf P, let P_h be the subquotient of P of pure weight h. We can find $h \in \mathbf{Z}$ such that $A \dashv {}^pH^j(\bar{K})_h$ for some $j \in \mathbf{Z}$; moreover, we may assume that h is maximum possible. Note that $A \dashv {}^pH^{j+4\mathbf{r}}(\bar{K}[-4\mathbf{r}](-2\mathbf{r}))_{h+2\mathbf{r}}$ and $A \not\dashv {}^pH^{j'}(\bar{K}[-e](2\mathbf{r}-e))_{h+2\mathbf{r}}$ for $2\mathbf{r} \le e < 4\mathbf{r}$ and any j'; hence from (a) we see that $A \dashv {}^pH^{j+4\mathbf{r}}(K)_{h+2\mathbf{r}}$. In particular, $A \dashv K$, and (b) is proved.

40.11. Let $w, w'\mathcal{L}, \mathcal{L}', X, \bar{X}, \tau$ be as in 40.6. We set $\mathbf{L} = \underline{\mathcal{L}}_w^\sharp * \underline{\mathcal{L}'}_{w'}{}^\sharp \in \mathcal{D}^{cs}(C)$. Let $A = \underline{\mathcal{L}''}_{w''}{}^\sharp[d_{w''}]$. We show that

(a) *if $A \dashv \mathbf{L}$, then $[w'', \lambda'']$ appears with nonzero coefficient in the expansion of the product $[w, \lambda] * [w', \lambda']$ in terms of the basis $([y, \nu])$ of $\mathfrak{K}(C)$.*

Let

$$\mathbf{X} = \{(h_1U^*, h_2U^*, h_3U^*) \in (G^0/U^*)^3; h_1^{-1}h_2 \in \overline{B^*\dot{w}B^*}, h_2^{-1}h_3 \in \overline{B^*\dot{w}'B^*}\},$$
$$\bar{\mathbf{X}} = \{(h_1U^*, h_2B^*, h_3U^*) \in G^0/U^* \times G^0/B^* \times G^0/U^*;$$
$$h_1^{-1}h_2 \in \overline{B^*\dot{w}B^*}, h_2^{-1}h_3 \in \overline{B^*\dot{w}'B^*}\}.$$

Note that X (resp. $\bar{X}$) is naturally an open dense subset of $\mathbf{X}$ (resp. $\bar{\mathbf{X}}$). Define $\sigma' : \mathbf{X} \to C$ by $(h_1U^*, h_2U^*, h_3U^*) \mapsto (h_1U^*, h_3U^*)$. Define $\bar{\sigma}' : \bar{\mathbf{X}} \to C$ by $(h_1U^*, h_2B^*, h_3U^*) \mapsto (h_1U^*, h_3U^*)$. Let $\mathcal{F} = \tau^*(\mathcal{L} \boxtimes \mathcal{L}')$, a local system on X. Then $\mathcal{F}^\sharp := IC(\mathbf{X}, \mathcal{F}) \in \mathcal{D}(\mathbf{X})$ is defined and we have $\mathbf{L} = \sigma'_!\mathcal{F}^\sharp$. We have a cartesian diagram

$$\begin{array}{ccccc}
\mathbf{X} & \xrightarrow{\tilde{f}} & \bar{\mathbf{X}} & \xrightarrow{\bar{\sigma}'} & C \\
\uparrow & & \uparrow & & \\
X & \xrightarrow{f} & \bar{X} & & \\
{\scriptstyle\tau}\downarrow & & {\scriptstyle\bar{\tau}}\downarrow & & \\
T \times T & \xrightarrow{f'} & T & &
\end{array}$$

where $X \to \mathbf{X}, \bar{X} \to \bar{\mathbf{X}}$ are the obvious imbeddings, $f, f', \bar{\tau}$ are as in 40.6 and $\tilde{f}$ is the obvious map.

Assume first that 40.6(i) holds. Let $m' : T \times \mathbf{X} \to \mathbf{X}$ be the free T-action $t_1 : (h_1U^*, h_2U^*, h_3U^*) \mapsto (h_1U^*, h_2t_1^{-1}U^*, h_3U^*)$. This restricts to a free T-action $m : T \times X \to X$. Define a free T action $m_0 : T \times (T \times T) \to T \times T$ by $t_1 : (t, t') \mapsto (t_1^{-1}t, \mathrm{Ad}(\dot{w}')^{-1}(t_1)t'$. Then m, m_0 are compatible with τ. By our assumption we have $m_0^*(\mathcal{L} \boxtimes \mathcal{L}') = \mathcal{L}_0 \boxtimes \mathcal{L} \boxtimes \mathcal{L}'$ where $\mathcal{L}_0 \in \mathfrak{s}(T)$, $\mathcal{L}_0 \ncong \bar{\mathbf{Q}}_l$. It follows that $m^*(\mathcal{F}) \cong \mathcal{L}_0 \boxtimes \mathcal{F}$. From the properties of intersection cohomology we then have $m'^*(\mathcal{F}^\sharp) \cong \mathcal{L}_0 \boxtimes \mathcal{F}^\sharp$. Let $r : T \times \mathbf{X} \to \mathbf{X}$ be the second projection. Since $\mathcal{L}_0 \in \mathfrak{s}(T)$, $\mathcal{L}_0 \ncong \bar{\mathbf{Q}}_l$, we have $r_!(\mathcal{L}_0 \boxtimes \mathcal{F}^\sharp) = 0$. Hence $r_!m'^*(\mathcal{F}^\sharp) = 0$. Since m', f', r, f' form a cartesian diagram we must have $f'^*f'(\mathcal{F}^\sharp) = 0$. Since f' is a principal T-bundle we deduce that $f'_!(\mathcal{F}^\sharp) = 0$. We have $\mathbf{L} = \bar{\sigma}'_!f'_!(\mathcal{F}^\sharp)$ hence $\mathbf{L} = 0$. In this case (a) is clear.

Assume next that 40.6(ii) holds. Then $\mathcal{L} \boxtimes \mathcal{L}' = f'^*\mathcal{L}'$ and $\mathcal{F} = \tau^*f'^*\mathcal{L}' = f^*\bar{\tau}^*\mathcal{L}'$. Since f' is a principal T-bundle and $X = f'^{-1}(\bar{X})$ it follows that $\mathcal{F}^\sharp =$

$f'^*IC(\bar{\mathbf{X}}, \bar\tau^*\mathcal{L}')$. Note that $f'_!\bar{\mathbf{Q}}_l \approx \{\mathcal{H}^e(f'_!\bar{\mathbf{Q}}_l)[-e], \mathbf{r} \le e \le 2\mathbf{r}\}$,

$$\mathcal{H}^e(f'_!\bar{\mathbf{Q}}_l) \approx \{\bar{\mathbf{Q}}_l(\mathbf{r}-e), \dots, \bar{\mathbf{Q}}_l(\mathbf{r}-e), (\binom{\mathbf{r}}{2\mathbf{r}-e} \text{ copies})\}.$$

Hence setting $\bar{\mathbf{L}} = \bar\sigma'_!(IC(\bar{\mathbf{X}}, \bar\tau^*\mathcal{L}'))$ we have

$$\mathbf{L} = \sigma'_! f'^* IC(\bar{\mathbf{X}}, \bar\tau^*\mathcal{L}') = \bar\sigma'_! f'_! f'^* IC(\bar{\mathbf{X}}, \bar\tau^*\mathcal{L}') = \bar\sigma'_!(IC(\bar{\mathbf{X}}, \bar\tau^*\mathcal{L}') \otimes f'_!\bar{\mathbf{Q}}_l),$$

$$\mathbf{L} \approx \{\bar{\mathbf{L}}(\mathbf{r}-e)[-e], \dots, \bar{\mathbf{L}}(\mathbf{r}-e)[-e], (\binom{\mathbf{r}}{2\mathbf{r}-e} \text{ copies}), \mathbf{r} \le e \le 2\mathbf{r}\}.$$

Since $A \dashv \mathbf{L}$, this shows that $A \dashv \bar{\mathbf{L}}$. We regard $\mathcal{L}'$ as a pure local system of weight 0. Then $\bar{\mathbf{L}} = \bar\sigma'_!(IC(\bar{\mathbf{X}}, \bar\tau^*\mathcal{L}'))$ is again pure of weight 0, since $\bar\sigma'$ is proper (see [BBD]). Hence the coefficient with which A appears in the expansion of $gr(\bar{\mathbf{L}})$ is a polynomial in $-v$ with coefficients given by the multiplicities of A in the various ${}^pH^j(\bar{\mathbf{L}})$; in particular, A appears with coefficient $\ne 0$ in $gr(\bar{\mathbf{L}})$. On the other hand, the arguments above show that $[w,\lambda] * [w',\lambda'] = (v^2-1)^{\mathbf{r}} gr(\bar{\mathbf{L}})$. It follows that A appears with coefficient $\ne 0$ in $[w,\lambda] * [w',\lambda']$. This proves (a).

41. Character sheaves and two-sided cells

41.1. In this section we preserve the notation of 40.3. We fix a connected component D of G and we pick $\delta \in N_D B^* \cap N_D T$. We write ϵ instead of $\epsilon_D : \mathbf{W} \to \mathbf{W}$. For $w \in \mathbf{W}$ we set

$$Z^w_{\emptyset,D} = \{(B, B', xU_B) \in Z_{\emptyset,D}; \text{pos}(B,B') = w\}.$$

(This is the same as ${}^{w^{-1}}Z_{\emptyset,D}$ in 36.2.) Define $\xi_D : C \to Z_{\emptyset,D}$ by $(hU^*, h'U^*) \mapsto (hB^*h^{-1}, h'B^*h'^{-1}, h'\delta h^{-1}U_{hB^*h^{-1}})$, a principal T-bundle for the free T-action on C given by $t : (hU^*, h'U^*) \to (htU^*, h'(\delta t\delta^{-1})U^*)$.

Since $\xi_D^{-1}(Z^w_{\emptyset,D}) = C_w$, ξ_D restricts to a principal T-bundle $\xi_{D,w} : C_w \to Z^w_{\emptyset,D}$. We have a commutative diagram

$$\begin{array}{ccccc}
T & \xleftarrow{\psi} & C_w & \xrightarrow{=} & C_w \\
\zeta\downarrow & & j'\uparrow & & \xi_{D,w}\downarrow \\
\mathbf{d} & \xleftarrow{pr_2} & G^0/(U^* \cap \dot wU^*\dot w^{-1}) \times \mathbf{d} & \xrightarrow{j} & Z^w_{\emptyset,D}
\end{array}$$

where ψ is as in 40.3,
$\mathbf{d} = \dot w\delta T$,
$j(f(U^* \cap \dot wU^*\dot w^{-1}), s) = (fB^*f^{-1}, f\dot wB^*\dot w^{-1}f^{-1}, fsf^{-1}U_{fB^*f^{-1}})$,
$j'(f(U^* \cap \dot wU^*\dot w^{-1}), s) = (fU^*, fs\delta^{-1}U^*)$,
$\zeta(t) = \dot w\delta(\delta^{-1}t\delta)$.
Note that the lower row in the diagram is as in 36.2(a).

Define $\iota : \mathbf{d} \to T$ by $\iota(\dot w\delta t) = t$ where $t \in T$. If $\mathcal{L} \in \mathfrak{s}$ is such that $\text{Ad}((\dot wd)^{-1})^*\mathcal{L} \cong \mathcal{L}$, then $pr_2^*\iota^*(\mathcal{L})$ is a local system on $G^0/(U^* \cap \dot wU^*\dot w^{-1}) \times \mathbf{d}$, equivariant for the T-action $t_0 : (f(U^* \cap \dot wU^*\dot w^{-1}), s) = (ft_0^{-1}(U^* \cap \dot wU^*\dot w^{-1}), t_0st_0^{-1})$ on $G^0/(U^* \cap \dot wU^*\dot w^{-1}) \times \mathbf{d}$, which makes j a principal T-bundle. It follows that there is a well-defined local system $\dot{\mathcal{L}}_w$ (of rank 1) on $Z^w_{\emptyset,D}$ such that $j^*\dot{\mathcal{L}}_w = pr_2^*\iota^*(\mathcal{L})$. We show that

(a) $\xi^*_{D,w}(\dot{\mathcal{L}}_w) = (\text{Ad}(\delta^{-1})^*\mathcal{L})_w$.

Since j' is an isomorphism, it is enough to show that $j'^*\xi^*_{D,w}(\dot{\mathcal{L}}_w)=j'^*(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})_w$ or that $j^*\dot{\mathcal{L}}_w = j'^*(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})_w$ or that $pr_2^*\iota^*\mathcal{L} = j'^*\psi^*(\mathrm{Ad}(\delta^{-1})^*)\mathcal{L})$ or that $j'^*\psi^*\zeta^*\iota^*\mathcal{L} = j'^*\psi^*(\mathrm{Ad}(\delta^{-1})^*)\mathcal{L})$. It is enough to show that $\zeta^*\iota^*\mathcal{L} = \mathrm{Ad}(\delta^{-1})^*\mathcal{L}$. This follows from $\mathrm{Ad}(\delta^{-1}) = \iota\zeta : T \to T$.

Let $h_w : Z^w_{\emptyset,D} \to Z_{\emptyset,D}$, $\bar{h}_w : \bar{Z}^w_{\emptyset,D} \to Z_{\emptyset,D}$ be the inclusions ($\bar{Z}^w_{\emptyset,D} = \bigcup_{w';w'\le w} Z^{w'}_{\emptyset,D}$ is the closure of $Z^w_{\emptyset,D}$ in $Z_{\emptyset,D}$). Let $\underline{\dot{\mathcal{L}}}_w = h_{w!}\dot{\mathcal{L}}_w, \underline{\dot{\mathcal{L}}}^\sharp_w = \bar{h}_{w!}\dot{\mathcal{L}}^\sharp_w$. Using (a) and the fact that ξ_D is a principal T-bundle we deduce

(b) $\xi^*_D(\underline{\dot{\mathcal{L}}}_w) = \underline{(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})}_w$,

(c) $\xi^*_D(\underline{\dot{\mathcal{L}}}^\sharp_w) = \underline{(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})}^\sharp_w$.

Now let D' be another connected component of G. We pick $\delta' \in N_{D'}B^* \cap N_{D'}T$. We have a commutative diagram with a cartesian right square

$$\begin{array}{ccccc}
C \times C & \xleftarrow{r} & (G^0/U^*)^3 & \xrightarrow{s} & C \\
{\scriptstyle \xi_D\times\xi_{D'}}\downarrow & & {\scriptstyle \xi_0}\downarrow & & {\scriptstyle \xi_{D'D}}\downarrow \\
Z_{\emptyset,D} \times Z_{\emptyset,D'} & \xleftarrow{b_1} & Z_0 & \xrightarrow{b_2} & Z_{\emptyset,D'D}
\end{array}$$

where r, s are as in 40.4, Z_0, b_1, b_2 are as in 32.5 (with $J = \emptyset$) and

$$\xi_0(h_1U^*, h_2U^*, h_3U^*)$$
$$= (h_1B^*h_1^{-1}, h_2B^*h_2^{-1}, h_3B^*h_3^{-1}, h_2\delta h_1^{-1}U_{h_1B^*h_1^{-1}}, h_3\delta' h_2^{-1}U_{h_2B^*h_2^{-1}}).$$

Hence, if $A \in \mathcal{D}(Z_{\emptyset,D})$, $A' \in \mathcal{D}(Z_{\emptyset,D'})$, then $\xi^*_{D'D}b_{2!}b_1^*(A \boxtimes A') = s_!r^*(\xi^*_D A \boxtimes \xi^*_{D'}A')$, or equivalently

(d) $\xi^*_{D'D}(A * A') = (\xi^*_D A) * (\xi^*_{D'}A')$.

41.2. Let $u \in \mathbf{W}$. Let

$$\Upsilon_u = \{(B, B', g(U_B \cap U_{B'});$$
$$B \in \mathcal{B}, B' \in \mathcal{B}, g(U_B \cap U_{B'}) \in D/(U_B \cap U_{B'}), \mathrm{pos}(B, B') = u\}$$

and let $\Phi_u : \mathcal{D}(Z_{\emptyset,D}) \to \mathcal{D}(Z_{\emptyset,D})$ be the composition $\mathfrak{h}_!\mathfrak{j}^*$ where $\mathfrak{j} : \Upsilon_u \to Z_{\emptyset,D}$ is $(B, B', g(U_B \cap U_{B'}) \mapsto (B, gBg^{-1}, gU_B)$ and $\mathfrak{h} : \Upsilon_u \to Z_{\emptyset,D}$ is

$$(B, B', g(U_B \cap U_{B'}) \mapsto (B', gB'g^{-1}, gU_{B'}).$$

(A special case of definitions in 37.1.) Let

$$\Upsilon' = \{(B', B, \tilde{B}, \tilde{B}', gU_{B'}); B' \in \mathcal{B}, B \in \mathcal{B}, \tilde{B} \in \mathcal{B}, \tilde{B}' \in \mathcal{B},$$
$$gU_{B'} \in D/U_{B'}, \mathrm{pos}(B', B) = u^{-1}, \mathrm{pos}(\tilde{B}, \tilde{B}') = \epsilon(u), gB'g^{-1} = \tilde{B}'\},$$
$$s : \Upsilon_u \to \Upsilon', (B, B', g(U_B \cap U_{B'}) \mapsto (B', B, gBg^{-1}, gB'g^{-1}, gU_{B'}).$$

Note that s is an isomorphism. (We show this only at the level of sets. Define $s' : \Upsilon' \to \Upsilon_u$ by $(B', B, \tilde{B}, \tilde{B}', gU_{B'}) \mapsto (B, B', x(U_B \cap U_{B'}))$ where $x \in D$ is such that $xBx^{-1} = \tilde{B}$, $xU_{B'} = gU_{B'}$. This is well defined and clearly an inverse of s.) It follows that $\mathfrak{h}_!\mathfrak{j}^* = \mathfrak{h}'_!\mathfrak{j}'^*$ where

$$\mathfrak{h}' = \mathfrak{h}s' : \Upsilon' \to Z_{\emptyset,D} \text{ is } (B', B, \tilde{B}, \tilde{B}', gU_{B'}) \mapsto (B', \tilde{B}', gU_{B'}),$$
$$\mathfrak{j}' = \mathfrak{j}s' : \Upsilon' \to Z_{\emptyset,D} \text{ is } (B', B, \tilde{B}, \tilde{B}', gU_{B'}) \mapsto (B, \tilde{B}, xU_B)$$

and $x \in D$ is such that $xBx^{-1} = \tilde{B}, xU_{B'} = gU_{B'}$ (then $x(U_B \cap U_{B'})$ is well defined). We have a commutative diagram with a cartesian right square

$$\begin{array}{ccccc} C & \xleftarrow{\tilde{j}} & \tilde{C} & \xrightarrow{\tilde{h}} & C \\ \xi_D \downarrow & & \xi' \downarrow & & \xi_D \downarrow \\ Z_{\emptyset,D} & \xleftarrow{\mathfrak{j}'} & \Upsilon'_u & \xrightarrow{\mathfrak{h}'} & Z_{\emptyset,D} \end{array}$$

where ξ_D is as in 41.1,

$$\tilde{C} = \{(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in (G^0/U^*)^4; \\ h_1^{-1}h_2 \in B^*\dot{u}^{-1}B^*, h_3^{-1}h_4 \in B^*\delta\dot{u}\delta^{-1}B^*\},$$

$\tilde{h}$ is $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto (h_1U^*, h_4U^*)$, ξ' is

$$(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \\ \mapsto (h_1B^*h_1^{-1}, h_2B^*h_2^{-1}, h_3B^*h_3^{-1}, h_4B^*h_4^{-1}, h_4\delta h_1^{-1}U_{h_1B^*h_1^{-1}}),$$

$\tilde{j}$ is $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto (h_2t^{-1}U^*, h_3\tilde{t}U^*)$ where $t, \tilde{t} \in T$ are given by $h_1^{-1}h_2 \in U^*\dot{u}^{-1}tU^*$, $h_3^{-1}h_4 \in U^*\tilde{t}\delta\dot{u}\delta^{-1}U^*$.

We see that for $A \in \mathcal{D}(Z_{\emptyset,D})$ we have

$$\xi_D^*\Phi_u(A) = \xi_D\mathfrak{h}_!\mathfrak{j}^*A = \xi_D^*\mathfrak{h}'_!\mathfrak{j}'^*A = \tilde{h}_!\xi'^*\mathfrak{j}'^*A = \tilde{h}_!\tilde{j}^*\xi_D^*A.$$

Taking here $A = \underline{\dot{\mathcal{L}}}_w^\sharp$ (with $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, \mathcal{L} \in \lambda$ with $w\underline{D}\lambda = \lambda$) and using 41.1(c) we obtain $\xi_D^*\Phi_u(\underline{\dot{\mathcal{L}}}_w^\sharp) = \tilde{h}_!\tilde{j}^*(\underline{(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})}_w^\sharp)$ or equivalently $\xi_D^*\Phi_u(\underline{\dot{\mathcal{L}}}_w^\sharp) = \bar{\sigma}_!\tilde{j}'^*((\mathrm{Ad}(\delta^{-1})^*\mathcal{L})_w^\sharp)$ where

$$\bar{X} = \{(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in \tilde{C}; h_2^{-1}h_3 \in \overline{B^*\dot{w}B^*}\}$$

and $\tilde{j}' : \bar{X} \to \bar{C}_w, \bar{\sigma} : \bar{X} \to C$ are the restrictions of $\tilde{j}, \tilde{h}$. Let

$$\bar{X}_0 = \{(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in \tilde{C}; h_2^{-1}h_3 \in B^*\dot{w}B^*\}$$

and let $\tilde{j}'_0 : \bar{X}_0 \to C_w$ be the restriction of $\tilde{j}$. Let $\mathcal{F}_0 = \tilde{j}'^*_0(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})$, a local system on $\bar{X}_0$. Since $\tilde{j}'$ is a fibration with smooth connected fibres, we have $\tilde{j}'^*((\mathrm{Ad}(\delta^{-1})^*\mathcal{L})_w^\sharp) = IC(\bar{X}, \mathcal{F}_0)$. Thus, $\xi_D^*\Phi_u(\underline{\dot{\mathcal{L}}}_w^\sharp) = \bar{\sigma}_!(IC(\bar{X}, \mathcal{F}_0))$. From the definitions we see that $\mathcal{F}_0 = \bar{\tau}^*\mathcal{L}''$, hence $\bar{\sigma}_!(IC(\bar{X}, \mathcal{F}_0)) = \bar{K}$ and

(a) $$\xi_D^*\Phi_u(\underline{\dot{\mathcal{L}}}_w^\sharp) = \bar{K}$$

where $\bar{\tau}^*\mathcal{L}', \bar{K}$ are given as in 40.10 in terms of

$$(u^{-1}, \mathcal{L}), (w, \mathrm{Ad}(\delta^{-1})^*\mathcal{L}), (\epsilon(u), \mathrm{Ad}(\delta\dot{u}\delta^{-1})^*\mathrm{Ad}(\delta^{-1})^*\mathcal{L})$$

instead of $(w, \mathcal{L}), (w', \mathcal{L}'), (w'', \mathcal{L}'')$.

41.3. For $J \subset \mathbf{I}$ let $\mathcal{D}_J^{cs}(C)$ be the subcategory of $\mathcal{D}^{cs}(C)$ whose objects are those $K \in \mathcal{D}(C)$ such that for any j, any simple subquotient of ${}^pH^jK$ is isomorphic to $\underline{\mathcal{L}}_w^\sharp$ for some $\mathcal{L} \in \mathfrak{s}$ and some $w \in \mathbf{W}_J$.

Let $J, J' \subset \mathbf{I}$. Let $K \in \mathcal{D}_J^{cs}(C), K' \in \mathcal{D}_{J'}^{cs}(C)$, and let $w', w'' \in \mathbf{W}$, $\lambda', \lambda'' \in \underline{\mathfrak{s}}$, $\mathcal{L}' \in \lambda', \mathcal{L}'' \in \lambda''$. Let $A = \underline{\mathcal{L}''}_{w''}^\sharp[d_{w''}]$. We show that

(a) *if (i)* $A \dashv K * \underline{\mathcal{L}'}_{w'}^\sharp[d_{w'}]$ *or (ii)* $A \dashv \underline{\mathcal{L}'}_{w'}^\sharp[d_{w'}] * K'$ *or (iii)* $A \dashv K * \underline{\mathcal{L}'}_{w'}^\sharp[d_{w'}] * K'$, *then* $(w'', \lambda'') \preceq_{J,J'} (w', \lambda')$.

For the proof we may assume that $\mathbf{k}$ is an algebraic closure of a finite field. Then the results in 40.7 are applicable. We first consider the case (i). In this case we can find $\mathcal{L} \in \mathfrak{s}, w \in \mathbf{W}_J$ such that $A \dashv \underline{\mathcal{L}}_w^\sharp[d_w] * \underline{\mathcal{L}'}_{w'}^\sharp[d_{w'}]$. By 40.11(a), $[w'', \lambda'']$ appears with nonzero coefficient in the expansion of the product $[w, \lambda] * [w', \lambda']$ in terms of the basis $([y, \nu])$ of $\mathfrak{K}(C)$. Applying ω (see 40.7(b)) we see that $c_{w'', \lambda''}$ appears with nonzero coefficient in the expansion of the product $c_{w,\lambda} c_{w',\lambda'}$ in terms of the basis $(c_{y,\nu})$ of H and the desired result follows. Case (ii) is treated in an entirely similar way. We now consider case (iii). In this case we must have $A \dashv A' * K'$ for some simple perverse sheaf A' such that $A' \dashv K * \underline{\mathcal{L}'}_{w'}^\sharp[d_{w'}]$. We have $A' = \underline{\mathcal{M}}_y^\sharp[d_y]$ where $y \in \mathbf{W}$, $\mathcal{M} \in \mathfrak{s}$. Let ν be the isomorphism class of $\mathcal{M}$. From case (ii) applied to $A \dashv A' * K'$ we see that $(w'', \lambda'') \preceq_{J,J'} (y, \nu)$. From case (i) applied to $A' \dashv K * \underline{\mathcal{L}'}_{w'}^\sharp[d_{w'}]$ we see that $(y, \nu) \preceq_{J,J'} (w', \lambda')$. Combining these two inequalities we obtain $(w'', \lambda'') \preceq_{J,J'} (w', \lambda')$, as desired.

41.4. Let $J \subset \mathbf{I}$. In the remainder of this section we write $\mathfrak{f}, \mathfrak{e}$ instead of $\mathfrak{f}_{\emptyset,J}$: $\mathcal{D}(Z_{\emptyset,D} \to \mathcal{D}(Z_{J,D})$, $\mathfrak{e}_{\emptyset,J} : \mathcal{D}(Z_{J,D} \to \mathcal{D}(Z_{\emptyset,D})$. We note that

(a) *if* $A \in \mathcal{D}(Z_{J,D})$, *then* $\mathfrak{f}\mathfrak{e}(A) \cong A[m] \oplus A'$ *for some* $m \in \mathbf{Z}$ *and some* $A' \in \mathcal{D}(Z_{J,D})$.

See [G], [MV] for the special case $D = G^0, J = \mathbf{I}$ and [L10, 6.6] for the general case. Now:

(b) *Let A be a simple perverse sheaf on $Z_{J,D}$. Then $A \dashv \mathfrak{f}({}^pH^j(\mathfrak{e}(A)))$ for some $j \in \mathbf{Z}$.*

Assume that this is not true. As in [BBD, p. 142], for any $n \in \mathbf{Z}$ we have a distinguished triangle $({}^p\tau_{\le n-1}\mathfrak{e}A, {}^p\tau_{\le n}\mathfrak{e}A, {}^pH^n(\mathfrak{e}A)[-n])$, hence a distinguished triangle

$$(\mathfrak{f}({}^p\tau_{\le n-1}\mathfrak{e}A), \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A), \mathfrak{f}({}^pH^n(\mathfrak{e}A))[-n]).$$

Using our assumption, we see that $A \dashv \mathfrak{f}({}^p\tau_{\le n-1}\mathfrak{e}A)$ if and only if $A \dashv \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A)$. Thus we have $A \dashv \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A)$ for some n if and only if $A \dashv \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A)$ for any n. Since ${}^p\tau_{\le n}\mathfrak{e}A = 0$ for some n, we see that $A \not\dashv \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A)$ for any n. Since ${}^p\tau_{\le n}\mathfrak{e}A = \mathfrak{e}A$ for some n, we deduce that $A \not\dashv \mathfrak{f}\mathfrak{e}A$. This contradicts (a); (b) is proved.

We show that

(c) *if A is a simple perverse sheaf on $Z_{J,D}$, then there exists a simple perverse sheaf A' on $Z_{\emptyset,D}$ such that $A \dashv \mathfrak{f}(A'), A' \dashv \mathfrak{e}(A)$.*

By (b) we can find $i, j \in \mathbf{Z}$ such that $A \dashv {}^pH^i(\mathfrak{f}(P))$ where $P = {}^pH^j(\mathfrak{e}(A))$.

Assume that $A \not\dashv {}^pH^i(\mathfrak{f}(A'))$ for any simple subquotient A' of P. We claim that $A \not\dashv {}^pH^i(\mathfrak{f}(P'))$ for any subobject P' of P. We argue by induction on the length of P'. If P' has length 1, the claim holds by assumption. If P' has length ≥ 2, we can find a simple subobject P'' of P'. We have a distinguished triangle $(\mathfrak{f}(P''), \mathfrak{f}(P'), \mathfrak{f}(P'/P''))$. Hence we have an exact sequence ${}^pH^i(\mathfrak{f}(P'')) \to {}^pH^i(\mathfrak{f}(P')) \to {}^pH^i(\mathfrak{f}(P'/P''))$. By the induction hypothesis, we have $A \not\dashv {}^pH^i(\mathfrak{f}(P''))$, $A \not\dashv {}^pH^i(\mathfrak{f}(P'/P''))$. Hence $A \not\dashv {}^pH^i(\mathfrak{f}(P'))$. This proves the claim. In particular, $A \not\dashv {}^pH^i(\mathfrak{f}(P))$, contradicting the definition of i, P.

We see that there exists a simple subquotient A' of P such that $A \dashv {}^pH^i(\mathfrak{f}(A'))$. Then A' is as required by (c).

Let $\bar{d}_w = \dim Z_{\emptyset,D}^w$. Let

(d) $A' = \underline{\dot{\mathcal{L}}}_w^\sharp[\bar{d}_w], A'' = \underline{\dot{\mathcal{M}}}_y^\sharp[\bar{d}_y] \in \hat{Z}_{\emptyset,D}$, $\mathcal{L} \in \lambda, \mathcal{M} \in \nu$.

Here $w\underline{D}\lambda=\lambda, y\underline{D}\nu=\nu$. Now:

(e) *Let A be a character sheaf on $Z_{J,D}$ such that $A\dashv\mathfrak{f}(A'), A''\dashv\mathfrak{e}(A)$. Then $(y,\underline{D}\nu)\preceq_{J,J'}(w,\underline{D}\lambda)$.*

Since $\mathfrak{f}$ is proper, $\mathfrak{f}(A')$ is a semisimple complex (see [BBD]). Hence $\mathfrak{f}(A')\cong A[m]\oplus A_1$ for some $m\in\mathbf{Z}, A'\in\mathcal{D}(Z_{J,D})$ and $\mathfrak{e}\mathfrak{f}(A')\cong\mathfrak{e}(A)[m]\oplus\mathfrak{e}(A_1)$. Hence from $A''\dashv\mathfrak{e}(A)$ we can deduce $A''\dashv\mathfrak{e}\mathfrak{f}(A')$. By 37.2 we have $\mathfrak{e}\mathfrak{f}(A')\approx\{\Phi_u(A')[[-m_u]];u\in\mathbf{W}_J\}$ where m_u are certain integers. Hence for some $u\in\mathbf{W}_J$ we have $A''\dashv\Phi_u(A')[[-m_u]]$, that is, $A''\dashv\Phi_u(A')$ and $\xi_D^*A''[\mathbf{r}]\dashv\xi_D^*\Phi_u(A')[\mathbf{r}]$. Hence using 41.2(a) we have $\xi_D^*A''[\mathbf{r}]\dashv\bar{K}$ where $\bar{K}$ is as in the end of 41.2. Thus, $\underline{\mathcal{M}}_y^\sharp[d_y]\dashv\bar{K}$. Using 40.10(b) we deduce that

$$\underline{\mathcal{M}}_y^\sharp[d_y]\dashv\underline{\mathrm{Ad}(\dot{w})^{-1})^*\mathrm{Ad}(\delta^{-1})^*\mathcal{L}}_{u^{-1}}*\underline{(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})}^\sharp_w*\underline{\mathrm{Ad}(\delta\dot{u}\delta^{-1})^*\mathrm{Ad}(\delta^{-1})^*\mathcal{L}}_{\epsilon(u)}.$$

Using this and 41.3(a) we see that (e) holds.

Now:

(f) *Let A be a character sheaf on $Z_{J,D}$. In the setup of (d) assume that $A\dashv\mathfrak{f}(A')$, $A'\dashv\mathfrak{e}(A)$, $A\dashv\mathfrak{f}(A'')$, $A''\dashv\mathfrak{e}(A)$. Then $(y,\underline{D}\nu)\sim_{J,J'}(w,\underline{D}\lambda)$.*

Applying (e) to A',A'' we see that $(y,\underline{D}\nu)\preceq_{J,J'}(w,\underline{D}\lambda)$. Applying (e) to A'',A' (instead of A',A'') we see that $(w,\underline{D}\lambda)\preceq_{J,J'}(y,\underline{D}\nu)$. Hence (f) holds.

From (c),(f) we see that there is a well-defined map $A\mapsto\mathbf{c}_A$ from the set of character sheaves on $Z_{J,D}$ (up to isomorphism) to the set of (J,J')-two-sided cells in $\mathbf{W}\times\underline{\mathcal{F}}$ where $\mathbf{c}_A$ is the unique (J,J')-two-sided cell that contains

$$\{(w,\underline{D}\lambda)\in\mathbf{W}\times\underline{\mathfrak{s}};w\underline{D}\lambda=\lambda, A\dashv\mathfrak{f}(\underline{\dot{\mathcal{L}}}_w^\sharp[\bar{d}_w]),\underline{\dot{\mathcal{L}}}_w^\sharp[\bar{d}_w]\dashv A\}$$

(a nonempty set); here $\mathcal{L}\in\lambda$.

41.5. In the setup of 41.4, let A be a character sheaf on $Z_{J,D}$. We show that:

(a) *There exists $(w,\underline{D}\lambda)\in\mathbf{c}_A$ such that $w\underline{D}\lambda=\lambda$, $A\dashv\mathfrak{f}(\underline{\dot{\mathcal{L}}}_w^\sharp[\bar{d}_w])$. If $(w',\underline{D}\lambda')\in\mathbf{W}\times\underline{\mathfrak{s}}$ is such that $w'\underline{D}\lambda'=\lambda'$, $A\dashv\mathfrak{f}(\underline{\dot{\mathcal{L}}'}_{w'}{}^\sharp[\bar{d}_{w'}])$, then $(w,\underline{D}\lambda)\preceq_{J,J'}(w',\underline{D}\lambda')$. Here $\mathcal{L}\in\lambda,\mathcal{L}'\in\lambda'$.*

(b) *There exists $(w,\underline{D}\lambda)\in\mathbf{c}_A$ such that $w\underline{D}\lambda=\lambda$, $\underline{\dot{\mathcal{L}}}_w^\sharp[\bar{d}_w]\dashv\mathfrak{e}(A)$. If $(w',\underline{D}\lambda')\in\mathbf{W}\times\underline{\mathfrak{s}}$ is such that $w'\underline{D}\lambda'=\lambda'$, $\underline{\dot{\mathcal{L}}'}_{w'}{}^\sharp[\bar{d}_{w'}]\dashv\mathfrak{e}(A)$, then $(w',\underline{D}\lambda')\preceq_{J,J'}(w,\underline{D}\lambda)$. Here $\mathcal{L}\in\lambda,\mathcal{L}'\in\lambda'$.*

Note that (a) follows immediately from 41.4(c),(e) and the definition of $\mathbf{c}_A$. Similarly, (b) follows from 41.4(c),(e) and the definition of $\mathbf{c}_A$.

41.6. In this subsection we assume that $J=\mathbf{I}$. The $\mathcal{A}$ linear map $H\to H$ given by

(a) $\tilde{T}_w1_\lambda\mapsto\tilde{T}_{\epsilon(w)}1_{\underline{D}\lambda}$ for $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}$

is an $\mathcal{A}$-algebra isomorphism. It carries $c_{w,\lambda}$ to $c_{\epsilon(w),\underline{D}\lambda}$ for any $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}$. It induces a bijection $\mathbf{c}\mapsto\mathbf{c}'$ from the set of two-sided cells in $\mathbf{W}\times\underline{\mathfrak{s}}$ onto itself. We show that

(b) *if A is a character sheaf on D, then $(\mathbf{c}_A)'=\mathbf{c}_A$.*

Consider the automorphism $\mathrm{Ad}(\delta):D\to D$. From the definitions we see that for $(w,\lambda)\in\mathbf{W}\times\underline{\mathfrak{s}}$ such that $w\underline{D}\lambda=\lambda$ we have $A\dashv\mathfrak{f}(\underline{\dot{\mathcal{L}}}_w^\sharp[\bar{d}_w])$ if and only if

$\mathrm{Ad}(\delta^{-1})^*A \dashv \mathfrak{f}(\underline{\mathrm{Ad}(\dot{\underline{D}}^{-1})^*\mathcal{L}^\sharp_{\epsilon(w)}}[\bar{d}_w])$. Using this and 41.5(a) we see that

$$\mathbf{c}_{\mathrm{Ad}(\delta^{-1})^*A} = (\mathbf{c}_A)'.$$

It is then enough to show that $\mathrm{Ad}(\delta^{-1})^*A \cong A$. By the G^0-equivariance of A we have $m^*A \cong q^*A$ where $m : G^0 \times D \to D$ is $(x,g) \mapsto xgx^{-1}$ and $q : G^0 \times D \to D$ is $(x,g) \mapsto g$. Define $r : D \to G^0 \times D$ by $r(g) = (\delta g^{-1}, g)$. Then $r^*m^*A \cong r^*q^*A$ that is, $(mr)^*A \cong (qr)^*A$. We have $mr = \mathrm{Ad}(\delta)$, $qr = 1$, hence $\mathrm{Ad}(\delta)^*A \cong A$ and $\mathrm{Ad}(\delta^{-1})^*A \cong A$, as required.

Note also that for (w,λ) as above we have

(c) $\mathfrak{f}(\underline{\mathrm{Ad}(\dot{\underline{D}}^{-1})^*\mathcal{L}^\sharp_{\epsilon(w)}}[\bar{d}_w]) \cong \mathfrak{f}(\underline{\dot{\mathcal{L}}^\sharp_w}[\bar{d}_w])$.

Indeed, let $K = \mathfrak{f}(\underline{\dot{\mathcal{L}}^\sharp_w}[\bar{d}_w])$. Clearly, we have $m^*K \cong q^*K$ with m, q as above. Then as in the proof of (b) we see that $\mathrm{Ad}(\delta)^*K \cong K$. From the definitions we see that $\mathfrak{f}(\underline{\mathrm{Ad}(\dot{\underline{D}}^{-1})^*\mathcal{L}^\sharp_{\epsilon(w)}}[\bar{d}_w]) = \mathrm{Ad}(\delta^{-1})^*K$. Since $\mathrm{Ad}(\delta^{-1})^*K \cong K$, (c) follows.

41.7. In this and the next subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field. From 41.1(c) we see that $\xi_D^* : \mathcal{D}(Z_{\emptyset,D}) \to \mathcal{D}(C)$ restricts to a functor $\mathcal{D}^{cs}(Z_{\emptyset,D}) \to \mathcal{D}^{cs}(C)$, hence, as in 36.8, the $\mathcal{A}$-linear map $gr(\xi_D^*) : \mathfrak{K}(Z_{\emptyset,D}) \to \mathfrak{K}(C)$ is well defined; from 41.1(c) we see also that

(a) $gr(\xi_D^*)(\underline{\dot{\mathcal{L}}^\sharp_w}[\bar{d}_w]) = (-v)^{\mathbf{r}}[w; \underline{D}\lambda]$

for $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}$ such that $w\underline{D}\lambda = \lambda$ and $\mathcal{L} \in \lambda$. From (a) we see that $gr(\xi_D^*)$ is injective with image equal to $\mathfrak{K}(C)^D$, the $\mathcal{A}$-submodule of $\mathfrak{K}(C)$ spanned by $\{[w; \underline{D}\lambda]; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda = \lambda\}$ or equivalently by $\{[w; \underline{D}\lambda]'; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda = \lambda\}$. Thus, $gr(\xi_D^*)$ defines an isomorphism $\eta' : \mathfrak{K}(Z_{\emptyset,D}) \xrightarrow{\sim} \mathfrak{K}(C)^D$. Let $\eta = \eta'^{-1}$.

Let $n \in \mathbf{N}_{\mathbf{k}}^*$. Let $\mathfrak{K}(C)^D_n$ be the $\mathcal{A}$-submodule of $\mathfrak{K}(C)$ spanned by $\{[w; \underline{D}\lambda]; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda\}$ or equivalently by $\{[w; \underline{D}\lambda]'; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda\}$.

Let $u, w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$ be such that $w\underline{D}\lambda = \lambda$ and let $\mathcal{L} \in \lambda$. From 37.3(c) we see that the $\mathcal{A}$-linear map $gr(\Phi_u) : \mathfrak{K}(Z_{\emptyset,D}) \to \mathfrak{K}(Z_{\emptyset,D})$ is well defined; we denote it again by Φ_u. From 40.10(a), 41.2(a) we have

$$[u^{-1}; \lambda]' * [w; \underline{D}\lambda]'^\sharp * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)]' = (v^2-1)^{2\mathbf{r}}\eta'\Phi_u\eta([w; \underline{D}\lambda]'^\sharp),$$

equality in $\mathfrak{K}(C)$. If $\lambda' \in \underline{\mathfrak{s}}_n, \lambda' \neq \lambda$, we have (from 40.7) that $[u^{-1}; \lambda']' * [w; \underline{D}\lambda]'^\sharp * [\epsilon_D(u); \underline{D}(u^{-1}\lambda')]' = 0$. It follows that

$$(v^2-1)^{2\mathbf{r}}\eta'\Phi_u\eta([w, \underline{D}\lambda]'^\sharp)) = \sum_{\lambda' \in \underline{\mathfrak{s}}_n} [u^{-1}; \lambda']' * [w; \underline{D}\lambda]'^\sharp * [\epsilon_D(u); \underline{D}(u^{-1}\lambda')]'.$$

Using this and the definition of $\mathfrak{K}(C)^D_n$ we see that

$$(v^2-1)^{2\mathbf{r}}\eta'\Phi_u\eta(x) = \sum_{\lambda' \in \underline{\mathfrak{s}}_n} [u^{-1}; \lambda']' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda')]'$$

for any $x \in \mathfrak{K}(C)^D_n$. Applying η to both sides we obtain

(b) $$(v^2-1)^{2\mathbf{r}}\Phi_u\eta'(x) = \sum_{\lambda' \in \underline{\mathfrak{s}}_n} \eta([u^{-1}; \lambda']' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda')]')$$

for any $x \in \mathfrak{K}(C)^D_n$.

41.8. In the setup of 41.4, let A be a character sheaf on $Z_{J,D}$. From 36.9(b) we see that the condition that, if $(w',\underline{D}\lambda')\in\mathbf{W}\times\underline{\mathfrak{s}}$ is such that $w'\underline{D}\lambda'=\lambda'$, then we have $A\dashv\mathfrak{f}(\underline{\dot{\mathcal{L}}}'_{w'}{}^\sharp[\bar d_{w'}])$ if and only if A appears with coefficient $\ne 0$ in the expansion of $\mathfrak{f}(\underline{\dot{\mathcal{L}}}'_{w'}{}^\sharp[\bar d_{w'}])\in\mathfrak{K}(Z_{J,D})$ as a linear combination of the canonical basis of $\mathfrak{K}(Z_{J,D})$. Hence from 41.5(a) we deduce:

(a) *There exists $(w,\underline{D}\lambda)\in\mathbf{c}_A$ such that $w\underline{D}\lambda=\lambda$ and A appears with nonzero coefficient in $\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w])\in\mathfrak{K}(Z_{J,D})$. If $(w',\underline{D}\lambda')\in\mathbf{W}\times\underline{\mathfrak{s}}$ is such that $w'\underline{D}\lambda'=\lambda'$ and A appears with nonzero coefficient in $\mathfrak{f}(\underline{\dot{\mathcal{L}}}'_{w'}{}^\sharp[\bar d_{w'}])\in\mathfrak{K}(Z_{J,D})$, then $(w,\underline{D}\lambda)\preceq_{J,J'}(w',\underline{D}\lambda')$. Here $\mathcal{L}\in\lambda,\mathcal{L}'\in\lambda'$.*

Clearly, property (a) characterizes $\mathbf{c}_A$.

41.9. Let $J\subset J'\subset\mathbf{I}$ and let D' be another connected component of G. Let $A_0\in\mathcal{D}(Z_{J,D})$, $A'\in\mathcal{D}(Z_{\epsilon_D(J'),D'})$. We show that

(a) $\mathfrak{f}_{J,J'}(A_0)*A'\cong\mathfrak{f}_{J,J'}(A_0*\mathfrak{e}_{\epsilon_D(J),\epsilon_D(J')}A')$ in $\mathcal{D}(Z_{J',D'D})$.

Indeed, from the definitions we see that both sides of (a) can be identified with $b_!c^*(A_0\boxtimes A')$ where b,c are as in the diagram

$$Z_{J,D}\times Z_{\epsilon_D(J'),D'}\xleftarrow{c}Y\xrightarrow{b}Z_{J',D'D}$$

where

$$\begin{aligned}Y=&\{(P,R,R',gU_R,g'U_{R'});P\in\mathcal{P}_J,R\in\mathcal{P}_{J'},R'\in\mathcal{P}_{\epsilon_D(J')},\\&gU_R\in D/U_R,g'U_{R'}\in D'/U_{R'},gRg^{-1}=R',P\subset R\},\end{aligned}$$

c is $(P,R,R',gU_R,g'U_{R'})\mapsto((P,gU_P),(R',g'U_{R'})$, b is $(P,R,R',gU_R,g'U_{R'})\mapsto(R,g'gU_R)$.

An entirely similar proof shows that, if $A\in\mathcal{D}(Z_{J',D})$, $A'_0\in\mathcal{D}(Z_{\epsilon_D(J),D'})$, then

(b) $A*\mathfrak{f}_{\epsilon_D(J),\epsilon_D(J')}(A'_0)\cong\mathfrak{f}_{J,J'}(\mathfrak{e}_{J,J'}A*A'_0)$ in $\mathcal{D}(Z_{J',D'D})$.

41.10. Let $\mathbf{c}$ be a two-sided cell in $\mathbf{W}\times\underline{\mathfrak{s}}$. Let $\bar{\mathbf{c}}$ be the set of all $(w,\lambda)\in\mathbf{W}\times\underline{\mathfrak{s}}$ such that $(w,\lambda)\preceq_{\mathbf{I},\mathbf{I}}(y,\nu)$ for some/any $(y,\nu)\in\mathbf{c}$.

If $K\in\mathcal{D}(Z_{\emptyset,D})$, we say that $K\in\mathcal{D}^{cs}_{\bar{\mathbf{c}}}(Z_{\emptyset,D})$ if for any $j\in\mathbf{Z}$ and simple subquotient A of ${}^pH^j(K)$ satisfies $\mathbf{c}_A\subset\bar{\mathbf{c}}$.

Let D' be another connected component of G. We show that

(a) *if $K\in\mathcal{D}^{cs}_{\bar{\mathbf{c}}}(Z_{\emptyset,D})$, $K'\in\mathcal{D}^{cs}(Z_{\epsilon_D(J'),D'})$, then $K*K'\in\mathcal{D}^{cs}_{\bar{\mathbf{c}}}(Z_{\emptyset,D'D})$.*

We may assume that $\mathbf{k}$ is an algebraic closure of a finite field. We may assume that $K\in\hat Z_{\emptyset,D}$ and $\mathbf{c}_K\subset\bar{\mathbf{c}}$. Then there exists $(w,\underline{D}\lambda)\in\mathbf{c}_K$ such that $w\underline{D}\lambda=\lambda$, $K\dashv\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w])$, $\mathcal{L}\in\lambda$. It is enough to show that, if $\tilde A\in\hat Z_{\emptyset,D'D}$ is such that $\tilde A\dashv K*K'$, then $\mathbf{c}_{\tilde A}\subset\bar{\mathbf{c}}$. Since $\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w])$ is a semisimple complex (see the line after 41.4(e)) we have $\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w])\cong K[m]\oplus\tilde K$ for some $m\in\mathbf{Z}$, $\tilde K\in\mathcal{D}(Z_{\emptyset,D'D})$. It follows that $\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w])*K'\cong K*K'[m]\oplus\tilde K*K'$ hence $\tilde A\dashv\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w])*K'$. By 41.9(a) we have $\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w])*K'\cong\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w]*\mathfrak{e}(K'))$ hence $\tilde A\dashv\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w]*\mathfrak{e}(K'))$. We deduce that there exists $K'_0\in\hat Z_{\emptyset,D'}$ such that $\tilde A\dashv\mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w]*K'_0)$ and $K''_0\in\hat Z_{\emptyset,D'D}$ such that $K''_0\dashv\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w]*K'_0$, $\tilde A\dashv\mathfrak{f}(K''_0)$. We then have $\xi^*_{D'D}K''_0[\mathbf{r}]\dashv\xi^*_{D'D}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w]*K'_0)[\mathbf{r}]$, hence, using 41.1(d), $\xi^*_{D'D}K''_0[\mathbf{r}]\dashv(\xi^*_D\underline{\dot{\mathcal{L}}}^\sharp_w[\bar d_w])*\xi_{D'}K'_0)$. Setting $gr(\xi^*_{D'D}K''_0[\mathbf{r}])=[w_1,\underline{D'}\underline{D}\lambda_1]\in\mathfrak{K}(C)$ with $(w_1,\lambda_1)\in\mathbf{W}\times\underline{\mathfrak{s}}$ we see, using 41.3(a) that $(w_1,\underline{D'}\underline{D}\lambda_1)\preceq_{\mathbf{I},\mathbf{I}}(w,\underline{D}\lambda)$. From $\tilde A\dashv\mathfrak{f}(K''_0)$ we see using 41.5(a) that

$\mathbf{c}_{\tilde{A}} \preceq_{\mathbf{I},\mathbf{I}} (w_1, \underline{D'}\,\underline{D}\lambda_1)$ (that is, some/any element of $\mathbf{c}_{\tilde{A}}$ is $\preceq_{\mathbf{I},\mathbf{I}} (w_1, \underline{D'}\,\underline{D}\lambda_1)$). Using the transitivity of $\preceq_{\mathbf{I},\mathbf{I}}$ we see that $\mathbf{c}_{\tilde{A}} \preceq_{\mathbf{I},\mathbf{I}} (w, \underline{D}\lambda)$. This proves (a).

An entirely similar argument shows that

(b) *if* $K \in \mathcal{D}^{cs}(Z_{\emptyset,D})$, $K' \in \mathcal{D}^{cs}_{\tilde{\mathbf{c}}}(Z_{\epsilon_D(J'),D'})$, *then* $K * K' \in \mathcal{D}^{cs}_{\tilde{\mathbf{c}}}(Z_{\emptyset,D'D})$.

42. Duality and the functor $\mathfrak{f}_{\emptyset,\mathbf{I}}$

42.1. In this section we fix a connected component D of G. We write ϵ instead of $\epsilon_D : \mathbf{W} \to \mathbf{W}$. We write $\mathfrak{f}$ instead of $\mathfrak{f}_{\emptyset,\mathbf{I}} : \mathcal{D}(Z_{\emptyset,D} \to \mathcal{D}(Z_{\mathbf{I},D})$. We assume that $\mathbf{k}$ is an algebraic closure of a finite field.

Let $J \subset \mathbf{I}$ be such that $\epsilon(J) = J$. Recall from 30.3 that $V_{J,D} = \{(P, gU_P); P \in \mathcal{P}_J, gU_P \in N_D P/U_P\}$. As in 30.4 (with $J' = \mathbf{I}$) we consider the diagram $V_{J,D} \xleftarrow{c} V_{J,\mathbf{I},D} \xrightarrow{d} D$ where $V_{J,\mathbf{I},D} = \{(P, g); P \in \mathcal{P}_J, g \in N_D P\}$, c is $(P, g) \mapsto (P, gU_P)$ and d is $(P, g) \mapsto g$. Define $\tilde{f}_J : \mathcal{D}(V_{J,D}) \to \mathcal{D}(D)$, $\tilde{e}_J : \mathcal{D}(D) \to \mathcal{D}(V_{J,D})$ by $\tilde{f}_J A = d_! c^* A$, $\tilde{e}_J A' = c_! d^* A'$. (In the notation of 30.4 we have $\tilde{f}_J = \tilde{f}_{J,\mathbf{I}}$, $\tilde{e}_J = \tilde{e}_{J,\mathbf{I}}$.) Define $f_J : \mathcal{D}(V_{J,D}) \to \mathcal{D}(D)$, $e_J : \mathcal{D}(D) \to \mathcal{D}(V_{J,D})$ by $f_J A = \tilde{f}_J A[[\alpha_J/2]]$, $e_J A = \tilde{e}_J A[[\alpha_J/2]]$ where $\alpha_J = \dim \mathcal{P}_J$. (In the notation of 30.4 we have $f_J A = f_{J,\mathbf{I}} A(\alpha_J/2)$, $e_J A = e_{J,\mathbf{I}} A(-\alpha_J/2)$. Thus, f_J, e_J are the same, up to a twist, as $f_{J,\mathbf{I}}, e_{J,\mathbf{I}}$.)

From 30.5 (with $J' = \mathbf{I}$) we see that for $A \in \mathcal{D}(V_{J,D})$, $A' \in \mathcal{D}(D)$ we have canonically

(a) $\mathrm{Hom}_{\mathcal{D}(V_{J,D})}(e_J A', A) = \mathrm{Hom}_{\mathcal{D}(D)}(A', f_J A)$.

Let $CS(V_{J,D}), CS(D)$ be as in 38.1. From 38.2, 38.3 we see that

(b) f_J, e_J *restrict to functors* $CS(V_{J,D}) \to CS(D)$, $CS(D) \to CS(V_{J,D})$ *denoted again by* f_J, e_J.

We show that

(c) *if* $A \in CS(V_{J,D})$ *comes from a pure complex of weight* 0 *with respect to a rational structure over a finite subfield of* $\mathbf{k}$, *then* $f_J A$ *(naturally regarded as a mixed complex) is pure of weight* 0.

Indeed, the functor c^* preserves pure complexes of weight 0 since c is smooth with connected fibres; the functor $d_!$ preserves pure complexes of weight 0 since d is proper (see [De, 6.2.6]) and $[[\alpha_J/2]]$ also preserves pure complexes of weight 0.

We show that

(d) *if* $A' \in CS(D)$ *comes from a pure complex of weight* 0 *with respect to a rational structure over a finite subfield of* $\mathbf{k}$, *then* $e_J A'$ *(naturally regarded as a mixed complex) is pure of weight* 0.

Using (b), it is enough to show that for any simple A as in (c), the natural action of Frobenius on the vector space $\mathrm{Hom}_{\mathcal{D}(V_{J,D})}(e_J A', A)$ has weight 0. Using (a) we see that it is enough to show that the natural action of Frobenius on the vector space $\mathrm{Hom}_{\mathcal{D}(D)}(A', f_J A)$ has weight 0. This follows from (c) using (b).

Define an imbedding $s : V_{J,D} \to Z_{J,D}$ by $(P, gU_P) \mapsto (P, P, gU_P)$. From the definitions we see that

(e) $\tilde{f}_J : \mathcal{D}(V_{J,D}) \to \mathcal{D}(D)$ *is the composition* $\mathcal{D}(V_{J,D}) \xrightarrow{s_!} \mathcal{D}(Z_{J,D}) \xrightarrow{\mathfrak{f}_{J,\mathbf{I}}} \mathcal{D}(D)$,

(f) $\tilde{e}_J : \mathcal{D}(D) \to \mathcal{D}(V_{J,D})$ *is the composition* $\mathcal{D}(D) \xrightarrow{\mathfrak{e}_{J,\mathbf{I}}} \mathcal{D}(Z_{J,D}) \xrightarrow{s^*} \mathcal{D}(V_{J,D})$.

Let $Y = \{(B, B', gU_B) \in Z_{\emptyset,D}; \mathrm{pos}(B, B') \in \mathbf{W}_J\}$ and let $r : Y \to Z_{\emptyset,D}$ be the inclusion. From the definitions we have

(g) $s_! s^* \mathfrak{f}_{\emptyset,J} = \mathfrak{f}_{\emptyset,J} r_! r^* : \mathcal{D}(Z_{\emptyset,D}) \to \mathcal{D}(Z_{J,D})$.

Note that $V_{J,D} = {}^1Z_{J,D}$ (see 36.2); hence the "character sheaves" on $V_{J,D} = {}^1Z_{J,D}$ are defined as in 36.8 and $\mathcal{D}^{cs}(V_{J,D} = \mathcal{D}^{cs}({}^1Z_{J,D}))$ is defined as 36.8. In particular, $\mathfrak{K}(V_{J,D}) = \mathfrak{K}({}^1Z_{J,D})$ is defined. Let $\mathfrak{K}_0(V_{J,D}) = \bigoplus_A \mathbf{Z}A \subset \mathfrak{K}(V_{J,D})$ where A runs through the character sheaves on $V_{J,D}$ (up to isomorphism).

From (b) we see that $\tilde{f}_J, \tilde{e}_J$ restrict to functors $\mathcal{D}^{cs}(V_{J,D}) \to \mathcal{D}^{cs}(D)$, $\mathcal{D}^{cs}(D) \to \mathcal{D}^{cs}(V_{J,D})$, hence the $\mathcal{A}$-linear maps $gr(\tilde{f}_J) : \mathfrak{K}(V_{J,D}) \to \mathfrak{K}(D)$, $gr(\tilde{e}_J) : \mathfrak{K}(D) \to \mathfrak{K}(V_{J,D})$ are well defined; we denote them by $\tilde{f}_J, \tilde{e}_J$. Define $f_J : \mathfrak{K}(V_{J,D}) \to \mathfrak{K}(D)$ by $f_J = (-v)^{-\alpha_J}\tilde{f}_J$ and $e_J : \mathfrak{K}(D) \to \mathfrak{K}(V_{J,D})$ by $e_J = (-v)^{-\alpha_J}\tilde{e}_J$. We show that

(h) $f_J : \mathfrak{K}(V_{J,D}) \to \mathfrak{K}(D)$, $e_J : \mathfrak{K}(D) \to \mathfrak{K}(V_{J,D})$ *restrict to group homomorphisms* $\mathfrak{K}_0(V_{J,D}) \to \mathfrak{K}_0(D)$, $\mathfrak{K}_0(D) \to \mathfrak{K}_0(V_{J,D})$ *denoted again by* f_J, e_J.

It is enough to prove the following statement. If x is a canonical basis element of $\mathfrak{K}(V_{J,D})$ (resp. $\mathfrak{K}(D)$), then $f_J(x)$ (resp. $e_J(x)$) is an $\mathbf{N}$-linear combination of canonical basis elements of $\mathfrak{K}(D)$ (resp. $\mathfrak{K}(V_{J,D})$). This is immediate from (c), (d).

Now, one checks easily that $r_!r^* : \mathcal{D}(Z_{\emptyset,D}) \to \mathcal{D}(Z_{\emptyset,D})$ restricts to a functor $\mathcal{D}^{cs}(Z_{\emptyset,D}) \to \mathcal{D}^{cs}(Z_{\emptyset,D})$. (Note that, if $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}$, $\mathcal{L} \in \lambda$ and $w\underline{D}\lambda = \lambda$, then $r_!r^*(\dot{\underline{\mathcal{L}}}_w) = \dot{\underline{\mathcal{L}}}_w$ for $w \in \mathbf{W}_J$ and $r_!r^*(\dot{\underline{\mathcal{L}}}_w) = 0$ for $w \in \mathbf{W} - \mathbf{W}_J$.) It follows that the $\mathcal{A}$-linear map $gr(r_!r^*) : \mathfrak{K}(Z_{\emptyset,D}) \to \mathfrak{K}(Z_{\emptyset,D})$ (denoted by ρ_J) is well defined.

Let $\mathfrak{K}(C)^D, \eta$ be as in 41.7. Define an $\mathcal{A}$-linear map $\tilde{\rho}_J : \mathfrak{K}(C)^D \to \mathfrak{K}(C)^D$ by $[w; \underline{D}\lambda]' \mapsto [w; \underline{D}\lambda]'$ if $w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda = \lambda$ and $[w; \underline{D}\lambda]' \mapsto 0$ if $w \in \mathbf{W} - \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda = \lambda$. From the definitions we see that

(i) $\rho_J\eta(x) = \eta\tilde{\rho}_J(x)$ for all $x \in \mathfrak{K}(C)^D$.

42.2. We define an $\mathcal{A}$-linear map $\mathbf{d} : \mathfrak{K}(D) \to \mathfrak{K}(D)$ by

$$\mathbf{d}(x) = \sum_{J; J \subset \mathbf{I}; \epsilon(J) = J} (-1)^{|J_\epsilon|} f_J e_J(x)$$

where f_J, e_J are as in 42.1(h) and J_ϵ is as in 38.1. Now:

(a) *Let A be a character sheaf on D. Then $\mathbf{d}(A) = \pm A'$ where A' is a character sheaf on D. Moreover, $\pm$ and A' are the same as in 38.11(a).*

For any $J \subset \mathbf{I}$ such that $\epsilon(J) = J$ let $\mathcal{K}(V_{J,D})$ be as in 38.9. We shall identify $\mathfrak{K}(V_{J,D})/(v-1)\mathfrak{K}(V_{J,D}) = \mathcal{K}(V_{J,D})$ as abelian groups in such a way that the image of A_1 (a character sheaf on $V_{J,D}$) in $\mathfrak{K}(V_{J,D})/(v-1)\mathfrak{K}(V_{J,D})$ is identified with the basis element A_1 of $\mathcal{K}(V_{J,D})$. From the definitions we see that the homomorphisms

$$\mathfrak{K}(D)/(v-1)\mathfrak{K}(D) \to \mathfrak{K}(V_{J,D})/(v-1)\mathfrak{K}(V_{J,D}) \to \mathfrak{K}(D)/(v-1)\mathfrak{K}(D)$$

induced by e_J, f_J in 42.1(h) are then identified with the homomorphisms

$$e_{J,\mathbf{I}} : \mathcal{K}(D) \to \mathcal{K}(V_{J,D}),\ f_{J,\mathbf{I}} : \mathcal{K}(V_{J,D}) \to \mathcal{K}(D)$$

in 38.2, 38.3. It follows that the endomorphism of $\mathfrak{K}(D)/(v-1)\mathfrak{K}(D)$ induced by $\mathbf{d} : \mathfrak{K}(D) \to \mathfrak{K}(D)$ is identified with the homomorphism $\mathcal{K}(D) \to \mathcal{K}(D)$ denoted in 38.10(a), 38.11 again by $\mathbf{d}$. Hence we have $\mathbf{d}(A) = \pm A' + (v-1)x$ (in $\mathfrak{K}(D)$) where $\pm, A'$ are as in 38.11(a) and $x \in \mathfrak{K}(D)$. From 42.1(h) we see that $\mathbf{d}(A) \in \mathfrak{K}_0(D)$. Since $\pm A' \in \mathfrak{K}_0(D)$, we see that $(v-1)x \in \mathfrak{K}_0(D)$. Since $\mathfrak{K}_0(D) \cap (v-1)\mathfrak{K}(D) = 0$, we have $(v-1)x = 0$ and $x = 0$. This proves (a).

42.3. We have $H = H_D \oplus H'_D$ where H_D (resp. H'_D) is the $\mathcal{A}$-submodule of H_n spanned by $\{\tilde{T}_w 1_{\underline{D}\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda = \lambda\}$ (resp. by $\{\tilde{T}_w 1_{\underline{D}\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda \neq \lambda\}$). Equivalently,

$$H_D = \sum_{\lambda \in \underline{\mathfrak{s}}} 1_\lambda H 1_{\underline{D}\lambda} \subset H, \ H'_D = \sum_{\lambda, \lambda' \in \underline{\mathfrak{s}}; \lambda \neq \lambda'} 1_{\lambda'} H 1_{\underline{D}\lambda} \subset H.$$

Recall that $\omega : \mathfrak{K}(C) \xrightarrow{\sim} H$ is defined in 40.7(b). Define an $\mathcal{A}$-linear map $\tilde{\omega} : H \to \mathfrak{K}(C)^D$ by

$$\tilde{\omega}(y) = \omega^{-1}(y) \ \text{ if } y \in H_D,$$
$$\tilde{\omega}(y) = 0 \ \text{ if } y \in H'_D.$$

Then $\eta\tilde{\omega}(y) \in \mathfrak{K}(Z_{\emptyset,D})$ is well defined for any $y \in H$. Here η is as in 41.7.

Let $n \in \mathbf{N}^*_{\mathbf{k}}$. Let $H_{n,D} = H_D \cap H_n$. Note that $H_{n,D}$ is the $\mathcal{A}$-submodule of H_n spanned by $\{\tilde{T}_w 1_{\underline{D}\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda\}$.

For $J \subset \mathbf{I}$ such that $\epsilon(J) = J$ we define an $\mathcal{A}$-linear map $\rho_{J,n} : H_{n,D} \to H_{n,D}$ by

$$\tilde{T}_w 1_{\underline{D}\lambda} \mapsto \tilde{T}_w 1_{\underline{D}\lambda} \ \text{ if } w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda,$$
$$\tilde{T}_w 1_{\underline{D}\lambda} \mapsto 0 \ \text{ if } w \in \mathbf{W} - \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda.$$

We have the following result.

Lemma 42.4. *For any $y \in H_{n,D}$ we have $\mathbf{d}(\mathfrak{f}\eta\tilde{\omega}(y)) = \mathfrak{f}\eta\tilde{\omega}(\delta(y))$ where*

$$\delta = \sum_{J \subset \mathbf{I}; \epsilon(J) = J} (-1)^{|J_\epsilon|} \delta_J$$

with $\delta_J : H_{n,D} \to H_{n,D}$ given by

$$\delta_J(y) = \rho_{J,n}(\sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}} y \tilde{T}_{\epsilon_D(u)})$$

(the sum in the right-hand side is computed in H_n but it belongs to $H_{n,D}$).

Applying 37.2 with K, K', J repaced by $\emptyset, J, \mathbf{I}$ and with $A' \in \mathcal{D}^{cs}(Z_{\emptyset,D})$ we obtain

$$\mathfrak{e}_{J,\mathbf{I}}\mathfrak{f}A' \approx \{\mathfrak{f}_{\emptyset,J}\Phi_u A'[[-m_u]]; u \in \mathbf{W}^J\}$$

(in $\mathcal{D}(Z_{J,D})$, with $\Phi_u : \mathcal{D}(Z_{\emptyset,D}) \to \mathcal{D}(Z_{\emptyset,D})$ as in 37.1 and $m_u = \alpha_J - \lambda(u)$ where $\alpha_J = \dim \mathcal{P}_J$. Applying here s^* we obtain

$$s^*\mathfrak{e}_{J,\mathbf{I}}\mathfrak{f}A' \approx \{s^*\mathfrak{f}_{\emptyset,J}\Phi_u A'[[-m_u]]; u \in \mathbf{W}^J\}.$$

We replace $s^*\mathfrak{e}_{J,\mathbf{I}}$ by $\tilde{e}_J$ (see 42.1(f)) and we apply $\tilde{f}_J = \mathfrak{f}_{J,\mathbf{I}}s_!$ (see 42.1(e)); we obtain

$$\tilde{f}_J\tilde{e}_J\mathfrak{f}A' \approx \{\mathfrak{f}_{J,\mathbf{I}}s_!s^*\mathfrak{f}_{\emptyset,J}\Phi_u A'[[-m_u]]; u \in \mathbf{W}^J\}.$$

Using now 42.1(g) we obtain

$$\tilde{f}_J\tilde{e}_J\mathfrak{f}A' \approx \{\mathfrak{f}_{J,\mathbf{I}}\mathfrak{f}_{\emptyset,J}r_!r^*\Phi_u A'[[-m_u]]; u \in \mathbf{W}^J\}.$$

Here we replace $\mathfrak{f}_{J,\mathbf{I}}\mathfrak{f}_{\emptyset,J}$ by $\mathfrak{f}$ (see 36.4(b)). This (or rather its mixed analogue) gives rise to the following equality in $\mathfrak{K}(D)$:

$$\tilde{f}_J\tilde{e}_J\mathfrak{f}(x') = \sum_{u \in \mathbf{W}^J} v^{2m_u}\mathfrak{f}\rho_J\Phi_u(x')$$

for any $x' \in \mathfrak{K}(Z_{\emptyset,D})$, or equivalently

$$f_J e_J \mathfrak{f}(x') = \sum_{u \in \mathbf{W}^J} v^{2m_u - 2\alpha_J} \mathfrak{f} \rho_J \Phi_u(x').$$

Taking $x' = \eta(x)$ where $x \in \mathfrak{K}(C)_n^D$ (see 41.7) and using 41.7(b) we obtain

$$(v^2-1)^{2\mathbf{r}} f_J e_J \mathfrak{f}\eta(x) = \sum_{u \in \mathbf{W}^J} \sum_{\lambda \in \underline{\mathfrak{s}}_n} v^{-2l(u)} \mathfrak{f} \rho_J \eta([u^{-1};\lambda]' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)]')$$

and using 42.1(i),

$$(v^2-1)^{2\mathbf{r}} f_J e_J \mathfrak{f}\eta(x) = \sum_{u \in \mathbf{W}^J} \sum_{\lambda \in \underline{\mathfrak{s}}_n} v^{-2l(u)} \mathfrak{f} \eta \tilde{\rho}_J([u^{-1};\lambda]' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)]')$$

for any $x \in \mathfrak{K}(C)_n^D$. Here we replace x by $\tilde{\omega}(y)$ where $y \in H_{n,D}$ and $\tilde{\rho}_J|_{\mathfrak{K}(C)_n^D}$ by $\tilde{\omega}|_{H_{n,D}} \rho_{J,n} \omega_{\mathfrak{K}(C)_n^D}$; using 40.7(b) we obtain:

$$f_J e_J \mathfrak{f}\eta\tilde{\omega}(y) = \sum_{u \in \mathbf{W}^J} \sum_{\lambda \in \underline{\mathfrak{s}}_n} \mathfrak{f}\eta\tilde{\omega}\rho_{J,n}(\tilde{T}_{u^{-1}} 1_\lambda y \tilde{T}_{\epsilon_D(u)} 1_{\underline{D}(u^{-1}\lambda)})$$
$$= \mathfrak{f}\eta\tilde{\omega}\rho_{J,n}(\sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}} y \tilde{T}_{\epsilon_D(u)}).$$

The lemma is proved.

42.5. As in 34.12 let $\mathfrak{U}$ be the subfield of $\bar{\mathbf{Q}}_l$ generated by the roots of 1. Let $\Phi : H_n^D \to \mathcal{A} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ be as in 34.12 (a special case of a definition in 34.1) and let $\Phi^1 : H_n^{D,1} \to \mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ be the specialization of Φ for $v = 1$ (see 34.12(b)). Let $\tilde{\mathcal{A}} = \mathfrak{U}[v, v^{-1}]$, let $H_n^{D,\tilde{\mathcal{A}}} = \tilde{\mathcal{A}} \otimes \mathcal{A} H_n^D$ and let $\Phi^{\tilde{\mathcal{A}}} : H_n^{D,\tilde{\mathcal{A}}} \to \tilde{\mathcal{A}} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ be the homomorphism obtained from Φ by extending the scalars from $\mathcal{A}$ to $\tilde{\mathcal{A}}$.

Let E be an $H_n^{D,1}$-module of finite dimension over $\mathfrak{U}$. Since Φ^1 is an isomorphism of $\mathfrak{U}$-algebras (see 34.12(b)) we may regard E as an $\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$-module E^∞ via $(\Phi^1)^{-1}$. By extension of scalars, $\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} E^\infty$ is naturally a module over

$$\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} (\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,iy}) = \tilde{\mathcal{A}} \otimes_{\mathbf{Z}} H_n^{D,iy}$$

and this can be regarded as an $H_n^{D,\tilde{\mathcal{A}}}$-module $E^{\tilde{\mathcal{A}}}$ via $\Phi^{\tilde{\mathcal{A}}}$.

Let $J \subset \mathbf{I}$ be such that $\epsilon(J) = J$. Let $H_{J,n}^D$ be the $\mathcal{A}$-algebra of H_n^D generated by $1_\lambda, \lambda \in \underline{\mathfrak{s}}_n$ by $\tilde{T}_w, w \in \mathbf{W}_J$ and by $\tilde{T}_{\underline{D}}$. Note that $\{\tilde{T}_{w\underline{D}'} 1_\lambda; w \in \mathbf{W}_J, \underline{D}' =$ power of $\underline{D}\}$ is an $\mathcal{A}$-basis of $H_{J,n}^D$. Let $H_{J,n}^{D,1} = \mathfrak{U} \otimes_{\mathcal{A}} H_{J,n}^D$ where $\mathfrak{U}$ is regarded as an $\mathcal{A}$-algebra via $v \mapsto 1$. Let $H_{J,n}^{D,\tilde{\mathcal{A}}} = \tilde{\mathcal{A}} \otimes_{\mathcal{A}} H_{J,n}^D$. Note that $H_{J,n}^{D,\tilde{\mathcal{A}}}$ is naturally a subalgebra of $H_n^{D,\tilde{\mathcal{A}}}$. Hence $E^{\tilde{\mathcal{A}}}$ may be regarded as an $H_{J,n}^{D,\tilde{\mathcal{A}}}$-module $(E^{\tilde{\mathcal{A}}})_J$. This $H_{J,n}^{D,\tilde{\mathcal{A}}}$-module may be induced to an $H_n^{D,\tilde{\mathcal{A}}}$-module

$$\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J) := H_n^{D,\tilde{\mathcal{A}}} \otimes_{H_{J,n}^{D,\tilde{\mathcal{A}}}} E_J^{\tilde{\mathcal{A}}}.$$

Next, $H_{J,n}^{D,1}$ is naturally a subalgebra of $H_n^{D,1}$. Hence E may be regarded as an $H_{J,n}^{D,1}$-module E_J. This $H_{J,n}^{D,1}$-module may be induced to an $H_n^{D,1}$-module $\mathrm{ind}(E_J) := H_n^{D,1} \otimes_{H_{J,n}^{D,1}} E_J$. Define an $H_{J,n}^{D,\tilde{\mathcal{A}}}$-module $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ in terms of $\mathrm{ind}(E_J)$

in the same way as $E^{\tilde{\mathcal{A}}}$ was defined in terms of E. By extension of scalars from $\tilde{\mathcal{A}}$ to $\mathfrak{U}(v)$ (the quotient field of $\tilde{\mathcal{A}}$), $\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$, $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ give rise to $\mathfrak{U}(v)\otimes_{\mathcal{A}} H_n^D$-modules $\mathfrak{U}(v)\otimes_{\tilde{\mathcal{A}}} \mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$, $\mathfrak{U}(v)\otimes_{\tilde{\mathcal{A}}} (\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$. We show that

(a) *these two* $\mathfrak{U}(v)\otimes_{\mathcal{A}} H_n^D$*-modules are isomorphic.*

Since $\mathfrak{U}(v)\otimes_{\mathcal{A}} H_n^D$, $H_n^{D,1}$ are (finite dimensional) semisimple algebras (see 34.12) it follows by standard arguments that it is enough to show that $\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$, $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ become isomorphic $H_n^{D,1}$-modules under the specialization $v=1$. First we note that under the specialization $v=1$, $E^{\tilde{\mathcal{A}}}$ becomes the $H_n^{D,1}$-module E. (This is because the specialization of $\Phi^{\tilde{\mathcal{A}}}$ at $v=1$ cancels $(\Phi_1)^{-1}$.) In particular, the specialization of $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ for $v=1$ is $\mathrm{ind}(E_J)$. Moreover, from the definition of induction, the specialization of $\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$ for $v=1$ is the same as $\mathrm{ind}(E'_J)$ where E' is the specialization of $E^{\tilde{\mathcal{A}}}$ for $v=1$, that is, $E'=E$. This proves (a).

Lemma 42.6. *We preserve the setup of 42.5. Let* $\mathfrak{U}(v)\otimes_{\tilde{\mathcal{A}}} E^{\tilde{\mathcal{A}}}$, $\mathfrak{U}(v)\otimes_{\tilde{\mathcal{A}}}(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ *be the* $\mathfrak{U}(v)\otimes_{\mathcal{A}} H_n^D$*-module obtained from* $E^{\tilde{\mathcal{A}}}$, $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ *by extension of scalars from* $\tilde{\mathcal{A}}$ *to* $\mathfrak{U}(v)$*. Let* $y\in H_{n,D}$*. We have*

$$\mathrm{tr}(\delta_J(y)\tilde{T}_{\underline{D}},\mathfrak{U}(v)\otimes_{\tilde{\mathcal{A}}} E^{\tilde{\mathcal{A}}})=\mathrm{tr}(y\tilde{T}_{\underline{D}},\mathfrak{U}(v)\otimes_{\tilde{\mathcal{A}}} (\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}).$$

Let $H_{J,n}$ be the $\mathcal{A}$-subalgebra of H_n defined in 31.8. Define an $\mathcal{A}$-linear map $p_J: H_n\to H_{J,n}$ by $p_J(\tilde{T}_z1_\lambda))=\tilde{T}_z1_\lambda$ if $z\in\mathbf{W}_J,\lambda\in\underline{\mathfrak{s}}_n$, $p_J(\tilde{T}_z1_\lambda)=0$ if $z\in\mathbf{W}-\mathbf{W}_J,\lambda\in\underline{\mathfrak{s}}_n$. We show that

(a) $p_J(\tilde{T}_uh')=\delta_{u,1}h'$ if $u\in\mathbf{W}^J,h'\in H_{J,n}$.

We may assume that $h'=\tilde{T}_b1_\lambda,b\in\mathbf{W}_J,\lambda\in\underline{\mathfrak{s}}_n$. Then $p_J(\tilde{T}_u\tilde{T}_b1_\lambda)=p_J(\tilde{T}_{ub}1_\lambda)=\delta_{u,1}\tilde{T}_{ub}1_\lambda=\delta_{u,1}\tilde{T}_b1_\lambda$, as required.

We show that

(b) $p_J(hh')=p_J(h)h'$ for any $h\in H_n,h'\in H_{J,n}$.

We may assume $h=\tilde{T}_u\tilde{T}_b1_\nu,h'=\tilde{T}_a1_\lambda$, $u\in\mathbf{W}^J,a,b\in\mathbf{W}_J$, $\lambda,\nu\in\underline{\mathfrak{s}}_n$. We must show that $p_J(\tilde{T}_u\tilde{T}_b1_\nu\tilde{T}_a1_\lambda)=p_J(\tilde{T}_u\tilde{T}_b1_\nu)\tilde{T}_a1_\lambda$. If $u\ne1$, both sides are zero by (a). If $u=1$, both sides are $\tilde{T}_b1_\nu\tilde{T}_a1_\lambda$. This proves (b).

By 34.13(a) we have

(c) $p_\emptyset(\tilde{T}_x\tilde{T}_{x'}1_\lambda)=\delta_{xx',1}$ for $x,x'\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n$.

For $u,u'\in\mathbf{W}^J,\lambda\in\underline{\mathfrak{s}}_n$ we write $\tilde{T}_{u^{-1}}\tilde{T}_{u'}1_\lambda=\sum_{a\in\mathbf{W}}f_a\tilde{T}_a1_\lambda$ where $f_a\in\mathcal{A}$. For $a'\in\mathbf{W}_J$ we have

$$\tilde{T}_{a'^{-1}u^{-1}}\tilde{T}_{u'}1_\lambda=\tilde{T}_{a'^{-1}}\tilde{T}_{u^{-1}}\tilde{T}_{u'}1_\lambda=\sum_{a\in\mathbf{W}}f_a\tilde{T}_{a'^{-1}}\tilde{T}_a1_\lambda.$$

Applying $p_\emptyset$ to this and using (c) gives $f_{a'}=\delta_{u',ua'}=\delta_{a',1}\delta_{u,u'}$ so that

$$p_J(\tilde{T}_{u^{-1}}\tilde{T}_{u'}1_\lambda)=\sum_{a\in\mathbf{W}_J}f_a\tilde{T}_a1_\lambda=\delta_{u,u'}\tilde{T}_11_\lambda.$$

Since this holds for any $\lambda\in\underline{\mathfrak{s}}_n$ we have

(d) $p_J(\tilde{T}_{u^{-1}}\tilde{T}_{u'})=\delta_{u,u'}\tilde{T}_1$.

Let $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n,u\in\mathbf{W}^J$. We have

$$\tilde{T}_w1_\lambda\tilde{T}_u=\sum_{u'\in\mathbf{W}^J,a\in\mathbf{W}_J}c_{w,u,u',a,\lambda}\tilde{T}_{u'}\tilde{T}_a1_{u^{-1}\lambda}$$

where $c_{w,u,u',a,\lambda} \in \mathcal{A}$ are uniquely determined. It follows that

$$\tilde{T}_{u^{-1}}\tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)} = \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda}\tilde{T}_{u^{-1}}\tilde{T}_{u'}\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}.$$

Applying p_J and using (b),(d) we obtain

$$p_J(\tilde{T}_{u^{-1}}\tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)}) = \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} p_J(\tilde{T}_{u^{-1}}\tilde{T}_{u'})\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}$$

(e) $$= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda}\delta_{u,u'}\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda} = \sum_{a \in \mathbf{W}_J} c_{w,\epsilon(u),u,a,\lambda}\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}.$$

Let $(e_i)_{i \in X}$ be a basis of the free $\tilde{\mathcal{A}}$-module $E^{\tilde{\mathcal{A}}}$. For $a \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n$ we have $\tilde{T}_a 1_\lambda \tilde{T}_{\underline{D}} e_i = \sum_{i' \in X} \tilde{c}_{a,\lambda,i,i'} e_{i'}$ where $\tilde{c}_{a,\lambda,i,i'} \in \tilde{\mathcal{A}}$.

Since $H_n^{D,\tilde{\mathcal{A}}}$ is a free right $H_{J,n}^{D,\tilde{\mathcal{A}}}$-module with basis $\{\tilde{T}_u; u \in \mathbf{W}^J\}$, we see that $\{\tilde{T}_u \otimes e_i; u \in \mathbf{W}^J, i \in X\}$ is a basis of the free $\tilde{\mathcal{A}}$-module $\mathrm{ind}((E^{\tilde{\mathcal{A}}})_J)$.

Let $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, u \in \mathbf{W}^J$ be such that $w\underline{D}\lambda = \lambda$. In $\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$ we have

$$\begin{aligned}
&\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}(\tilde{T}_u \otimes e_i) = (\tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)}\tilde{T}_{\underline{D}}) \otimes e_i \\
&= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda}(\tilde{T}_{u'}\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}\tilde{T}_{\underline{D}}) \otimes e_i \\
&= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda}\tilde{T}_{u'} \otimes (\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}\tilde{T}_{\underline{D}} e_i) \\
&= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J, i' \in X} c_{w,\epsilon(u),u',a,\lambda}\tilde{c}_{a,\epsilon(u)^{-1}\lambda,i,i'}\tilde{T}_{u'} \otimes e_{i'}.
\end{aligned}$$

Hence, using (e),

$$\begin{aligned}
&\mathrm{tr}(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}, \mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)) = \sum_{u \in \mathbf{W}^J, a \in \mathbf{W}_J, i \in X} c_{w,\epsilon(u),u,a,\lambda}\tilde{c}_{a,\epsilon(u)^{-1}\lambda,i,i} \\
&= \sum_{u \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u,a,\lambda}\mathrm{tr}(\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) \\
&= \sum_{u \in \mathbf{W}^J} \mathrm{tr}(\sum_{a \in \mathbf{W}_J} c_{w,\epsilon(u),u,a,\lambda}\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) \\
&= \mathrm{tr}(p_J(\sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}}\tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)})\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) \\
&= \mathrm{tr}(\rho_{J,n}(\sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}}\tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)})\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) = \mathrm{tr}(\delta_J(\tilde{T}_w 1_\lambda)\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}).
\end{aligned}$$

Thus we have

$$\mathrm{tr}(\delta_J(\tilde{T}_w 1_\lambda)\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) = \mathrm{tr}(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}, \mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)) = \mathrm{tr}(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}, (\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}})$$

where the second equality follows from 42.5(a). Since the elements $\tilde{T}_w 1_\lambda$ as above generate the $\mathcal{A}$-module $H_{n,D}$, the lemma follows.

42.7. Let $\mathcal{V}$ be the $\mathbf{Q}$-vector subspace of $\mathbf{Q} \otimes \mathrm{Hom}(\mathbf{k}^*, \mathbf{T})$ spanned by the coroots. Let $\mathcal{V}_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}$. The kernels of the roots $\mathcal{V}_{\mathbf{R}} \to \mathbf{R}$ a hyperplane arrangement which defines a partition of $\mathcal{V}_{\mathbf{R}}$ into facets in a standard way. Let $\mathcal{F}$ be the set of facets. Now the orbits of $\mathbf{W}$ on $\mathcal{F}$ are naturally indexed by the various subsets J

of $\mathbf{I}$. This gives a partition $\mathcal{F} = \bigsqcup_{J\subset\mathbf{I}} \mathcal{F}_J$. For example, $\mathcal{F}_\emptyset$ consists of all Weyl chambers. If $F \in \mathcal{F}_J$, then F is homeomorphic to a real affine space of dimension $|\mathbf{I} - J|$ hence we have $H^i_c(F) = 0$ if $i \neq |\mathbf{I} - J|$ and $H^{|\mathbf{I}-J|}_c(F) = \Lambda^{|\mathbf{I}-J|}[F]$; here we write $H^i_c(?)$ instead of $H^i_c(?, \mathbf{R})$, $[F]$ denotes the vector subspace of $\mathcal{V}_\mathbf{R}$ in which F is open dense and $\Lambda^{|\mathbf{I}-J|}[F]$ is the top exterior power of $[F]$. Note that $[F] = \mathbf{R} \otimes_\mathbf{Q} ([F]_\mathbf{Q})$ for a well-defined $\mathbf{Q}$-subspace $[F]_\mathbf{Q}$ of $\mathcal{V}$. For any $\underline{D}$-orbit η on the set of subsets of $\mathbf{I}$ let $\mathcal{V}^\eta_\mathbf{R} = \bigcup_{J\in\eta} \bigcup_{F\in\mathcal{F}_J} F \subset \mathcal{V}_\mathbf{R}$ and let $r_\eta = |\mathbf{I} - J|$ for some/any $J \in \eta$. We have $H^i_c(\mathcal{V}^\eta_\mathbf{R}) = 0$ if $i \neq r_\eta$, $H^{r_\eta}_c(\mathcal{V}^\eta_\mathbf{R}) = \bigoplus_{J\in\eta} \bigoplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F]$. Note also that $H^i_c(\mathcal{V}_\mathbf{R}) = 0$ if $i \neq |\mathbf{I}|$ and $H^{|\mathbf{I}|}_c(\mathcal{V}_\mathbf{R}) = \Lambda^{|\mathbf{I}|}\mathcal{V}_\mathbf{R}$. The $\mathbf{W}^D$-action on $\mathbf{T}$ induces a linear action of $\mathbf{W}^D$ on $\mathcal{V}_\mathbf{R}$. This action restricts for any η to a $\mathbf{W}^D$-action on $\mathcal{V}^\eta_\mathbf{R}$ and this induces a $\mathbf{W}^D$-action on $H^{r_\eta}_c(\mathcal{V}^\eta_\mathbf{R})$. It also induces a natural $\mathbf{W}^D$-action on $H^{|\mathbf{I}|}_c(\mathcal{V}_\mathbf{R}) = \Lambda^{|\mathbf{I}|}\mathcal{V}_\mathbf{R}$. The long cohomology exact sequences attached to the partition $\mathcal{V}_\mathbf{R} = \bigcup_\eta \mathcal{V}^\eta_\mathbf{R}$ show that $(-1)^{|\mathbf{I}|}H^{|\mathbf{I}|}_c(\mathcal{V}_\mathbf{R}) = \sum_\eta (-1)^{r_\eta} H^{r_\eta}_c(\mathcal{V}^\eta_\mathbf{R})$ in the Grothendieck group of representations of $\mathbf{W}^D$ over $\mathbf{R}$, that is,

$$\Lambda^{|\mathbf{I}|}\mathcal{V}_\mathbf{R} \oplus \bigoplus_{\eta; r_\eta = |\mathbf{I}|+1 \mod 2} \Big(\bigoplus_{J\in\eta} \bigoplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F]\Big)$$
$$\cong \bigoplus_{\eta; r_\eta = |\mathbf{I}| \mod 2} \Big(\bigoplus_{J\in\eta} \bigoplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F]\Big)$$

as representations of $\mathbf{W}^D$ over $\mathbf{R}$. All real representations in this formula come naturally from representations of $\mathbf{W}^D$ over $\mathbf{Q}$. Hence the previous formula remains valid (as representations of $\mathbf{W}^D$ over $\mathbf{Q}$) if $\mathcal{V}_\mathbf{R}, [F]$ are replaced by $\mathcal{V}, [F]_\mathbf{Q}$ and the exterior powers are taken over $\mathbf{Q}$. Tensoring both sides (over $\mathbf{Q}$) by $\mathfrak{U}$ (as in 42.5) we obtain

$$\Lambda^{|\mathbf{I}|}\mathcal{V}_\mathfrak{U} \oplus \bigoplus_{\eta; r_\eta = |\mathbf{I}|+1 \mod 2} \Big(\bigoplus_{J\in\eta} \bigoplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F]_\mathfrak{U}\Big)$$
$$\text{(a)} \qquad \cong \bigoplus_{\eta; r_\eta = |\mathbf{I}| \mod 2} \Big(\bigoplus_{J\in\eta} \bigoplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F]_\mathfrak{U}\Big)$$

as representations of $\mathbf{W}^D$ over $\mathfrak{U}$; here $\mathcal{V}_\mathfrak{U} = \mathfrak{U} \otimes_\mathbf{Q} \mathcal{V}$, $[F]_\mathfrak{U} = \mathfrak{U} \otimes_\mathbf{Q} [F]_\mathbf{Q}$ and the exterior powers are taken over $\mathfrak{U}$. We may view (a) as an isomorphism of $H^{D,1}_n$-modules: the $\mathbf{W}^D$-modules in (a) may be viewed as $H^{D,1}_n$-modules via the algebra homomorphism $H^{D,1}_n \to \mathfrak{U}[\mathbf{W}^D]$ given by $\tilde{T}_w \mapsto w$ for $w \in \mathbf{W}^D$, $1_\lambda \mapsto 0$ for $\lambda \neq \lambda_0$, $1_{\lambda_0} \mapsto 1$ (here λ_0 is the neutral element of the abelian group $\underline{\mathfrak{s}}_n$; see 28.1).

We define an $\mathfrak{U}$-linear map $\Delta : H^{D,1}_n \to H^{D,1}_n \otimes H^{D,1}_n$ by $\Delta(\tilde{T}_w) = \tilde{T}_w \otimes \tilde{T}_w$ for $w \in \mathbf{W}^D$ and $\Delta(1_\lambda) = \sum_{\lambda_1,\lambda_2\in\underline{\mathfrak{s}}_n; \lambda_1\lambda_2=\lambda} 1_{\lambda_1} \otimes 1_{\lambda_2}$ for $\lambda \in \underline{\mathfrak{s}}_n$. (Here we use the abelian group structure on $\underline{\mathfrak{s}}_n$, a subgroup of $\underline{\mathfrak{s}}$; see 28.1.) This makes $H^{D,1}_n$ into a Hopf algebra. (Note that the analogous formulas do not make H^D_n into a Hopf algebra.) It follows that for any two $H^{D,1}_n$-modules E_1, E_2, the $\mathfrak{U}$-vector space $E_1 \otimes E_2$ is naturally an $H^{D,1}_n$-module.

Now let E be an $H^{D,1}_n$-module of finite dimension over $\mathfrak{U}$. Then we can take the tensor product of each $H^{D,1}_n$-module in (a) with E and we obtain an isomorphism of $H^{D,1}_n$-modules

$$E \otimes \Lambda^{|\mathbf{I}|}\mathcal{V}_\mathfrak{U} \oplus \bigoplus_{\eta; r_\eta = |\mathbf{I}|+1 \mod 2} X_\eta \cong \bigoplus_{\eta; r_\eta = |\mathbf{I}| \mod 2} X_\eta$$

where $X_\eta = E \otimes \bigoplus_{J\in\eta} \bigoplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F]_{\mathfrak{U}}$. Applying to this the functor $E \mapsto E^{\tilde{\mathcal{A}}}$ (see 42.5), we deduce an isomorphism of $H_n^{D,\tilde{\mathcal{A}}}$-modules

$$(E\otimes\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}})^{\tilde{\mathcal{A}}} \oplus \bigoplus_{\eta;r_\eta=|\mathbf{I}|+1 \mod 2} X_\eta^{\tilde{\mathcal{A}}} \cong \bigoplus_{\eta;r_\eta=|\mathbf{I}| \mod 2} X_\eta^{\tilde{\mathcal{A}}}.$$

We deduce that for $y \in H_{n,D}$ we have

(b) $$\mathrm{tr}(y\tilde{T}_{\underline{D}}, (E\otimes\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}})^{\tilde{\mathcal{A}}}) = \sum_\eta (-1)^{r_\eta+|\mathbf{I}|}\mathrm{tr}(y\tilde{T}_{\underline{D}}, X_\eta^{\tilde{\mathcal{A}}}).$$

We have $X_\eta = \bigoplus_{J\in\eta} X^J$ where $X^J = E \otimes (\bigoplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F]_{\mathfrak{U}})$.

Assume first that η consists of at least two subsets of $\mathbf{I}$. Then each X_J is stable under $H_n^{D,1}$ and is mapped by $\tilde{T}_{\underline{D}}$ into $X_{J'}$ with $J \neq J'$. From the definitions we have $X_\eta^{\tilde{\mathcal{A}}} = \bigoplus_{J\in\eta} \tilde{\mathcal{A}} \otimes_{\mathfrak{U}} X_J$ as an $\tilde{\mathcal{A}}$-module and each summand $\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} X_J$ is stable under H_n and is mapped by $\tilde{T}_{\underline{D}}$ into a summand $\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} X_{J'}$ with $J \neq J'$. It follows that for our η we have

(c) $\mathrm{tr}(y\tilde{T}_{\underline{D}}, X_\eta^{\tilde{\mathcal{A}}}) = 0$.

Next assume that η consists of a single subset J of $\mathbf{I}$. We have $\underline{D}(J) = J$. Let F_J be the unique facet in $\mathcal{F}_J$ such that F_J is contained in the closure of the dominant Weyl chamber. Then F_J is stable under the the subgroup $\mathbf{W}_J^D$ of $\mathbf{W}^D$ generated by $\mathbf{W}_J$ and $\underline{D}$ and X_η may be identified with $E \otimes (H_n^{D,1} \otimes_{H_{J,n}^{D,1}} (\Lambda^{|\mathbf{I}-J|}[F_J]_{\mathfrak{U}}))$. Here $\Lambda^{|\mathbf{I}-J|}[F_J]_{\mathfrak{U}}$ is regarded as a WW_J^D-module and then is viewed as a $H_{J,n}^{D,1}$-module via the canonical algebra homomorphism $H_{J,n}^{D,1} \to \mathfrak{U}[\mathbf{W}_J^D]$; thus 1_λ acts on it as 1 if $\lambda = \lambda_0$ and as 0 if $\lambda \neq \lambda_0$. Note that in the $\mathbf{W}_J^D$-module $\Lambda^{|\mathbf{I}-J|}[F_J]_{\mathfrak{U}}$, $\mathbf{W}_J$ acts trivially (since $\mathbf{W}_J$ acts trivially on $[F_J]_{\mathfrak{U}}$) and $\underline{D}$ acts as multiplication by $(-1)^{|\mathbf{I}-J|-|(\mathbf{I}-J)_\epsilon|}$. Let $X'_\eta = E \otimes (H_n^{D,1} \otimes_{H_{J,n}^{D,1}} \mathfrak{U})$ where $\mathfrak{U}$ is regarded as a $H_{J,n}^{D,1}$-module coming from the trivial representation of $\mathbf{W}_J^D$. We see that we may identify X_η, X'_η in a way compatible with the H_n^1-module structures and so that the action of $\tilde{T}_{\underline{D}}$ on X_η corresponds to $(-1)^{|\mathbf{I}-J|-|(\mathbf{I}-J)_\epsilon|}$ times the action of $\tilde{T}_{\underline{D}}$ on X'_η. Using the definitions we see that we may identify $X_\eta^{\tilde{\mathcal{A}}}, X'^{\tilde{\mathcal{A}}}_\eta$ in a way compatible with the H_n-module structures and so that the action of $\tilde{T}_{\underline{D}}$ on $X_\eta^{\tilde{\mathcal{A}}}$ corresponds to $(-1)^{|\mathbf{I}-J|-|(\mathbf{I}-J)_\epsilon|}$ times the action of $\tilde{T}_{\underline{D}}$ on $X'^{\tilde{\mathcal{A}}}_\eta$. From the definitions we have $X'_\eta = \mathrm{ind}(E_J)$ (notation of 42.5). We see that for our η we have

(d) $$\mathrm{tr}(y\tilde{T}_{\underline{D}}, X_\eta^{\tilde{\mathcal{A}}}) = (-1)^{|\mathbf{I}-J|-|(\mathbf{I}-J)_\epsilon|}\mathrm{tr}(y\tilde{T}_{\underline{D}}, (\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}).$$

From the definitions (34.4) we see that there is a unique $\tilde{\mathcal{A}}$-algebra homomorphism $\vartheta : H_n^{D,\tilde{\mathcal{A}}}D \to H_n^{D,\tilde{\mathcal{A}}}$ such that

$\vartheta(1_\lambda) = 1_\lambda$ for any $\lambda \in \underline{\mathfrak{s}}_n$,

$\vartheta(\tilde{T}_w) = (-1)^{l(w)}\tilde{T}_{w^{-1}}^{-1}$ for any $w \in \mathbf{W}$,

$\vartheta(\tilde{T}_{\underline{D}}) = (-1)^{|\mathbf{I}|-|\mathbf{I}_\epsilon|}\tilde{T}_{\underline{D}}$.

We have $\vartheta^2 = 1$.

Using ϑ and $E^{\tilde{\mathcal{A}}}$ we can define a new $H_n^{D,\tilde{\mathcal{A}}}$-module $E^{\tilde{\mathcal{A}},\vartheta}$ with the same underlying $\tilde{\mathcal{A}}$-module as $E^{\tilde{\mathcal{A}}}$ but with $x \in H_n^{D,\tilde{\mathcal{A}}}$ acting on $E^{\tilde{\mathcal{A}},\vartheta}$ in the same way that $\vartheta(x)$ acts on $E^{\tilde{\mathcal{A}}}$. We show that

(e) *under extension of scalars from $\tilde{\mathcal{A}}$ to $\mathfrak{U}(v)$, the $H_n^{D,\tilde{\mathcal{A}}}$-modules $E^{\tilde{\mathcal{A}},\vartheta}$ and $(E\otimes\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}})^{\tilde{\mathcal{A}}}$ become isomorphic $\mathfrak{U}(v)\otimes_{\mathcal{A}} H_n^D$-modules.*

As in the proof of 42.5(a) it is enough to show that these $H_n^{D,\tilde{\mathcal{A}}}$-modules become isomorphic $H_n^{D,1}$-modules under the specialization $v=1$. Thus it is enough to show that $E^{\tilde{\mathcal{A}},\vartheta}|_{v=1}\cong E\otimes\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}}$ as $H_n^{D,1}$-modules. Now the underlying $\mathfrak{U}$-vector space of $E^{\tilde{\mathcal{A}},\vartheta}|_{v=1}$ is E but the action of $x\in H_n^{D,1}$ on $E^{\tilde{\mathcal{A}},\vartheta}|_{v=1}$ is the same as the action of $\vartheta_1(x)$ on E. Here $\vartheta_1:H_n^{D,1}\to H_n^{D,1}$ is the specialization of ϑ_1 for $v=1$. Note that $\vartheta_1(1_\lambda)=1_\lambda$ for any $\lambda\in\underline{\mathfrak{s}}_n$ and $\vartheta_1(\tilde{T}_w)=\gamma_w\tilde{T}_w$ for any $w\in\mathbf{W}^D$, where $\gamma_w=\pm1$ is the scalar by which w acts in the $\mathbf{W}^D$-module $\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}}$. The desired result follows.

Combining (b),(c),(d),(e) we see that for any $y\in H_{n,D}$ we have

$$(-1)^{|\mathbf{I}|+|\mathbf{I}_\epsilon|}\mathrm{tr}(\vartheta(y\tilde{T}_{\underline{D}}),E^{\tilde{\mathcal{A}}})=\sum_{J\subset\mathbf{I};\epsilon(J)=J}(-1)^{|J_\epsilon|}\mathrm{tr}(y\tilde{T}_{\underline{D}},(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}).$$

Replacing here $(-1)^{|\mathbf{I}|+|\mathbf{I}_\epsilon|}\vartheta(y\tilde{T}_{\underline{D}})$ by $\vartheta(y)\tilde{T}_{\underline{D}}$ and using Lemma 42.6 we may rewrite this as

$$\mathrm{tr}(\vartheta(y)\tilde{T}_{\underline{D}},E^{\tilde{\mathcal{A}}})=\sum_{J\subset\mathbf{I};\epsilon(J)=J}(-1)^{|J_\epsilon|}\mathrm{tr}(\delta_J(y)\tilde{T}_{\underline{D}},E^{\tilde{\mathcal{A}}})$$

or equivalently (see 42.4) $\mathrm{tr}(\vartheta(y)\tilde{T}_{\underline{D}},E^{\tilde{\mathcal{A}}})=\mathrm{tr}(\delta(y)\tilde{T}_{\underline{D}},E^{\tilde{\mathcal{A}}})$. Since any simple $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$-module can be obtained by extension of scalars (from $\tilde{\mathcal{A}}$ to $\mathfrak{U}(v)$) from some $E^{\tilde{\mathcal{A}}}$ as above, we deduce that

$$\mathrm{tr}((\delta(y)-\vartheta(y))\tilde{T}_{\underline{D}},\mathbf{E})=0$$

for any simple $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$-module $\mathbf{E}$. Since $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$ is a semisimple algebra, it follows that $(\delta(y)-\vartheta(y))\tilde{T}_{\underline{D}}$ belongs to the $\mathfrak{U}(v)$-subspace of $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$ spanned by commutators $xx'-x'x$ with $x,x'\in\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$. Hence we have

$$g(\delta(y)-\vartheta(y))\tilde{T}_{\underline{D}}=\sum_{i=1}^m g_i(x_i\tilde{T}_{\underline{D}}^{s_i}x_i'\tilde{T}_{\underline{D}}^{1-s_i}-x_i'\tilde{T}_{\underline{D}}^{1-s_i}x_i\tilde{T}_{\underline{D}}^{s_i})$$

with $g\in\mathcal{A}-\{0\},g_i\in\mathcal{A},x_i\in H_n,x_i'\in H_n,s_i\in\mathbf{Z}$, that is,

(f) $$g(\delta(y)-vt(y))=\sum_{i=1}^m g_i(x_i\tilde{T}_{\underline{D}}^{s_i}x_i'\tilde{T}_{\underline{D}}^{-s_i}-x_i'\tilde{T}_{\underline{D}}^{1-s_i}x_i\tilde{T}_{\underline{D}}^{s_i-1}).$$

42.8. We show that for any $y,y'\in H_n$ we have

(a) $\mathfrak{f}\eta\tilde{\omega}(yy'-y'\tilde{T}_{\underline{D}}y\tilde{T}_{\underline{D}}^{-1})=0$.

Let $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n$. Let $\mathcal{L}\in\lambda$. If $w\underline{D}\lambda=\lambda$, using notation and results in 31.6, 31.7, we have

$$\mathfrak{f}\eta\tilde{\omega}(v^{l(w)}\tilde{T}_w1_{\underline{D}\lambda})=gr(K_{\mathbf{I},D}^{w,\mathcal{L}})$$
$$=\sum_A\chi_v^A(K_{\mathbf{I},D}^{w,\mathcal{L}})=\sum_A\tilde{\zeta}^A(v^{l(w)}\tilde{T}_w1_{\underline{D}\lambda}\tilde{T}_{\underline{D}})=\sum_A\zeta^A(v^{l(w)}\tilde{T}_w1_{\underline{D}\lambda}\tilde{T}_{\underline{D}})$$

(the last equation comes from 31.7(e); A runs over the objects in $\hat{D}$ up to isomorphism such that $\zeta^A\neq0$.) The equation

$$\mathfrak{f}\eta\tilde{\omega}(v^{l(w)}\tilde{T}_w1_{\underline{D}\lambda})=\sum_A\zeta^A(v^{l(w)}\tilde{T}_w1_{\underline{D}\lambda}\tilde{T}_{\underline{D}})$$

holds also if $w\underline{D}\lambda \neq \lambda$ (in this case both sides are 0). It follows that

$$\mathfrak{f}\eta\tilde{\omega}(x) = \sum_A \zeta^A(x\tilde{T}_{\underline{D}}) \text{ for any } x \in H_n.$$

We deduce

$$\mathfrak{f}\eta\tilde{\omega}(yy' - y'\tilde{T}_{\underline{D}} y \tilde{T}_{\underline{D}}^{-1}) = \sum_A (\zeta^A(yy'\tilde{T}_{\underline{D}}) - \zeta^A(y\epsilon(y)\tilde{T}_{\underline{D}})) = 0$$

where the last equality follows from 31.8. This proves (a).

Proposition 42.9. *Let $y \in H$. We have $\mathbf{d}(\mathfrak{f}\eta\tilde{\omega}(y)) = \mathfrak{f}\eta\tilde{\omega}(\vartheta(y)) \in \mathfrak{K}(D)$ with $\mathbf{d} : \mathfrak{K}(D) \to \mathfrak{K}(\Delta)$ as in 42.2.*

If $y \in H'_D$ (see 42.3), both sides of the desired equality are 0. (Note that ϑ maps H_D into itself and H'_D into itself.) Hence we may assume that $y \in H_D$. We can assume that $y \in H_n$ where $n \in \mathbf{N}^*_{\mathbf{k}}$. Then $y \in H_{n,D}$. By 42.4 it is enough to show that $\mathfrak{f}\eta\tilde{\omega}(\delta(y) - \vartheta(y)) = 0$. Let g, g_i, x_i, x'_i, s_i be as in 42.7(f). Since $g \neq 0$, it is enough to show that $g\mathfrak{f}\eta\tilde{\omega}(\delta(y) - \vartheta(y)) = 0$ or that $\mathfrak{f}\eta\tilde{\omega}(g(\delta(y) - \vartheta(y))) = 0$. Using 42.7 it is enough to show that

$$\mathfrak{f}\eta\tilde{\omega}(\sum_{i=1}^m g_i(x_i\tilde{T}_{\underline{D}}^{s_i} x'_i \tilde{T}_{\underline{D}}^{-s_i} - x'_i\tilde{T}_{\underline{D}}^{1-s_i} x_i \tilde{T}_{\underline{D}}^{s_i-1}) = 0.$$

Hence it is enough to show that

$$\mathfrak{f}\eta\tilde{\omega}(x\tilde{T}_{\underline{D}}^{s} x' \tilde{T}_{\underline{D}}^{-s} - x'\tilde{T}_{\underline{D}}^{1-s} x \tilde{T}_{\underline{D}}^{s-1}) = 0$$

for any $x, x' \in H_n$ and any $s \in \mathbf{Z}$. We have

$$x\tilde{T}_{\underline{D}}^{s} x' \tilde{T}_{\underline{D}}^{-s} - x'\tilde{T}_{\underline{D}}^{1-s} x \tilde{T}_{\underline{D}}^{s-1} = (z - \tilde{T}_{\underline{D}}^{-s} z \tilde{T}_{\underline{D}}^{s}) + (z'x' - x'\tilde{T}_{\underline{D}} z' \tilde{T}_{\underline{D}}^{-1})$$

where $z = x\tilde{T}_{\underline{D}}^{s} x' \tilde{T}_{\underline{D}}^{-s} \in H_n$ and $z' = \tilde{T}_{\underline{D}}^{-s} x \tilde{T}_{\underline{D}}^{s} \in H_n$. Hence it is enough to show that $\mathfrak{f}\eta\tilde{\omega}(z'x' - x'\tilde{T}_{\underline{D}} z' \tilde{T}_{\underline{D}}^{-1}) = 0$ (see 42.8(a)) and

(a) $\mathfrak{f}\eta\tilde{\omega}(z - \tilde{T}_{\underline{D}}^{-s} z \tilde{T}_{\underline{D}}^{s}) = 0$

for any $z \in H_n$. This follows from 41.6(c).

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