

# CHARACTER SUMS AND SMALL EIGENVALUES FOR $\Gamma_0(p)$

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*Dedicated to Robert Rankin*

**1. Introduction. Statement of results.** Let  $\Delta$  denote the Laplace operator acting on the space  $L^2(\Gamma/H)$  of automorphic functions with respect to a congruence group  $\Gamma$ , square integrable over the fundamental domain  $F = \Gamma/H$ . It is known that  $\Delta$  has a point spectrum

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$$

with (Weyl's law)

$$\lambda_n \sim \frac{4\pi}{|F|} n \quad \text{as } n \rightarrow \infty$$

and it has a purely continuous spectrum on  $[\frac{1}{4}, \infty)$  of finite multiplicity equal to the number of inequivalent cusps. The eigenpacket of the continuous spectrum is formed by the Eisenstein series  $E_a(z, s)$  on  $s = \frac{1}{2} + it$  where  $a$  ranges over inequivalent cusps. The eigenfunctions  $u_i(z)$  with positive eigenvalues are Maass cusp forms.

A. Selberg's celebrated conjecture [9] asserts that all positive eigenvalues lie on the continuous spectrum, i.e.

$$\lambda_1 \geq \frac{1}{4}. \tag{1.1}$$

Selberg [9] succeeded to show that

$$\lambda_1 \geq \frac{3}{16} \tag{1.2}$$

by using A. Weil's upper bound for Kloosterman sums

$$|\mathcal{S}(m, n; c)| \leq (m, n, c)^{1/2} c^{1/2} \tau(c), \tag{1.3}$$

and S. S. Gelbart and H. Jacquet [2] have proved the strict inequality  $\lambda_1 > 3/16$  by a different method (lifting from  $GL(2)$  to  $GL(3)$ ). The conjecture (1.1) is known to be true for subgroups of small index of the modular group, cf. Huxley [3].

Let us call exceptional the eigenvalues which do not satisfy the Selberg conjecture, i.e. those with

$$0 < \lambda_j < \frac{1}{4}.$$

They play a similar role to the real zeros of Dirichlet's  $L$ -series in the multiplicative number theory. In fact letting

$$\lambda_j = s_j(1 - s_j)$$

it turns out that  $s_j$  are zeros of the Selberg zeta-function; thus the exceptional eigenvalues correspond to the real zeros in the segment

$$\frac{1}{2} < s_j < 1.$$

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The remaining zeros  $s_j$  satisfy the Riemann hypothesis, i.e. they lie on the line

$$s_j = \frac{1}{2} + it_j, \quad t_j \text{ real.}$$

Being unable to prove the Selberg eigenvalue conjecture J.-M. Deshouillers and H. Iwaniec [1] began to establish statistical results showing a rarity of the  $s_j$  in much the same form as the density theorems about the zeros of Dirichlet’s  $L$ -series. Some of their results proved to be powerful enough to go around the conjecture in a number of important applications. It is not surprising that the matter has something to do with character sums. The first transparent connection was pointed out in [6] where the following kind of density theorems were established

$$\sum_{1/2 < s_j < 1} |F|^{A(s_j - 1/2)} \ll |F|^{1+\varepsilon}, \tag{1.4}$$

the constant implied in  $\ll$  depending on  $\varepsilon$  alone. The larger  $A$  is the less often exceptional eigenvalues of  $\Gamma$  may occur. J. Szmidt and H. Iwaniec [6] considered the Hecke congruence group  $\Gamma = \Gamma_0(q)$  of level  $q$  (for technical reason we assumed  $q$  be prime) showing (1.4) with  $A = 24/11$ . Here the point is that  $A = 24/11 > 2$  because the result with  $A = 2$  follows simply by applying Selberg’s trace formula with an appropriate test function, see M. N. Huxley [4] for example. It is natural to conjecture that (1.4) holds with  $A = 4$  (density conjecture). This would contain the Selberg lower bound (1.2) for an individual eigenvalue.

The character sums in question are of the type

$$\sum_a \sum_b \left( \frac{a^2 - 4}{b} \right). \tag{1.5}$$

In order to estimate them in [6] we used A. Weil’s (see (3.3)) and D. Burgess’ bounds for character sums. The first replaces (1.3) while the second is vital and it yields the desired saving to effect  $A > 2$ . If the Lindelöf hypothesis for Dirichlet’s  $L$ -series was used instead of Burgess’ bound then we could get the density theorem with  $A = 3$ .

The problem is also related with the Lindelöf hypothesis for the Rankin zeta-functions. Let us define them. Given a cusp  $a$  of  $\Gamma$  take  $\sigma_a \in \text{SL}(2, \mathbb{R})$  (once and for all) such that

$$\sigma_a^\infty = a \quad \text{and} \quad \sigma_a^{-1} \Gamma_a \sigma_a = \Gamma_\infty \tag{1.6}$$

where  $\Gamma_a$  is the stabilizer of  $a$  in  $\Gamma$ . Each cusp form  $u_j(z)$ , being an eigenfunction of  $\Delta$

$$\Delta u_j = \lambda_j u_j,$$

has the Fourier expansion at  $a$  of type

$$u_j(\sigma_a z) = \sqrt{y} \sum_{n \neq 0} \rho_{ja}(n) K_{s_j - 1/2}(2\pi |n| y) e(nx) \tag{1.7}$$

where the numbers  $\rho_{ja}(n)$  are called the Fourier coefficients and  $K_\nu(y)$  is the McDonald-Bessel function. We assume that the cusp forms  $u_j$  form an orthonormal system

$$\langle u_{j_1}, u_{j_2} \rangle = \int_F u_{j_1}(z) \bar{u}_{j_2}(z) dz = \delta_{j_1 j_2}.$$

The Rankin zeta-functions are defined by

$$R_{j_a}(s) = \sum_1^\infty |\rho_{j_a}(n)|^2 n^{-s}, \quad \text{Re } s > 1.$$

They possess meromorphic continuation to the whole complex plane and they satisfy a (vector) functional equation which connects values at  $s$  with those at  $1-s$ . It is reasonable to expect that

$$\rho_{j_a}(1) \ll \left(\frac{\text{ch } \pi t_j}{q}\right)^{1/2} (\lambda_j q)^e$$

and that the analogue of the Lindelöf hypothesis is true

$$R_{j_a}(s) \ll \left(\frac{\text{ch } \pi t_j}{q}\right)^{1/2} (\lambda_j q |s|)^e \tag{1.8}$$

on  $\text{Re } s = \frac{1}{2}$ . This would imply the density theorem with  $A = 3$ . What we actually need is a consequence of (1.8), namely that

$$\Lambda_{j_a}(N) = \sum_{n \leq N} |\rho_{j_a}(n)|^2 \gg \frac{N}{|F|} \tag{1.9}$$

for  $N \geq q^e$ . If (1.9) is true for  $N = q^\theta$  with  $0 < \theta < 2$  then the density theorem holds with  $A = 2(2 - \theta)$ . Therefore the density conjecture is a consequence of another conjecture, that (1.9) is true for all  $N \geq q^e$ . It is disappointing that by present means we are able to show (1.9) only when  $N \gg q^{1+\epsilon}$ , compare with Theorem 7.

In this paper we give another treatment of the character sums (1.5) which yields the following improvement over [6].

**THEOREM 1.** *The density theorem (1.4) holds for groups  $\Gamma_0(p)$  with  $A = 12/5$ .*

The present method of estimating the relevant character sums does not depend on the Burgess inequality and is more general.

I benefited a lot from discussions on the subject with H. L. Montgomery to whom I wish to express my thanks as well as to the Mathematics Department of the University of Michigan in Ann Arbor for financial support and a nice atmosphere to work.

**2. Estimates for character sums.** Let  $\mathcal{D}$  be a finite sequence of positive integers (not necessarily distinct) from the interval  $[D, 4D]$  with some  $D \geq 1$ . For any sequence of complex numbers  $\beta = (\beta_b)_{1 \leq b \leq B}$  we consider the sum

$$\mathcal{M}(\beta, \mathcal{D}) = \sum_{d \in \mathcal{D}} \left| \sum_{1 \leq b \leq B} \beta_b \left(\frac{d}{b}\right) \right|^2$$

with the aim of showing that

$$\mathcal{M}(\beta, \mathcal{D}) \leq \mathfrak{d}(B, D, \Delta_1, \Delta_2) \|\beta\|^2 \tag{2.1}$$

where  $\mathfrak{d}(B, D, \Delta_1, \Delta_2)$  depends at most on  $B, D$  and two other parameters  $\Delta_1, \Delta_2$  defined by

$$\Delta_1 = \sum_1^\infty v^{1/2} |\mathcal{D}_{v^2}|$$

and

$$\Delta_2 = \max_r r^{1/2} |\mathcal{D}_r|.$$

Here  $\mathcal{D}_r$  stands for the subsequence of those elements in  $\mathcal{D}$  which are divisible by  $r$  and  $|\mathcal{D}_r|$  denotes its cardinality. While the first parameter  $\Delta_1$  measures how much  $\mathcal{D}$  differs from the sequence of squares (on which the characters are trivial) the second one  $\Delta_2$  controls the multiplicity  $\lambda(d)$  of elements  $d$  in  $\mathcal{D}$ , namely it yields

$$\lambda(d) \leq d^{-1/2} \Delta_2 \leq D^{-1/2} \Delta_2.$$

Our main result in this section is

**THEOREM 2.** *We have (2.1) with*

$$\mathfrak{d}(B, D, \Delta_1, \Delta_2) = c(\varepsilon)(BD)^\varepsilon \Delta_1^{2/3} \{B + \Delta_2^{1/6} D^{1/3} + \Delta_2^{1/3} D^{-1/6} B\}$$

where  $\varepsilon$  is any positive number and  $c(\varepsilon)$  depends on  $\varepsilon$  alone.

As a corollary to Theorem 2 we shall deduce

**THEOREM 3.** *For any  $A, B \geq 1$  and  $\varepsilon > 0$  we have*

$$\sum_{1 \leq a \leq A} \left| \sum_{1 \leq b \leq B} \beta_b \left( \frac{a^2 - 4}{b} \right) \right|^2 \ll (AB)^\varepsilon (A^{3/2} + A^{2/3} B) \|\beta\|^2,$$

the constant implied in  $\ll$  depending on  $\varepsilon$  alone.

By Cauchy's inequality Theorem 3 yields

**COROLLARY.** *For any  $A, B \geq 1$  and  $\varepsilon > 0$  we have*

$$\sum_{1 \leq a \leq A} \left| \sum_{1 \leq b \leq B} \beta_b \left( \frac{a^2 - 4}{b} \right) \right| \ll (AB)^\varepsilon (A^{5/4} + A^{5/6} B^{1/2}) \|\beta\|,$$

the constant implied in  $\ll$  depending on  $\varepsilon$  alone.

In the proof of Theorem 2 we shall appeal to the following simpler result.

**THEOREM 4.** *Let  $\mathcal{D}$  be a sequence of squarefree positive integers  $d \leq D$  (not necessarily distinct). We then have*

$$\sum_{1 \leq b \leq B} \left| \sum_{d \in \mathcal{D}} \gamma_d \left( \frac{d}{b} \right) \right|^2 \leq c(\varepsilon)(BD)^\varepsilon (|\mathcal{D}| D^{1/2} + |\mathcal{D}|^{1/2} B) \|\gamma\|^2.$$

For clarity we split up the arguments into several lemmas.

LEMMA 1 (Polya–Vinogradov). *If  $\chi$  is a nonprincipal character (mod  $q$ ) then*

$$\sum_{1 \leq n \leq N} \chi(n) \ll q^{1/2} \log q.$$

LEMMA 2 (Poisson summation formula). *Let  $f(x)$  be a smooth function on  $\mathbb{R}$  such that  $xf'(x)$  is bounded. We then have*

$$\sum_{n \equiv a \pmod{q}} f(n) = \frac{1}{q} \sum_m e\left(-\frac{am}{q}\right) \hat{f}\left(\frac{m}{q}\right)$$

where  $e(z) = e^{2\pi iz}$  and  $\hat{f}(y)$  is the Fourier transform of  $f(x)$ .

LEMMA 3. *For  $q > 1$ ,  $q \equiv 1 \pmod{8}$ ,  $q$  squarefree, put*

$$G(q, m) = \sum_{a \pmod{q}} \left(\frac{q}{a}\right) e\left(-\frac{am}{q}\right).$$

We have

$$G(q, 0) = 0$$

and for  $m \neq 0$ , we have

$$G(q, m) = \left(\frac{q}{m}\right) \sqrt{q}.$$

*Proof.* This follows immediately from the quadratic reciprocity law and the well known formula for the Gaussian sum  $G(q, 1) = \sqrt{q}$ .

Combining Lemmas 2 and 3 we infer

LEMMA 4. *Let  $f(x)$  be a smooth function on  $\mathbb{R}$  such that  $xf'(x)$  is bounded,  $r \geq 1$ ,  $q > 1$ ,  $q \equiv 1 \pmod{8}$ ,  $q$  squarefree. We then have*

$$\sum_{\substack{(n,r)=1 \\ n \equiv a \pmod{q}}} f(n) \left(\frac{q}{n}\right) = \frac{1}{\sqrt{q}} \sum_{k|r} \frac{\mu(k)}{k} \sum_{m \neq 0} \left(\frac{q}{km}\right) \hat{f}\left(\frac{m}{kq}\right). \tag{2.2}$$

*Proof.* By Möbius inversion formula our sum is equal to

$$\sum_{k|r} \mu(k) \left(\frac{q}{k}\right) \sum_n f(kn) \left(\frac{q}{n}\right).$$

By Lemma 2 the innermost sum is equal to

$$\sum_{a \pmod{q}} \left(\frac{q}{a}\right) \sum_{n \equiv a \pmod{q}} f(kn) = (kq)^{-1} \sum_m G(q, m) \hat{f}(m/kq).$$

On applying Lemma 3 we complete the proof.

For the purpose of the proof of Theorem 2 it is convenient to take

$$f(x) = \exp\left(-\pi \left(\frac{x}{N}\right)^2\right) \tag{2.3}$$

with some  $N \geq 1$ , so

$$\hat{f}(y) = N \exp(-\pi(yN)^2) < y^{-2} \exp(-(yN)^2).$$

Hence, for  $|m| > \tau k q N^{-1}$  with some  $\tau \geq 1$  to be chosen later, we have

$$\hat{f}\left(\frac{m}{kq}\right) < \left(\frac{\tau k q}{m}\right)^2 \exp(-\tau^2).$$

For the remaining  $m$ 's we want to separate the variables in  $\hat{f}(m/kq)$ , so we write  $\hat{f}(y)$  as the Mellin transform of the gamma function

$$\hat{f}(y) = \frac{1}{2\pi i} \int_{(\epsilon)} \pi^{-s} \Gamma(s) N^{1-2s} y^{-2s} ds.$$

At this occasion notice that by Stirling's formula

$$|\Gamma(s)| \ll \epsilon^{-1} \exp\left(-\frac{\pi}{2}|s|\right).$$

Now gathering together the above results we obtain a truncated form of (2.2).

LEMMA 5. Let  $q > 1$ ,  $q \equiv 1 \pmod{8}$   $q$  square free,  $f(x)$  be given by (2.3) and  $M_k \geq \tau k q N^{-1}$ . We then have (with some  $|\theta| \leq 1$ )

$$\sum_{(n,r)=1} f(n)\left(\frac{q}{n}\right) = 4\theta(rq)^{3/2} \tau^2 \exp(-\tau^2) + \frac{1}{2\pi i} \int_{(2\epsilon)} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) N^{1-s} \sum_{k|r} \mu(k) k^{s-1} q^{s-1/2} \sum_{1 \leq m \leq M_k} m^{-s} \left(\frac{q}{km}\right) ds.$$

By Lemma 5 we immediately obtain

LEMMA 6. Let  $f(x)$  be given by (2.4),  $Q \geq 1$ ,  $M_k = \tau k Q^2 N^{-1}$  and  $\alpha_q$  be any complex numbers supported in one of the four arithmetic progressions  $q = 1, 3, 5, 7 \pmod{8}$ . We then have (with some  $|\theta| \leq 1$ )

$$\begin{aligned} \sum_{1 < q_1, q_2 \leq Q} \mu^2(q_1 q_2) \alpha_{q_1} \alpha_{q_2} \sum_{(n,r)=1} f(n)\left(\frac{q_1 q_2}{n}\right) &= 4\theta \tau^2 \exp(-\tau^2) r^{3/2} Q^4 \|\alpha\|^2 \\ &+ \frac{1}{2\pi i} \int_{(\epsilon)} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) N^{1-s} \sum_{k|r} \mu(k) k^{s-1} \sum_{\substack{1 \leq e \leq Q \\ (e,k)=1}} \mu(e) e^{2s-1} \\ &\times \sum_{\substack{1 \leq m \leq M_k \\ (m,e)=1}} m^{-s} \left( \sum_{1 < eh \leq Q} \mu^2(eh) \alpha_{eh} h^{s-1/2} \left(\frac{h}{km}\right) \right)^2 ds. \end{aligned}$$

LEMMA 7 (duality principle). The following two statements are equivalent

(i) for all complex numbers  $\alpha_m$

$$\sum_n \left| \sum_m \alpha_m f(m, n) \right|^2 \leq \nu_f \sum_m |\alpha_m|^2$$

(ii) for all complex numbers  $\beta_n$

$$\sum_m \left| \sum_n \beta_n f(m, n) \right|^2 \leq \nu_f \sum_n |\beta_n|^2.$$

I. Proof of Theorem 4. We have

$$\mathcal{M}(\beta, \mathcal{D}) \leq \sum_{1 \leq b_1, b_2 \leq B} |\beta_{b_1} \beta_{b_2}| \left| \sum_{d \in \mathcal{D}} \left( \frac{d}{b_1 b_2} \right) \right|.$$

By Cauchy’s inequality and since  $\tau(n) \ll n^\epsilon$  we get

$$\begin{aligned} \mathcal{M}^2(\beta, \mathcal{D}) &\ll \|\beta\|^4 B^\epsilon \sum_{1 \leq n \leq B^2} \left| \sum_{d \in \mathcal{D}} \left( \frac{d}{n} \right) \right|^2 \\ &\leq \|\beta\|^4 B^\epsilon \sum_{d_1, d_2 \in \mathcal{D}} \left| \sum_{1 \leq n \leq B^2} \left( \frac{d_1 d_2}{n} \right) \right|. \end{aligned}$$

If  $d_1 = d_2$  then we use the trivial bound  $\sum_n \ll B^2$  and if  $d_1 \neq d_2$  then  $d_1 d_2$  is not a square, so by Lemma 1  $\sum_n \ll D \log 2D$ . Gathering these results together we obtain

$$\mathcal{M}(\beta, \mathcal{D}) \leq c(\epsilon)(BD)^\epsilon (|\mathcal{D}| D^{1/2} + |\mathcal{D}|^{1/2} B) \|\beta\|^2.$$

The dual form of the above (see Lemma 7) is just the assertion of Theorem 4.

II. Proof of Theorem 2. Every  $d \in \mathcal{D}$  can be factored uniquely as  $d = uv^2$  or  $2uv^2$  where  $u$  is odd and squarefree. Therefore  $\mathcal{D}$  can be split up into 8 disjoint subsequences according to the residue class  $u \pmod{8}$ . Clearly it is enough to show (2.1) for each of such subsequences separately. The case of 4 subsequences of numbers  $2uv^2$  can be reduced to the case of 4 subsequences of numbers  $uv^2$  simply by changing the coefficients  $\beta_b$  into  $\left(\frac{2}{b}\right)\beta_b$ . In other words we may assume, without loss of generality, that all elements  $d$  in  $\mathcal{D}$  have the squarefree parts odd and congruent  $(\text{mod } 8)$ .

Now, we can write

$$\mathcal{M}(\beta, \mathcal{D}) = \sum_v \sum_{u \in \mathcal{D}(v)} |c(u, v)|^2$$

where  $\mathcal{D}(v) = \{u; uv^2 \in \mathcal{D}, u \text{ squarefree}\}$  and

$$c(u, v) = \sum_{\substack{1 \leq b \leq B \\ (b, v) = 1}} \beta_b \left( \frac{u}{b} \right).$$

Hence by Cauchy’s inequality we get

$$\begin{aligned} \mathcal{M}^2(\beta, \mathcal{D}) &\leq \Delta_1 \sum_v v^{-1/2} \sum_{u \in \mathcal{D}(v)} |c(u, v)|^4 \\ &\leq \Delta_1 \sum_v v^{-1/2} \sum_{1 \leq b_1, b_2, b_3 \leq B} |\beta_{b_1} \beta_{b_2} \beta_{b_3}| \left| \sum_{u \in \mathcal{D}(v)} c(u, v) \left( \frac{u}{b_1 b_2 b_3} \right) \right|. \end{aligned}$$

By Cauchy’s inequality again we obtain

$$M^4(\beta, \mathcal{D}) \leq \|\beta\|^6 \Delta_1^2 \left( \sum_{1 \leq v \leq 2D} \frac{1}{v} \right) \mathcal{S}(\beta, \mathcal{D}) \tag{2.4}$$

where

$$\mathcal{S}(\beta, \mathcal{D}) = \sum_{1 \leq v \leq 2D} \sum_{1 \leq n \leq B^3} \tau_3(n) \left| \sum_{u \in \mathcal{D}(v)} c(u, v) \left( \frac{u}{n} \right) \right|^2.$$

Here we have  $\tau_3(n) \ll n^\epsilon f(n)$  where  $f(x)$  is given by (2.3) with  $N = B^3$ . Accordingly we obtain

$$\mathcal{S}(\beta, \mathcal{D}) \ll B^{3\epsilon} \mathcal{S}_f(\beta, \mathcal{D}), \tag{2.5}$$

say. Squaring out the innermost sum in  $\mathcal{S}_f(\beta, \mathcal{D})$  and changing the order of summation we get

$$\mathcal{S}_f(\beta, \mathcal{D}) = \sum_v \sum_{\substack{u_1, u_2 \in \mathcal{D}(v) \\ u_1 \neq u_2}} c(u_1, v) c(u_2, v) \sum_n f(n) \left( \frac{u_1 u_2}{n} \right) + O(B^3 M(\beta, \mathcal{D})). \tag{2.6}$$

Put  $r = (u_1, u_2)$ ,  $u_1 = r q_1$ ,  $u_2 = r q_2$ , so  $1 < q_1, q_2 \leq Q = Q(r, v) = 4D/rv^2$ . By Lemma 6 we deduce that

$$\begin{aligned} & \sum_{\substack{u_1, u_2 \in \mathcal{D}(v) \\ u_1 \neq u_2}} c(u_1, v) c(u_2, v) \sum_n f(n) \left( \frac{u_1 u_2}{n} \right) \\ &= \sum_r \sum_{\substack{1 < q_1, q_2 \leq Q(r, v) \\ r q_1, r q_2 \in \mathcal{D}(v)}} \mu^2(q_1 q_2) c(r q_1, v) c(r q_2, v) \sum_{(n, r)=1} f(n) \left( \frac{q_1 q_2}{n} \right) \\ &\ll \tau^2 \exp(-\tau^2) \sum_r r^{3/2} Q^4(r, v) \sum_{r q \in \mathcal{D}(v)} |c(r q, v)|^2 \\ &+ D^\epsilon B^3 \sum_{1 \leq r \leq D} \sum_{k|r} \frac{1}{k} \sum_{1 \leq e \leq D} \frac{1}{e} \sum_{1 \leq m \leq M} \left| \sum_{\substack{1 \leq h \leq H \\ e h r \in \mathcal{D}(v)}} \lambda_h \left( \frac{h}{m} \right) \right|^2, \end{aligned} \tag{2.7}$$

with

$$M = M(k/r^2 v^4) = \tau k r^{-2} v^{-4} D^2 B^{-3},$$

$$H = H(erv^2) = 4D/erv^2$$

and some  $\lambda_h$  independent of  $m$  such that

$$|\lambda_h| \leq h^{-1/2} |c(ehr, v)|.$$

For the innermost sum we apply Theorem 4 giving

$$\sum_m \left| \sum_h \lambda_h \left( \frac{h}{m} \right) \right|^2 \ll D^\epsilon \sum_{ehr \in \mathcal{D}(v)} |\lambda_h|^2 \{ |\mathcal{D}_{erv^2}| H^{1/2}(erv^2) + |\mathcal{D}_{erv^2}|^{1/2} M(k/r^2 v^4) \}.$$

If  $ehr \in \mathcal{D}(v)$  then  $ehrv^2 \in \mathcal{D}$  so  $D < ehrv^2 \leq 4D$ . This yields

$$\sum_{ehr \in \mathcal{D}(v)} |\lambda_h|^2 \ll erv^2 D^{-1} \sum_{ehr \in \mathcal{D}(v)} |c(ehr, v)|^2.$$



Moreover we have  $|\mathcal{D}_{erv^2}| \leq (erv^2)^{-1/2} \Delta_2$ . Hence we conclude that

$$\sum_m \left| \sum_h \lambda_h \left( \frac{h}{m} \right) \right|^2 \ll D^\epsilon \{ \Delta_2 D^{-1/2} + \tau k \Delta_2^{1/2} D B^{-3} \} \sum_{ehr \in \mathcal{D}(v)} |c(ehr, v)|^2.$$

Inserting this into (2.7) by (2.6) we infer

$$\begin{aligned} \mathcal{S}_f(\beta, \mathcal{D}) &\ll \tau^2 \exp(-\tau^2) D^4 \sum_v \sum_{rq \in \mathcal{D}(v)} |c(rq, v)|^2 \\ &\quad + D^\epsilon \{ \Delta_2 D^{-1/2} B^3 + \tau \Delta_2^{1/2} D \} \sum_v \sum_{ehr \in \mathcal{D}(v)} |c(ehr, v)|^2 \\ &\ll D^\epsilon \{ \Delta_2 D^{-1/2} B^3 + \Delta_2^{1/2} D \} \mathcal{M}(\beta, \mathcal{D}) \end{aligned}$$

by taking  $\tau = \log 4D$ . Finally combining (2.4)–(2.6) we complete the proof of Theorem 2.

*III. Proof of Theorem 3.* We apply Theorem 2 for the sequence

$$\mathcal{D} = \{a^2 - 4; A < a \leq 2A\}.$$

Therefore  $D = A^2$  and it remains to determine the parameters  $\Delta_1$  and  $\Delta_2$ .

If  $a^2 \equiv 4 \pmod r$  then there exists a decomposition  $r = r_1 r_2$  such that  $a \equiv 2 \pmod{r_1}$  and  $a \equiv -2 \pmod{r_2}$ . Hence  $(r_1, r_2) | 4$  and if  $r = v^2$  then  $r_1 = v_1^2$  or  $2v_1^2$  and  $r_2 = v_2^2$  or  $2v_2^2$ . This yields

$$\begin{aligned} \sum v^{1/2} |\mathcal{D}_{v^2}| &\leq \sum_{r_1, r_2} (r_1 r_2)^{1/4} \sum_{\substack{a \equiv 2 \pmod{r_1} \\ a \equiv -2 \pmod{r_2} \\ A < a \leq 2A}} 1 \\ &\leq \sum_{r_1} r_1^{1/2} \sum_{\substack{a \equiv 2 \pmod{r_1} \\ A < a \leq 2A}} \tau(a+2) + \sum_{r_2} r_2^{1/2} \sum_{\substack{a \equiv -2 \pmod{r_2} \\ A < a \leq 2A}} \tau(a-2) \\ &\ll A^{1+\epsilon} \left( \sum_{r_1} r_1^{-1/2} + \sum_{r_2} r_2^{-1/2} \right) \ll A^{1+\epsilon} \log 2A. \end{aligned}$$

We also have

$$|\mathcal{D}_r| \ll \tau(r) \left( \frac{A}{r} + 1 \right) \ll r^{-1/2} A^{1+\epsilon}.$$

Therefore  $\Delta_1, \Delta_2 \ll A^{1+\epsilon}$  and the rest of the proof follows from Theorem 2.

**3. Quadratic congruences.** Let  $a$  and  $c \geq 1$  be integers and let  $\rho(c, a)$  stand for the number of incongruent solutions  $x \pmod c$  of

$$x^2 - ax + 1 \equiv 0 \pmod c. \tag{3.1}$$

Our aim here is to evaluate  $\rho(c, a)$  on average with respect to  $a$  and  $c$ . We have

$$\sum_{a \pmod c} \rho(c, a) = \phi(c), \tag{3.2}$$

so trivially

$$\sum_{1 \leq a \leq A} \rho(c, a) = \frac{\phi(c)}{c} A + O(c).$$

The error term  $O(c)$  proves to be too big for our applications in mind. On applying A. Weil’s bounds for character sums (see Lemma 8) we can reduce the error term to  $O(c^{1/2}\tau(c))$  which is still not satisfactory. In two papers [5], [6] sharper results were established on average with respect to  $c$  by means of D. Burgess’ inequality.

In this section we improve the result (25) of [6] by an appeal to the corollary to Theorem 3.

**THEOREM 5.** *Let  $A, C \geq 1$  and  $q \geq 1, q$  squarefree. For any  $\varepsilon > 0$  we have*

$$\begin{aligned} \mathfrak{B}(A, C; q) &= \sum_{\substack{1 \leq c \leq C \\ c \equiv 0 \pmod{q}}} \left| \sum_{2 < a \leq A} \rho(c, a) - \frac{\phi(c)}{c} A \right| \\ &\ll (AC)^\varepsilon \{A^{5/6} + q^{1/4} A^{5/8} + A^{1/3} C^{1/6}\} \frac{C}{q}, \end{aligned}$$

the constant implied in  $\ll$  depending on  $\varepsilon$  alone.

*Proof.* Every  $c$  can be uniquely factored as  $c = kl$  where  $k$  is squarefree,  $4l$  is squareful and  $(k, 4l) = 1$ . For notational simplicity in the sequel we do not repeat these properties of  $k$  and  $l$ , so the reader should keep them in mind to the end of the proof. Since  $\rho(c, a)$  is multiplicative in  $c$  and  $k$  is squarefree and odd we have

$$\rho(c, a) = \rho(l, a)\rho(k, a) = \rho(l, a) \sum_{b|k} \left(\frac{a^2-4}{b}\right).$$

For a parameter  $X$  to be chosen later we partition  $\rho(c, a) = \rho_1(c, a) + \rho_2(c, a)$  where

$$\rho_1(c, a) = \rho(l, a) \sum_{b|k, bl \leq X} \left(\frac{a^2-4}{b}\right)$$

and

$$\rho_2(c, a) = \rho(l, a) \sum_{b|k, bl > X} \left(\frac{a^2-4}{b}\right).$$

The first term  $\rho_1(c, a)$  contributes to the main term. We have

$$\begin{aligned} \sum_{2 < a \leq A} \rho_1(c, a) &= \sum_{\substack{b|k \\ bl \leq X}} \sum_{2 < a \leq A} \rho(l, a) \left(\frac{a^2-4}{b}\right) \\ &= \sum_{\substack{b|k \\ bl \leq X}} \sum_{\lambda \pmod{l}} \rho(l, \lambda) \sum_{\substack{2 < a \leq A \\ a \equiv \lambda \pmod{l}}} \left(\frac{a^2-4}{b}\right). \end{aligned}$$

We evaluate the innermost sum by

LEMMA 8. *If  $b, l \geq 1, (b, l) = 1, b$  squarefree then*

$$\sum_{\substack{2 < a \leq A \\ a \equiv \lambda \pmod{l}}} \left( \frac{a^2 - 4}{b} \right) = \frac{\mu(b)}{bl} A + O(\tau(b)b^{1/2} \log 2A).$$

*Proof.* It follows in a standard way from A. Weil’s bounds for character sums, precisely from (see [8])

$$\left| \sum_{a \pmod{b}} \left( \frac{a^2 - 4}{b} \right) e\left(\frac{ah}{b}\right) \right| \leq b^{1/2} \tau(b) \tag{3.3}$$

and that for  $h = 0$  the sum is equal to  $\mu(b)$ .

By Lemma 8 and (3.2) we further infer

$$\begin{aligned} \sum_{2 < a \leq A} \rho_1(c, a) &= A \frac{\phi(l)}{l} \sum_{\substack{b|k \\ bl \leq X}} \mu(b)b^{-1} + O((lX)^{1/2}(AC)^\epsilon) \\ &= \frac{\phi(c)}{c} A + O\left\{ \left( \frac{lA}{X} + (lX)^{1/2} \right) (AC)^\epsilon \right\}. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{B}_1 &= \sum_{\substack{1 \leq c \leq C \\ c \equiv 0 \pmod{q}}} \left| \sum_{2 < a \leq A} \rho_1(c, a) - \frac{\phi(c)}{c} A \right| \\ &\ll (AC)^\epsilon \sum_{\substack{kl \leq C \\ kl \equiv 0 \pmod{q}}} \{lAX^{-1} + (lX)^{1/2}\} \\ &\ll (AC)^\epsilon C \sum_{l \leq C} \frac{(l, q)}{lq} \{lAX^{-1} + (lX)^{1/2}\} \\ &\ll (AC)^\epsilon \{AC^{1/2}X^{-1} + X^{1/2}\} \frac{C}{q}. \end{aligned} \tag{3.4}$$

Now it remains to estimate

$$\mathcal{B}_2 = \sum_{\substack{1 \leq c \leq C \\ c \equiv 0 \pmod{q}}} \left| \sum_{2 < a \leq A} \rho_2(c, a) \right|.$$

Let  $L \geq 1$  be a parameter to be chosen later. We split up the summation over  $c$  into two sums

$$\mathcal{B}_3 = \sum_{\substack{kl \leq C, l \leq L \\ kl \equiv 0 \pmod{q}}} \left| \sum_{2 < a \leq A} \rho_2(kl, a) \right|$$

and

$$\mathcal{B}_4 = \sum_{\substack{kl \leq C, l > L \\ kl \equiv 0 \pmod{q}}} \left| \sum_{2 < a \leq A} \rho_2(kl, a) \right|.$$

First we estimate  $\mathcal{B}_4$  by essentially elementary means. We have  $|\rho_2(kl, 0)| \leq \rho(l, a)\tau(k)$  and

$$\sum_{\substack{k \leq C/l \\ k \equiv 0 \pmod{q/(l, q)}}} \tau(k) \ll \frac{(l, q)}{lq} C^{1+\epsilon}.$$

Hence

$$\mathcal{B}_4 \ll \frac{C^{1+\epsilon}}{q} \sum_{L < l \leq C} \frac{(l, q)}{l} \sum_{2 < a \leq A} \rho(l, a).$$

Since  $l$  is squareful the smallest  $n$  such that  $l | n^2$  satisfies  $l^{1/2} \leq n \leq l^{3/4}$ . Moreover  $\rho(l, a) \leq \rho(n^2, a) \ll (a^2 - 4, n^2)^{1/2} n^\epsilon$ , so

$$\begin{aligned} \mathcal{B}_4 &\ll q^{-1} C^{1+\epsilon} \sum_{L^{1/2} < n \leq C^{3/4}} (n, q) n^{-4/3} \sum_{2 < a \leq A} (a^2 - 4, n^2)^{1/2} \\ &\ll q^{-1} C^{1+\epsilon} L^{-1/6} A. \end{aligned} \tag{3.5}$$

This bound is admissible for  $L = A$ .

Now it remains to estimate  $\mathcal{B}_3$ . Letting  $\nu(c)$  be the sign of the innermost sum we obtain

$$\mathcal{B}_3 = \sum_{\substack{kl \leq C, l < L \\ kl \equiv 0 \pmod{q}}} \sum_{2 < a \leq A} \nu(kl) \sum_{\substack{b | k \\ b > X/l}} \rho(l, a) \sum_{b | k} \left( \frac{a^2 - 4}{b} \right).$$

Then writing  $k = br$  we get by Cauchy's inequality

$$\begin{aligned} \mathcal{B}_3 &= \sum_{\substack{rl \leq C \\ l < L, r < C/X}} \sum_{2 < a \leq A} \rho(l, a) \sum_{\substack{X/l < b < C/lr \\ brl \equiv 0 \pmod{q}}} \nu(brl) \left( \frac{a^2 - 4}{b} \right) \\ &\ll \sum_{\substack{rl \leq C \\ l < L, r < C/X}} \left\{ A(l) \sum_{2 < a \leq A} \left| \sum_{B_1 < b \leq B_2} \nu(bqr/l(rl, q)) \left( \frac{a^2 - 4}{b} \right) \right|^2 \right\}^{1/2} \end{aligned} \tag{3.6}$$

where  $B_1 = X(rl, q)/lq$ ,  $B_2 = C(rl, q)/lq$  and

$$A(l) = \sum_{2 < a \leq A} \rho^2(l, a) \leq \frac{2A}{l} \sum_{a \pmod{l}} \rho^2(l, a).$$

For any  $c \geq 1$  we have

$$\sum_{a \pmod{c}} \rho^2(c, a) = \#\{x, y \pmod{c}; (xy, c) = 1, (x - y)(xy + 1) \equiv 0 \pmod{c}\} \ll c^{1+\epsilon}.$$

Therefore

$$A(l) \ll A^{1+\epsilon}. \tag{3.7}$$

By (3.6), (3.7) and Theorem 3 we obtain

$$\begin{aligned} \mathcal{B}_3 &\ll (AC)^\epsilon \sum_{l < L} \sum_{r < C/X} \left\{ A^{5/4} C^{1/2} \left( \frac{rl, q}{rlq} \right)^{1/2} + A^{5/6} C \frac{(rl, q)}{rlq} \right\} \\ &\ll (AC)^\epsilon \sum_{l < L} \left\{ A^{5/4} C X^{-1/2} \left( \frac{l, q}{lq} \right)^{1/2} + A^{5/6} C \frac{(l, q)}{lq} \right\} \\ &\ll (AC)^\epsilon \{ A^{5/4} C X^{-1/2} q^{-1/2} + A^{5/6} C q^{-1} \}. \end{aligned} \tag{3.8}$$

Combining (3.4)–(3.8) we obtain

$$\mathcal{B}(A, C; q) \ll (AC)^{\epsilon} \{A^{5/6} + q^{1/2} A^{5/4} X^{-1/2} + X^{1/2} + AC^{1/2} X^{-1}\} \frac{C}{q}$$

for any  $X > 0$ . On putting  $X = q^{1/2} A^{5/4} + A^{2/3} C^{1/3}$  we complete the proof.

By partial summation we infer from Theorem 5 the following

COROLLARY. Let  $0 < \alpha \leq 1$ ,  $\xi \geq 0$ ,  $q \geq 1$ ,  $q$  squarefree and  $C \geq 1$ . We then have

$$\sum_{\substack{1 \leq c \leq C \\ c \equiv 0 \pmod{q}}} \left| \sum_{|a| \leq \alpha c} e\left(\frac{a}{c} \xi\right) \left(\rho(c, a) - \frac{\phi(c)}{c}\right) \right| \ll (1 + \alpha \xi) \left\{ \left(\frac{C}{q}\right)^{1/2} + (\alpha C)^{5/6} + q^{1/4} (\alpha C)^{5/8} + (\alpha C)^{1/3} C^{1/6} \right\} \frac{C^{1+\epsilon}}{q}. \quad (3.9)$$

REMARK. We included the terms with  $|a| \leq 2$  using trivial bounds  $\rho(qc_0, a) \ll c_0^{1/2} (c_0, q)^{1/2} (qc_0)^{\epsilon}$  for  $q$  squarefree.

**4. Estimates for sums of Kloosterman sums.** In this section we apply the corollary to Theorem 5 to estimate sums of Kloosterman sums  $\mathcal{S}(n, n; c)$  over moduli  $c \equiv 0 \pmod{q}$  as well as over the coefficients  $n$ . It is the latter parameter which yields an extra saving compared to the Weil upper bound (1.3). By contrast the conjecture of Y. V. Linnik [7] and A. Selberg [9] predicts a cancellation of terms  $\mathcal{S}(m, n; c)$  in sums over the moduli  $c$ . In fact the analogue of the Linnik–Selberg conjecture, namely the following statement

$$\sum_{c \equiv 0 \pmod{q}} g\left(\frac{x}{c}\right) \mathcal{S}(m, n; c) \ll X^{1+\epsilon}$$

for a smooth function  $g(\xi)$  compactly supported in  $\mathbb{R}^+$ , and any  $X \geq 1$  is equivalent to the eigenvalue conjecture (1.1).

We first prove the following general result.

THEOREM 6. Let  $f_0(\xi)$  be a smooth function supported in  $[1, 2]$ ,  $N \geq 1$ ,  $C \geq 4N$ ,  $q \geq 1$ ,  $q$  squarefree. Put  $f(n) = f_0(n/N)$ . We then have

$$\sum_{\substack{C < c \leq 2C \\ c \equiv 0 \pmod{q}}} \left| \sum_n f(n) \mathcal{S}(n, n; c) \right| \ll \left\{ \left(\frac{C}{q}\right)^{1/2} + \left(\frac{C}{N}\right)^{5/6} + q^{1/4} \left(\frac{C}{N}\right)^{5/8} + \left(\frac{C}{N}\right)^{1/3} C^{1/6} + \frac{C}{N^2} \right\} \frac{(CN)^{1+\epsilon}}{q}.$$

Proof. We have

$$\mathcal{S}(n, n; c) = \sum_{a \pmod{c}} e\left(\frac{a}{c} n\right) \rho(c, a).$$

By Poisson’s formula

$$\sum_n f(n) e\left(\frac{a}{c} n\right) = \sum_h \hat{f}\left(h + \frac{a}{c}\right).$$

Suppose that  $|a/c| \leq \frac{1}{2}$ , then

$$\hat{f}\left(h + \frac{a}{c}\right) \ll (hN)^{-2} \quad \text{if } h \neq 0$$

and

$$\hat{f}\left(\frac{a}{c}\right) \ll N^{-2} \quad \text{if } |a/c| \geq \alpha = N^{\varepsilon-1}.$$

Hence

$$\sum_n f(n) \mathcal{P}(n, n; c) = \int f(\xi) \sum_{|a| < \alpha c} e\left(\frac{a}{c} \xi\right) \rho(c, a) d\xi + O(cN^{-2}).$$

Now, by (3.9) our sum is equal to

$$\sum_{\substack{C < c \leq 2C \\ c \equiv 0 \pmod{q}}} \frac{\phi(c)}{c} \left| \int f(\xi) \sum_{|a| \leq \alpha c} e\left(\frac{a}{c} \xi\right) d\xi \right| + O\left(\left\{\left(\frac{C}{q}\right)^{1/2} + \left(\frac{C}{N}\right)^{5/6} + q^{1/4} \left(\frac{C}{N}\right)^{5/8} + \left(\frac{C}{N}\right)^{1/3} C^{1/6} + \frac{C}{N^2}\right\} \frac{(CN)^{1+\varepsilon}}{q}\right).$$

Here, the innermost sum is equal to

$$\frac{e(a\xi) - e(-a\xi)}{e(\xi/c) - 1} + O(1).$$

Notice that  $\xi/c \leq 2N/C \leq \frac{1}{2}$  and  $\alpha\xi \geq N^\varepsilon$ . Therefore integrating over  $\xi$  yields

$$\left| \int f(\xi) \sum_{|a| \leq \alpha c} e\left(\frac{a}{c} \xi\right) d\xi \right| \ll N.$$

Gathering the above results together we complete the proof.

From Theorem 6 it is easy to deduce the following

**COROLLARY.** *Let  $g(\xi)$  be a smooth function supported in  $[1, \sqrt{2}]$ . For  $q \geq 1$ ,  $q$  squarefree and  $X \geq 2q$  we have*

$$\sum_{c \equiv 0 \pmod{q}} \frac{1}{c} \left| \sum_n \frac{1}{n} \exp\left(-\frac{n}{q}\right) g\left(\frac{kn}{c} X\right) \mathcal{P}(n, n; c) \right| \ll (X^{5/6} + q^{1/4} X^{5/8}) \frac{X^\varepsilon}{q},$$

the constant implied in  $\ll$  depending on  $g(\xi)$  and  $\varepsilon$  at most.

*Proof.* The partial sum with  $c \leq C_1 = X^{4/3}$  by Weil's upper bound (1.3) is

$$\sum_{c \leq C_1} \ll q^{-1} X^{2/3+\varepsilon}$$

and the partial sum with  $c > C_2 = qX \log X$  by the trivial estimate  $|\mathcal{P}(n, n; c)| \leq c$  is

$$\sum_{c > C_2} \leq \frac{1}{q} \sum_{c > X \log X} \sum_{nX > c} \exp(-n) \ll \frac{1}{q}.$$

We split the remaining range of summation over  $c$  into  $\ll \log X$  subintervals of the type  $(C, \sqrt{2}C)$  with  $C_1 < C \leq C_2$ . For each of the resulting sums separately Theorem 6 is

applicable with  $N = C/4\pi X$  and

$$f(x) = \frac{1}{x} \exp\left(-\frac{xC}{qX}\right) g\left(\frac{4\pi xC}{c}\right)$$

giving

$$\sum_{\substack{C < c \leq 2C \\ c \equiv 0 \pmod{q}}} \frac{1}{c} \left| \sum_n \frac{1}{n} \exp\left(-\frac{n}{q}\right) g\left(\frac{4\pi n}{c} X\right) \mathcal{F}(n, n; c) \right| \ll \left\{ \left(\frac{C}{q}\right)^{1/2} + X^{5/6} + q^{1/4} X^{5/8} + X^{1/3} C^{1/6} + \frac{X^2}{C} \right\} \frac{C^e}{q}.$$

Gathering the above results together we complete the proof.

**5. Lower bounds for Fourier coefficients of cusp forms.** We shall show a prototype of (1.9). Our method is so special that it requires  $q$  be prime. Thus  $\Gamma_0(q)$  has two inequivalent cusps  $\infty$  and  $0$ . Let  $u_j(z)$  be a Maass cusp form whose Fourier expansions at  $a = 0$  and  $a = \infty$  are given by (1.7). Put

$$c_{j0} = \sum_1^\infty \frac{q}{n} \exp\left(-\frac{n}{q}\right) |\rho_{j0}(n)|^2$$

and

$$c_{j\infty} = \sum_1^\infty \frac{1}{n} \exp(-n) |\rho_{j\infty}(n)|^2.$$

**THEOREM 7.** *If  $\lambda_j$  is an exceptional eigenvalue then*

$$c_j = c_{j0} + c_{j\infty} \geq \sqrt{3}.$$

*Proof.* This result is Lemma 3 of [6]. Let  $P(Y)$  stand for the euclidean strip

$$P(Y) = \{z = x + iy; |x| \leq \frac{1}{2}, y \geq Y\}.$$

One can find positive numbers  $Y_\alpha$  such that

$$F \subseteq \bigcup_\alpha \sigma_\alpha P(Y_\alpha). \tag{5.1}$$

Hence and by the Fourier expansions (1.7) we obtain

$$\begin{aligned} 1 &= \int_F |u_j(z)|^2 dz \leq \sum_\alpha \int_{\sigma_\alpha P(Y_\alpha)} |u_j(z)|^2 dz \\ &= \sum_\alpha \int_{P(Y_\alpha)} |u_j(\sigma_\alpha z)|^2 dz = 2 \sum_\alpha \sum_1^\infty |\rho_{j\alpha}(n)|^2 \int_{2\pi n Y_\alpha}^\infty K_{it_j}^2(y) \frac{dy}{y} \end{aligned}$$

because  $\rho_{j\alpha}(n) = \rho_{j\alpha}(-n)$ . We have  $0 < it_j < \frac{1}{2}$ , so

$$\int_A^\infty K_{it_j}^2(y) \frac{dy}{y} \leq \int_A^\infty K_{1/2}^2(y) \frac{dy}{y} = \frac{\pi}{2} \int_A^\infty e^{-2y} \frac{dy}{y^2} \leq \frac{\pi}{2A} e^{-2A}.$$

This yields

$$\sum_a \frac{1}{Y_a} \sum_1^\infty \frac{1}{n} \exp(-4\pi n Y_a) |\rho_{ja}(n)|^2 \geq 2. \tag{5.2}$$

Now notice that  $\sigma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\sigma_0 = \begin{pmatrix} 0 & -1/\sqrt{q} \\ \sqrt{q} & 0 \end{pmatrix}$  satisfy (1.6) and that  $Y_\infty = \sqrt{3}/2$  and  $Y_0 = \sqrt{3}/2q$  satisfy (5.1) completing the proof of Theorem 7.

**6. Proof of the density theorem.** We begin by applying Kuznetsov’s formula for the Hecke group  $\Gamma = \Gamma_0(q)$ , see [1]. Let  $\{u_j(z)\}$  be the orthonormal basis of Maass cusp forms whose Fourier expansions at a cusp  $a$  are given by (1.7). Let  $E_c(z, s)$  be the Eisenstein series associated with the cusp  $c$  whose Fourier expansion at  $a$  is given by

$$E_c(\sigma_a z, s) = \text{constant term} + \sqrt{y} \sum_{n \neq 0} \phi_{can}(s) K_{s-1/2}(2\pi |n| y) e(nx).$$

Let  $\{\Psi_{jk}(z)\}_{1 \leq j \leq \theta_k}$  be an orthonormal basis of the space  $\mathcal{M}_k^0(\Gamma)$  of holomorphic cusp forms of weight  $k$  whose Fourier expansion at  $a$  is given by

$$\psi_{jk}(\sigma_a z) = j(\sigma_a, z)^k \sum_1^\infty \psi_{jk}(a, n) e(nz).$$

Let  $f(x)$  be a smooth function supported in  $(0, \infty)$ . Define

$$\hat{f}(t) = \frac{\pi}{2i \operatorname{sh} \pi t} \int_0^\infty [J_{2it}(x) - J_{-2it}(x)] f(x) \frac{dx}{x},$$

$$\tilde{f}(k) = \int_0^\infty J_k(x) f(x) \frac{dx}{x},$$

$$V_0(a, n) = \sum_{0 < \lambda_j < 1/4} \frac{\hat{f}(t_j)}{\operatorname{ch} \pi t_j} |\rho_{ja}(n)|^2,$$

$$V_1(a, n) = \sum_{\lambda_j \geq 1/4} \frac{\hat{f}(t_j)}{\operatorname{ch} \pi t_j} |\rho_{ja}(n)|^2,$$

$$V_2(a, n) = \frac{1}{2\pi} \sum_{k \text{ even}} i^k \tilde{f}(k-1) \frac{(k-1)!}{(4\pi n)^{k-1}} \sum_{1 \leq j \leq \theta_k} |\psi_{jk}(a, n)|^2,$$

$$V_3(a, n) = \frac{1}{\pi} \sum_c \int_{-\infty}^\infty \hat{f}(t) |\phi_{can}(\frac{1}{2} + it)|^2 dt,$$

$$\mathcal{S}(a, n) = \sum_{c > 0} \frac{1}{c} \mathcal{S}_{aa}(n, n; c) f\left(\frac{4\pi n}{c}\right).$$

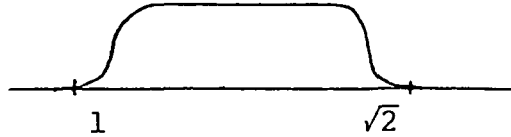
Here  $\mathcal{S}_{aa}(n, n; c)$  is the generalized Kloosterman sum. In case of  $\Gamma = \Gamma_0(q)$ ,  $a = 0$  or  $\infty$  we have  $c \equiv 0 \pmod{q}$  and the Kloosterman sums  $\mathcal{S}_{aa}(n, n; c)$  coincide with the classical ones  $\mathcal{S}(n, n; c)$ .



The sum formula of Kuznetsov says that

$$\sum_{i=0}^3 V_i(a, n) = \mathcal{S}(a, n). \tag{6.1}$$

We take  $f(x) = g(xX)$  where  $g(\xi)$  is a smooth function whose graph is



and  $X \geq 2q$ . We have  $\tilde{f}(k-1) \ll k^{-2}$  and for real  $t$ ,  $\hat{f}(t) \ll (t^2 + 1)^{-1} \log X$ . This together with Theorem 2 of Deshouillers and Iwaniec [1] shows that the series  $V_i(a, n)$  with  $i = 1, 2, 3$  converges rapidly and that

$$V_i(a, n) \ll \left(1 + \frac{n}{q}\right) (nX)^{\epsilon}, \quad i = 1, 2, 3. \tag{6.2}$$

Hence by (6.1)

$$V_0(a, n) = \mathcal{S}(a, n) + O\left(\left(1 + \frac{n}{q}\right) (nX)^{\epsilon}\right). \tag{6.3}$$

Multiply both sides of (6.3) by

$$\frac{q}{n} \exp\left(-\frac{n}{q}\right) \quad \text{if } a = 0$$

and by

$$\frac{1}{n} \exp(-n) \quad \text{if } a = \infty$$

and sum over  $n = 1, 2, \dots$ , and  $a = 0, \infty$  getting (see Theorem 7)

$$\begin{aligned} \sum_{0 < \lambda_j < 1/4} c_j \frac{\hat{f}(t_j)}{\text{ch } \pi t_j} &= \sum_1^{\infty} \frac{q}{n} \exp\left(-\frac{n}{q}\right) \mathcal{S}(0, n) \\ &\quad + \sum_1^{\infty} \frac{1}{n} \exp(-n) \mathcal{S}(\infty, n) \\ &\quad + O(qX^{\epsilon}) \\ &\ll (X^{5/6} + q^{1/4} X^{5/8} + qX^{\epsilon}), \end{aligned}$$

the last inequality following from the Corollary to Theorem 6.

On the left hand side the arguments  $t_j$  of  $\hat{f}(t_j)$  are purely imaginary. Using the power series expansion of the Bessel functions  $J_{2it_j}(x)$  we deduce that

$$\hat{f}(t_j) \gg X^{2|t_j|} = X^{2(s_j - 1/2)}.$$

Combining this with (6.4) and Theorem 6 we conclude that

$$\sum_{1/2 < s_j < 1} X^{2(s_j - 1/2)} \ll (X^{5/6} + q^{1/4} X^{5/8} + q) X^\varepsilon$$

Putting  $X = q^{6/5}$  we complete the proof.

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