Character Theory of Finite Groups

January–March 2009

EXERCISE SHEET 3

Starred questions are somewhat harder. Do not attempt questions 6 and 7 until character triples and the Schur multiplier have been introduced in lectures.

1. (i) Suppose $\mathcal{K}$ is a conjugacy class of S_n contained in A_n; then $\mathcal{K}$ is called *split* if $\mathcal{K}$ is a union of two conjugacy classes of A_n. Show that the number of split conjugacy classes contained in A_n is equal to the number of characters $\chi \in \mathrm{Irr}(S_n)$ such that χ_{A_n} is not irreducible. (*Hint.* Consider the vector space of class functions on A_n which are invariant under conjugation by the transposition (12).)

   (ii) Let $g \in A_n$ have a cyclic decomposition with cycle lengths

   $$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0.$$

   Show that the conjugacy class of g in S_n is split if and only if the numbers μ_i are all distinct and odd. Deduce that the number of partitions λ of n such that $\lambda = \lambda'$ is equal to the number of partitions $(\mu_1, \ldots, \mu_k)$ of n with all parts μ_i distinct and odd.

   (iii)* Find an explicit combinatorial one-to-one correspondence between the set of self-conjugate partitions of n and the set of partitions of n with all parts distinct and odd.

2. Let χ be a character of G and let p be a prime. For $g \in G$, write $g = g_p g_{p'} = g_{p'} g_p$ where g_p is a p-element and $g_{p'}$ is p-regular. Define $\theta(g) = \chi(g_{p'})$. Show that θ is a generalised character of G.

3. Let $|G| = p^k m$ where the prime p does not divide m. Suppose $\chi \in \mathrm{Irr}(G)$ and $\chi(1)$ is divisible by p^k. Define a class function $\theta : G \to \mathbb{C}$ by

   $$\theta(g) = \begin{cases} \chi(g) & \text{if } g \text{ is } p\text{-regular}, \\ 0 & \text{otherwise.} \end{cases}$$

   (i) Let $g \in G$ and write $|C_G(g)| = p^l s$ where p does not divide s. Prove that $p^{-l}\chi(g)$ is an algebraic integer. (*Hint.* Consider the map ω_χ. Also, use the following easy fact: if $a \in \mathbb{C}$ and both au and av are algebraic integers for some coprime $u, v \in \mathbb{Z}$ then a is an algebraic integer.)

   (ii) Let E be a q-elementary subgroup of G where q is any prime. Write $E = PQ$ where P is a p-group, Q is a p'-group and $P \cap Q = [P, Q] = \{1\}$. Prove that $|P|^{-1}\chi_Q$ is a character of Q. Hence show that θ_E is a character of E.

   (iii) Deduce that θ is a generalised character of G.

   (iv) Show that $\chi(g) = 0$ for all $g \in G$ satisfying $p|o(g)$. (*Hint.* Consider $\langle \theta, \chi \rangle$.)

4. (Mackey formula) Let H and K be subgroups of G. Let T be a set of representatives of double H–K cosets; that is,

$$G = \bigcup_{t \in T} HtK$$

is a disjoint union. Let θ be a class function on H. Show that

$$(\theta^G)_K = \sum_{t \in T} (\theta^t_{H^t \cap K})^K,$$

where as usual θ^t is the class function on H^t given by $\theta^t(x) = \theta(txt^{-1})$, $x \in H^t$.

5.* Consider the group $\mathrm{GL}_2(q)$ of all invertible 2×2 matrices with entries in $\mathbb{F}_q$. Let α and β be distinct linear characters of the multiplicative group $\mathbb{F}_q^\times$. Let B be the subgroup of $\mathrm{GL}_2(q)$ consisting of all invertible upper-triangular matrices. Define the linear character $\phi = \phi_{\alpha,\beta}$ of B by

$$\phi\left(\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}\right) = \alpha(x)\beta(y).$$

Prove that ϕ^G is irreducible. Show that $\phi^G_{\alpha,\beta} = \phi^G_{\alpha',\beta'}$ if and only if either $\alpha = \alpha'$ and $\beta = \beta'$ or $\alpha = \beta'$ and $\beta = \alpha'$.

Hint. Use the Mackey formula.

6. Suppose (Id_G, σ) is a strong isomorphism of the character triple $(G, \{1\}, 1_{\{1\}})$ onto itself, where $\mathrm{Id}_G : G \to G$ is the identity map. Show that there exists a linear character λ of G such that, for all subgroups $H \leq G$ and all characters χ of H, $\sigma_H(\chi) = \chi\lambda_H$.

7. Let C be a cyclic group. Prove that the Schur multiplier $M(C)$ is trivial.

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