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CHARACTERISTIC CLASSES AND TRANSFER RELATIONS IN COBORDISM

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ABSTRACT. Decompositions of products of the Ray elements by free generators of small dimensions in the symplectic cobordism ring are obtained. In particular it is stated that most of the $4n$-dimensional generators, for n small, after multiplication by the Ray elements ϕ_i, $i \geqslant 0$, land in the ideal generated by Ray elements of low dimension.

1. INTRODUCTION

Immediately after its first appearence in the papers of J. Milnor [12] and S. P. Novikov [14], the symplectic cobordism attracted attention of many homotopy theorists. However, unlike the cobordism theories corresponding to other classical Lie groups — e.g. nonoriented ($\mathrm{O}(n)$), oriented ($\mathrm{SO}(n)$) and complex ($\mathrm{U}(n)$) — the structure of its coefficient ring remains largely unknown. In the study of symplectic cobordism various methods have been applied: the classical Adams spectral sequence [14], the Adams-Novikov spectral sequence [18, 20], the Atiyah-Hirzebruch spectral sequence [16], the use of characteristic classes and generalizations of formal groups [15, 5], and cobordism with singularities [19]. In this paper we apply the transfer maps to the study of the symplectic cobordism ring. Transfers first appeared in group theory at the beginning of the twentieth century in the works of I. Schur, as natural maps from the abelianization of a group to abelianizations of its subgroups, and then were generalized to other homologies and cohomologies of groups (see, e.g. [4]). In the work of J. C. Becker and P. H. Gottlieb [2], transfer maps were constructed as morphisms in the stable category and since then have been widely used in homotopy theory.

Since the work of S. P. Novikov [14] it is known that rationally the symplectic cobordism ring $M\mathrm{Sp}_*$ is isomorphic to the polynomial ring on an infinite number of generators which appear in dimensions $4n$ for all natural n. In the torsion part the key role is played by the family of elements $\phi_i \in M\mathrm{Sp}_{8i-3}$ of order 2 defined by Nigel Ray [15].

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Most of the relations between the Ray elements ϕ_i and free generators in the torsion part of the symplectic cobordism ring up to dimension 32 [18, 19] can be conventionally subdivided into three types: to the first type correspond relations which mainly follow from relations in the integral part coming from $M\mathrm{Sp}_{4n}$. In more detail, relations of the first type have form $(x+y)\phi_i = 0$, and these relations follow from the fact that the sum of free generators $x+y$ is divisible by 2, whereas the Ray elements are of order 2. Relations of the second type have form $z\phi_i = 0$ where z is again a $4n$-dimensional generator from the free part. The aim of the present paper is to elucidate origins of relations of the third type mentioned in the abstract above.

Let ζ denote the universal Sp(1)-bundle. Then $\zeta_1 \otimes_{\mathbb{C}} \zeta_2 \otimes_{\mathbb{C}} \zeta_3$ is a symplectic bundle over $B\mathrm{Sp}(1) \times B\mathrm{Sp}(1) \times B\mathrm{Sp}(1)$. Also $\zeta_1 \otimes_{\mathbb{C}} \zeta_2^2$ and $\zeta_1 \otimes_{\mathbb{R}} \zeta_2$ are symplectic bundles over $B\mathrm{Sp}(1) \times B\mathrm{Sp}(1)$.

Section 2 is devoted to the calculation of transfers [1, 6, 2, 11]. In Section 3 we prove the following main result:

Theorem 1.1. *Let $x_i = p_1(\zeta_i)$, $i = 1, 2$, be the first Conner-Floyd symplectic Pontyagin class. Let ϕ_j, $j \geqslant 0$, be the Ray elements, and let n be such that $M\mathrm{Sp}_{4m}$ is torsion free for $m \leqslant 2n-1$. Then:*

a) *the element $\phi_j p_1(\zeta_1 \otimes_{\mathbb{C}} \zeta_2^2)$ is divisible by $\phi_0 x_1 + \phi_1 x_1^2 + \ldots + \phi_{[n/2]} x_1^{2[n/2]}$;*
b) $\phi_j p_1(\zeta_1 \otimes_{\mathbb{R}} \zeta_2) = 0$

in the ring $M\mathrm{Sp}^*(\mathbb{H}P(n)^2) = M\mathrm{Sp}^*[\![x_1, x_2]\!]/(x_i^{n+1})$.

In Section 4 we will see that in terms of the coefficients a_{klm} of the first Conner-Floyd symplectic Pontryagin class

$$p_1(\zeta_1 \underset{\mathbb{C}}{\otimes} \zeta_2 \underset{\mathbb{C}}{\otimes} \zeta_3) = \sum_{k+l+m \geqslant 1} a_{klm} p_1^k(\zeta_1) p_1^l(\zeta_2) p_1^m(\zeta_3),$$

the structure of $M\mathrm{Sp}_{4k}$, $k \leqslant 4$, can be interpreted as follows:

k	$M\mathrm{Sp}_{4k}$	generators
1	$\mathbb{Z}$	a_{011}
2	$\mathbb{Z}+\mathbb{Z}$	$a_{012},\ a_{111}$
3	$\mathbb{Z}+\mathbb{Z}+\mathbb{Z}$	$a_{022},\ a_{011}a_{111},\ a_{211}$
4	$\mathbb{Z}+\mathbb{Z}+\mathbb{Z}+\mathbb{Z}+\mathbb{Z}$	$a_{014},\ a_{011}a_{211},\ a_{122},\ a_{111}^2,\ 2y_4$

Then Theorem 1.1 implies

Corollary 1.2. *For $i \geqslant 0$ one has:*

a) $\phi_i a_{001} = \phi_i a_{012} = \phi_i a_{022} = \phi_i a_{014} = 0$;
b) $\phi_i a_{111}$ *and* $\phi_i a_{122}$ *belong to the ideal* $\phi_0 M\mathrm{Sp}^*$;
c) $\phi_i a_{211}$ *belongs to the ideal* $\phi_0 M\mathrm{Sp}^* + \phi_1 M\mathrm{Sp}^*$.

Relations of Corollary 1.2 imply that multiplication by the elements ϕ_i, $i \geqslant 0$, carries most of the low-dimensional generators from the free part of $M\mathrm{Sp}_{4n}$ to the ideal generated by the elements ϕ_0 and ϕ_1.

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2. Preliminaries and calculations with transfer

Let ξ and Λ be, respectively, the universal U(1)-bundle and the universal Spin(3)-bundle. Thus the sphere bundle of Λ is $\pi : B\mathrm{U}(1) \to B\mathrm{Sp}(1)$, and one has

$$\pi^*(\zeta) = \xi + \bar{\xi}, \tag{2.1}$$

$$\pi^*(\Lambda) = \xi^2 + \mathbb{R}, \tag{2.2}$$

$$\zeta \underset{\mathbb{H}}{\otimes} \zeta = \Lambda + \mathbb{R}, \tag{2.3}$$

where ζ is the universal Sp(1)-bundle as above. Let N denote the normalizer of the torus U(1) in Sp(1). The classifying space BN coincides with the orbit space of the complex projective space $\mathbb{C}P(\infty)$ under the free involution I, which acts via

$$I : [z_0, z_1, ...] \mapsto [-\bar{z}_1, \bar{z}_0, ...]$$

in homogeneous coordinates.

The bundle $p : B\mathrm{N} \to B\mathrm{Sp}(1)$ coincides with the projective bundle of Λ and one has the canonical splitting

$$p^*(\Lambda) = \mu + \nu, \tag{2.4}$$

defined by projectivisation p, where μ and ν denote real plane and line bundles, respectively. Of course for the double covering $q : B\mathrm{U}(1) \to B\mathrm{N}$ one has $q^*(\mu) = \xi^2$ and $q^*(\nu) = \mathbb{R}$.

Let τ_π and τ_p be the transfer maps of the bundles π and p [2, 6, 11]. The next lemma follows from [7].

Lemma 2.1. $\pi^*\tau_\pi^* = 1 + I^*$ *and* $\pi^*\tau_p^* = q^*$.

The next lemma follows from the definitions.

Lemma 2.2. $(\xi_1\xi_2^2 + \bar{\xi}_1\bar{\xi}_2^2)_! = (\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu$, *where '$(\_)_!$' denotes the Atiyah transfer for the double covering* $1_{B\mathrm{U}(1)} \times q$.

Consider the map $f : B\mathrm{N} \to B\mathbb{Z}/2$ induced by the projection of N onto the Weyl group $\mathbb{Z}/2$ and let $\tau_{1\times q}^*$ be the transfer homomorphism for the above double covering $1_{B\mathrm{U}(1)} \times q$.

Lemma 2.3. *For some elements* $\alpha_i \in \widetilde{M\mathrm{Sp}}^*(B\mathbb{Z}/2)$ *the following formula holds*:

$$\tau_{1\times q}^*(p_1(\xi_1\xi_2^2 + \bar{\xi}_1\bar{\xi}_2^2)) = p_1((\xi_1 + \bar{\xi}_1) \underset{\mathbb{R}}{\otimes} \mu) + \sum_{i\geqslant 0} f^*(\alpha_i) p_2^i((\xi_1 + \bar{\xi}_1) \underset{\mathbb{R}}{\otimes} \mu).$$

Proof. Taking into account Lemma 2.2 the proof follows from the following formula [17]: let q be the double covering $q : X \to B$, let $\eta \to X$ be the symplectic line bundle, $\eta_! \to B$ the Atiyah transfer bundle, τ_q the transfer map of the covering q and $f : X \to B\mathbb{Z}/2$ the classifying map of the real line bundle associated with q. Then for some elements α_i from $\widetilde{M\mathrm{Sp}}^*(B\mathbb{Z}/2)$ the following formula holds:

$$\tau_q^*(p_1(\eta)) = p_1(\eta_!) + \sum_{i\geqslant 0} f^*(\alpha_i) p_2^i(\eta_!).$$

∎

Lemma 2.4. *Let* τ *be the transfer of the sphere bundle of a* Spin(3)*-bundle. Then* $\phi_j \operatorname{Im} \tau^* = 0$, $j \geqslant 0$.

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Proof. Of course it suffices to prove this for the universal Spin(3)-bundle Λ, that is, $\phi_j\tau_\pi^*(a)=0$ for all $a\in M\mathrm{Sp}^*(B\mathrm{U}(1))$.

Let δ_π be the Boardman map [3]. Then as it is known from [2],

$$\tau_\pi^*(a)=\delta_\pi(ae(\xi_2^2)).$$

Here $e(\xi_2^2)$ is the Euler class of the bundle ξ_2^2 which is the bundle of tangents along the fibers. Then from [13, 8], $\phi_je(\xi_2^2)=0$. This proves Lemma 2.4. □

Recall from [13, 8, 9] that the bundle Λ is $M\mathrm{Sp}$-orientable and the corresponding Euler class has the form

$$e(\Lambda)=\phi_0p_1(\zeta)+\sum_{j\geqslant 1}\phi_jp_1^{2j}(\zeta). \tag{2.5}$$

The restrictions of π and p to the symplectic projective space $\mathbb{H}P(n)$ will be denoted by the same symbols. Total spaces of these bundles coincide, respectively, with the complex projective space $\mathbb{C}P(2n+1)$ and with the orbit space $\mathbb{C}P(2n+1)/I$ under the free involution I which acts via

$$[z_0,z_1,...,z_{2n},z_{2n+1}]\mapsto[-\bar z_1,\bar z_0,...,-\bar z_{2n+1},\bar z_{2n}]$$

in homogeneous coordinates.

Proposition 2.5. *$\phi_j\tau_{\pi\times 1}^*(p_i(r\xi\otimes_{\mathbb{R}}\zeta))=0$ for $\pi\times 1=\pi\times 1_{B\mathrm{Sp}(1)}:B\mathrm{U}(1)\times B\mathrm{Sp}(1)\to B\mathrm{Sp}(1)^2$, $j\geqslant 0$, and $i=1,2$.*

Proof. In $M\mathrm{Sp}^*(B\mathrm{U}(1)\times B\mathrm{Sp}(1))=M\mathrm{Sp}^*(B\mathrm{U}(1))[\![p_1(\zeta)]\!]$ one has $p_i(r\xi\otimes_{\mathbb{R}}\zeta)=\sum_{k\geqslant 0}\omega_k^{(i)}p_1^k(\zeta)$. Then it follows from Lemma 2.4 that

$$\phi_j\tau^*(\sum_{k\geqslant 0}\omega_k^{(i)}p_1^k(\zeta))=\sum_{k\geqslant 0}\phi_j\tau^*(\omega_k^{(i)}p_1^k(\zeta))=0.$$

□

3. Proof of Theorem 1.1

The bundle $\pi\times 1:\mathbb{C}P(2n+1)\times\mathbb{H}P(n)\to\mathbb{H}P(n)\times\mathbb{H}P(n)$ coincides with the sphere bundle of the pullback of $\Lambda\to\mathbb{H}P(n)$ along the projection on the first factor $\mathbb{H}P(n)\times\mathbb{H}P(n)\to\mathbb{H}P(n)$. So taking into account the formula (2.5) we have to prove that

$$(\pi\times 1_{\mathbb{H}P(n)})^*(\phi_jp_1(\zeta_1\underset{\mathbb{C}}{\otimes}\zeta_2^2))=0$$

in $M\mathrm{Sp}^*(\mathbb{C}P(2n+1)\times\mathbb{H}P(n))$. The transfer $\tau^*=\tau_{1\times\pi}^*$ of the bundle $1_{\mathbb{C}P(2n+1)}\times\pi$ is a composite of two transfers, namely

$$\tau^*=\tau_1^*(\tau_2^*)$$

where τ_1 is the transfer of the bundle $1_{\mathbb{C}P(2n+1)}\times q$ and τ_2 is the transfer of $1_{\mathbb{C}P(2n+1)}\times p$, where the bundles p, π and q are the bundles defined above; that is,

$$1\times q:\mathbb{C}P(2n+1)\times\mathbb{C}P(2n+1)\to\mathbb{C}P(2n+1)\times\mathbb{C}P(2n+1)/I,$$
$$1\times p:\mathbb{C}P(2n+1)\times\mathbb{C}P(2n+1)/I\to\mathbb{C}P(2n+1)\times\mathbb{H}P(n).$$

Using (2.2) one obtains $(\xi_1+\bar\xi_1)\otimes_{\mathbb{C}}\zeta_2^2=(\xi_1+\bar\xi_1)\otimes_{\mathbb{R}}(\Lambda+\mathbb{R})$, hence

$$p_1((\xi_1+\bar\xi_1)\underset{\mathbb{C}}{\otimes}\zeta_2^2)=p_1((\xi_1+\bar\xi_1)\underset{\mathbb{R}}{\otimes}\Lambda)+p_1(\xi_1+\bar\xi_1). \tag{3.1}$$

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Applying Lemma 2.2 and Lemma 2.3 one obtains

$$\tau_1^*(p_1(\xi_1\xi_2^2+\bar{\xi_1}\bar{\xi_2^2}))=p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\mu)+\sum_{i\geqslant 0} f^*(\alpha_i)p_2^i((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\mu).$$

Then by (2.4)

$$p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\mu)=(1\times p)^*p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\Lambda)-p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\nu),$$

hence

$$\begin{aligned}
&\tau^*(p_1(\xi_1\xi_2^2+\bar{\xi_1}\bar{\xi_2^2}))\\
&\quad=\tau_2^*(p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\mu))\\
&\qquad+\tau_2^*(\sum_{i\geqslant 0} f^*(\alpha_i)p_2^i((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\mu))\\
&\quad=\tau_2^*((1\times p)^*(p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\Lambda)))-\tau_2^*(p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\nu))\\
&\qquad+\tau_2^*(\sum_{i\geqslant 0} f^*(\alpha_i)p_2^i((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\mu))\\
&\quad=p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\Lambda)\tau_2^*(1)-\tau_2^*(p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\nu))\\
&\qquad+\tau_2^*(\sum_{i\geqslant 0} f^*(\alpha_i)p_2^i((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\mu)).
\end{aligned}\tag{3.2}$$

Now we have to prove that $\tau_2^*(1)=1$, the second summand in (3.2) coincides with $x_1=p_1(\xi_1+\bar{\xi_1})$, and the third summand is zero.

Note that the bundle $(\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\nu$ is the pullback of the bundle $\zeta\otimes\eta\to B\mathrm{Sp}(1)\times B\mathbb{Z}/2$ along the map (π,f). Thus $p_1(\zeta\otimes\eta)$ is an element from $M\mathrm{Sp}^*(B\mathbb{Z}/2)[\![p_1(\zeta)]\!]$, hence

$$p_1(\zeta\otimes\eta)=p_1(\zeta)+\sum_{i\geqslant 0}\beta_i p_1^i(\zeta)$$

for some elements $\beta_i\in\widetilde{M\mathrm{Sp}}^*(B\mathbb{Z}/2)$. This implies

$$p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\nu)=p_1(\xi_1+\bar{\xi_1})+\sum_{i\geqslant 0}f^*(\beta_i)p_1^i(\xi_1+\bar{\xi_1}).$$

Similarly the bundle $(\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\mu$ is the pullback of the bundle $\zeta\otimes\eta(2)\to B\mathrm{Sp}(1)\times B\mathrm{O}(2)$, where $\eta(2)\to B\mathrm{O}(2)$ is the universal $\mathrm{O}(2)$-bundle. Hence

$$p_2((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\mu)\in M\mathrm{Sp}^*(B\mathrm{N})[\![p_1(\xi_1+\bar{\xi_1})]\!]$$

and for the third summand of (3.2) one has

$$\sum_{i\geqslant 0}f^*(\alpha_i)p_2^i((\xi_1+\bar{\xi_1})\underset{\mathbb{R}}{\otimes}\mu)=\sum_{i\geqslant 0}\gamma_i p_1^i(\xi_1+\bar{\xi_1})$$

for some $\gamma_i\in\widetilde{M\mathrm{Sp}}^*(B\mathrm{N})$.

So using (3.1) one has

$$\tau^*(p_1(\xi_1\xi_2^2+\bar{\xi_1}\bar{\xi_2^2}))=p_1((\xi_1+\bar{\xi_1})\underset{\mathbb{C}}{\otimes}\zeta^2)\tau_2^*(1)-x_1(\tau_2^*(1)+1)+\sum_{i=0}^{n}\tau_2^*(\delta_i)x_1^i\tag{3.3}$$

for some $\delta_i\in\widetilde{M\mathrm{Sp}}^*(\mathbb{C}P(2n+1)/I)$.

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It is known from [18, 19] that up to dimension 32, $M\mathrm{Sp}_{4n}$ is torsion free. Motivated by this fact let us assume that $M\mathrm{Sp}_{4m}$ is torsion free when $m \leqslant 2n-1$. Then it follows that $M\mathrm{Sp}^{4k}(\mathbb{H}P(n))$ is torsion free when $k \geqslant 1-n$. Then since the minimal dimension of the elements δ_i from (3.3) is $4-4n$, it follows from Lemma 2.1 that the third summand of (3.3) restricts to zero in $M\mathrm{Sp}^*(\mathbb{H}P(n))$. Also $\tau^*(p)(1) = 1$ and $\tau^*(\pi)(1) = 2$.

Thus one obtains from (3.3) and then Lemma 2.4

$$\phi_j p_1((\xi_1 + \bar{\xi_1}) \underset{\mathbb{C}}{\otimes} \zeta_2^2) = \phi_j \tau^*(p_1(\xi_1\xi_2^2 + \bar{\xi_1}\bar{\xi_2}^2)) = 0.$$

This proves Theorem 1.1a).

For the proof of b) note that it follows from Lemma 2.1 that for the bundle $\pi \times 1 = \pi \times 1_{\mathbb{H}P(n)})$ one has

$$\begin{aligned}(\pi \times 1)^* \tau^*_{\pi\times 1}(p_1(r\xi_1 \underset{\mathbb{R}}{\otimes} \zeta_2)) &= (1+I)^*(p_1((\xi_1 + \bar{\xi_1}) \underset{\mathbb{C}}{\otimes} \zeta_2)) \\ &= 2p_1((\xi_1 + \bar{\xi_1}) \underset{\mathbb{C}}{\otimes} \zeta_2) = (\pi \times 1)^* p_1((\zeta_1 \underset{\mathbb{R}}{\otimes} \zeta_2)).\end{aligned}$$

Then by (2.5) any element from $\ker(1 \times \pi)^*$ is divisible by $e(\Lambda)$. On the other hand by hypothesis $M\mathrm{Sp}^{4k}(\mathbb{H}P(n))$ is torsion free for $k \geqslant 1-n$. Hence one concludes that restriction of the homomorphism $(\pi \times 1)^*$ to $M\mathrm{Sp}^*(\mathbb{H}P(n)^2)$ is a monomorphism, thus in $M\mathrm{Sp}^4(\mathbb{H}P(n)^2)$ one has

$$p_1(\zeta_1 \underset{\mathbb{R}}{\otimes} \zeta_2) = \tau^*_{\pi\times 1}(p_1(r\xi_1 \underset{\mathbb{R}}{\otimes} \zeta_2)).$$

Now since Proposition 2.5 says that the right-hand side is zero after multiplication by ϕ_j, this completes the proof of Theorem 1.1.

4. Proof of Corollary 1.2

Let $h : \pi_*(M\mathrm{Sp}) \to H_*(M\mathrm{Sp}) = \mathbb{Z}[q_1, q_2, ...]$ be the Hurevicz homomorphism. Since $\pi_{4n}(M\mathrm{Sp})$ is torsion free for small n (see [16, 18, 19]), the Hurevicz homomorphism is a monomorphism in these dimensions. So in low dimensions $4n$ the Hurevicz homomorhism determines all relations. Our aim here is to express the coefficients a_{klm} from the Introduction through the generators x-es.

Values of the Hurevicz homomorphism on these a_{klm} are calculated in [10]. In low dimensions one has

$$\begin{aligned}
&h(a_{100}) = h(a_{010}) = h(a_{001}) = 4,\\
&h(a_{200}) = h(a_{020}) = h(a_{002}) = 0,\\
&h(a_{110}) = h(a_{101}) = h(a_{011}) = 24q_1,\\
&h(a_{111}) = 360q_2,\\
&h(a_{210}) = ... = h(a_{012}) = 60q_2 - 24q_1^2,\\
&h(a_{300}) = ... = h(a_{003}) = 0,\\
&h(a_{220}) = ... = h(a_{022}) = 280q_3 - 120q_1q_2 + 24q_1^3,\\
&h(a_{310}) = ... = h(a_{013}) = 112q_3 - 96q_1q_2 + 48q_1^3,\\
&h(a_{211}) = ... = h(a_{112}) = 1680q_3 - 360q_1q_2,\\
&h(a_{122}) = ... = h(a_{122}) = 75600q_4 - 3360q_1q_3 + 360q_1^2q_2,\\
&h(a_{410}) = ... = h(a_{140}) = 180q_4 - 360q_1q_3 + 420q_1^2q_2 - 120q_2^2 - 120q_1^4.
\end{aligned}$$

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Further, the Hurevicz images of generators of $M\mathrm{Sp}$ are calculated in [16]. Namely, $h(M\mathrm{Sp}_{4k}) \subset H_{4k}(M\mathrm{Sp})$ has the following generators:

$k = 1 :$ $24q_1$,

$k = 2 :$ $20q_2 - 8q_1^2,\ 144q_1^2$,

$k = 3 :$ $56q_3 - 72q_1q_2 + 24q_1^3,\ 120q_1q_2 - 48q_1^3,\ 3456q_1^3$,

$k = 4 :$ $12q_4 - 24q_1q_3 - 8q_2^2 + 28q_1^2q_2 - 8q_1^4,\ 50q_2^2 + 168q_1q_3 - 256q_1^2q_2 + 80q_1^4,$
$100q_2^2 - 80q_1^2q_2 + 16q_1^4,\ 2880q_1^2q_2 - 1152q_1^4,\ 20736q_1^4$.

Thus one concludes that the elements $a_{011}, a_{111}, a_{022}, a_{122}, a_{112}, a_{120}, a_{140}$ are generators as in the Introduction.

Remark 1. In terms of $2x_i$, the generators of $M\mathrm{Sp}_{4n}$ from [16], one has modulo $2M\mathrm{Sp}_*$: $2x_1 = a_{011}$, $2x_2 = a_{012}$, $2x_3 = a_{022}$, $x_4 = a_{014}$, $x_1^2 = a_{111}$, etc.

Remark 2. Alternatively, images of the elements a_{ijk} in complex cobordism $M\mathrm{U}_*$ can be calculated in terms of two-valued formal groups:

$$\mu^*(p_1(\zeta_1 \underset{\mathbb{C}}{\otimes} \zeta_2 \underset{\mathbb{C}}{\otimes} \zeta_3)) = \Theta_1(x_1, Y^+) + \Theta_1(x_1, Y^-),$$

where $Y^+ + Y^- = \Theta_1(x_2, x_3)$, $Y^+Y^- = \Theta_2(x_2, x_3)$; Θ_1 and Θ_2 are the coefficients of the two-valued formal group [5] and μ^* is the obvious map from the symplectic cobordism theory to the complex cobordism theory.

Let us now consider Corollary 1.2. From Theorem 1.1a), in $M\mathrm{Sp}^*(\mathbb{H}P(4)\times\mathbb{H}P(4)$ one has a relation of the form

$$\begin{aligned}\phi_j(a_{011}x_2^2 + a_{111}x_1x_2^2 + a_{022}x_2^4 + a_{211}x_1^2x_2^2 + a_{122}x_1x_2^4 + ...)\\ = (\phi_0 x_1 + \sum_{1\leqslant i\leqslant n} \phi_i x_1^{2i}) b(x_1, x_2)\end{aligned}$$

for some element $b(x_1, x_2) \in M\mathrm{Sp}^*(\mathbb{H}P(n)^2)$. Then by the equality of the coefficients at the monomials $x_1x_2^2$, $x_1^2x_2^2$ and $x_1x_2^4$ one obtains assertions b) and c) of Corollary 1.2.

Similarly from Proposition 2.5 one has

$$\phi_j(a_{110}x_2^2 + a_{120}x_1x_2^2 + a_{220}x_1^2x_2^2 + a_{140}x_1x_2^4 + ...) = 0,$$

and hence assertion a) of Corollary 1.2 is valid.

Proposition 4.1. *In dimension* 32 *there is an element* y_4^2 *such that* $\phi_2 y_4^2$ *does not belong to the ideal generated by* ϕ_0 *and* ϕ_1. *Moreover* $\phi_{2^i} y_4^2$ *does not belong to the ideal generated by* ϕ_0, ϕ_1, ... , $\phi_{2^{i-1}}$, $i \geqslant 1$.

Proof. It follows from the calculations of the symplectic cobordism ring made in [18, 19]. □

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