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CHARACTERISTIC CLASSES OF STABLE BUNDLES OF RANK 2 OVER AN ALGEBRAIC CURVE

BY

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ABSTRACT. Let X be a complete nonsingular algebraic curve over C and L a line bundle of degree 1 over X. It is well known that the isomorphism classes of stable bundles of rank 2 and determinant L over X form a nonsingular projective variety $S(X)$. The Betti numbers of $S(X)$ are also known. In this paper we define certain distinguished cohomology classes of $S(X)$ and prove that these classes generate the rational cohomology ring. We also obtain expressions for the Chern character and Pontrjagin classes of $S(X)$ in terms of these generators.

Introduction. Let X be a complete nonsingular algebraic curve of genus $g \geq 2$ over the complex numbers and let L be a line bundle of degree 1 over X. The isomorphism classes of stable bundles of rank 2 and determinant L over X form a nonsingular projective variety $S(X)$ (see [2], [6]), whose Betti numbers were calculated in [3]. The main object of this paper is to show that certain naturally occurring elements generate the rational cohomology ring of $S(X)$ (Theorem 1). We shall also obtain expressions for the Chern character and Pontrjagin classes of $S(X)$ in terms of these generators.

My thanks are due to S. Ramanan for informing me of his work on this topic in advance of publication (see [5]). He has obtained generators and relations for the rational cohomology ring of $S(X)$ in the case $g = 3$. He has also obtained some information for the spaces of stable bundles of rank n (in particular a generalisation of Corollary 1 to Theorem 2) by methods similar to those used in the proof of Proposition 2.2.

Unless the contrary is indicated, all cohomology groups in this paper will have integral coefficients. Also, if E is any bundle over $V \times W$ (or $W \times V$) and $v \in V$, we shall denote by E_v the bundle over W obtained by restricting E to $\{v\} \times W$ (or $W \times \{v\}$).

1. **Statement of the main theorem.** We recall [1, §1] that there exists an algebraic vector bundle U over $S(X) \times X$ with the property that, for all $s \in S(X)$,

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U_s is in the isomorphism class s. Since $S(X)$ is simply-connected [3, Theorem 1, Corollary 2], the Chern classes of U can be expressed in the form

$$c_1(U) = \phi + f, \qquad c_2(U) = \chi + \psi + \omega \otimes f,$$

where f is the positive generator of $H^2(X)$ and

$$\phi, \omega \in H^2(S(X)), \qquad \chi \in H^4(S(X)), \qquad \psi \in H^3(S(X)) \otimes H^1(X).$$

We choose a basis $a_1, \cdots, a_{2g}$ of $H^1(X)$ and write

$$\psi = \psi_1 \otimes a_1 + \cdots + \psi_{2g} \otimes a_{2g},$$

where $\psi_i \in H^3(S(X))$ $(1 \le i \le 2g)$. Finally we write $\alpha = 2\omega - \phi$; $\beta = \phi^2 - 4\chi$.

Theorem 1. *$H^*(S(X); \mathbf{Q})$ is generated by $\alpha, \beta, \psi_1, \cdots, \psi_{2g}$.*

Theorem 1 will be proved in §§2 and 3. In §2 we shall use properties of U to obtain results about the cohomology of $S(X)$ in low dimensions and to reduce Theorem 1 to a similar theorem concerning the manifold $S_0^{(g)}$ of [3]. This theorem will be proved in §3 using the methods of [3].

Remark 1. Almost all the subsidiary results in §§2 and 3 are stated in terms of integral cohomology; rational coefficients are needed only in Proposition 2.6, the deduction of Theorem 1 from Theorem 1′ and the deduction of Theorem 1′ from Proposition 3.4.

Remark 2. U can be replaced by $U \otimes \pi_1^* M$, where M is any line bundle over $S(X)$, but this leaves α, β and ψ unchanged. In fact, these are the only permitted modifications of U, so $\alpha, \beta, \psi_1, \cdots, \psi_{2g}$ are well-determined elements of $H^*(S(X))$. It therefore seems reasonable to call them *characteristic classes*. Note also that End U is independent of the choice of U and that

$$c(\text{End } U) = 1 - \beta + 4\psi + 2\alpha \otimes f. \tag{1}$$

2. Preliminary results.

Proposition 2.1. *$\psi_1, \cdots, \psi_{2g}$ form a basis of $H^3(S(X))$.*

Proof. This is Proposition 1 of [1].

To obtain information about α and β, we first construct a family of stable bundles over X. We consider the nontrivial extensions

$$0 \longrightarrow I \longrightarrow E \longrightarrow L \longrightarrow 0 \tag{2}$$

over X. These are classified (up to isomorphism) by the projective space $P = \mathbf{P}(H^1(X; L^*))$. Moreover, there exists a bundle F over $P \times X$ such that, for each p in P, F_p is isomorphic to the bundle obtained as the extension (2) given by p. In fact, let H be the hyperplane bundle over P and note that

$$H^1(P \times X;\ \pi_1^*(H) \otimes \pi_2^*(L^*)) \cong H^0(P;\ H) \otimes H^1(X;\ L^*) \cong \operatorname{End}(H^1(X;\ L^*)).$$

We define F by means of the extension

$$0 \longrightarrow \pi_1^*(H) \longrightarrow F \longrightarrow \pi_2^*(L) \longrightarrow 0$$

corresponding to the identity endomorphism of $H^1(X;\ L^*)$. It is easy to check that F has the required property.

Proposition 2.2. *For every nontrivial extension* (2), E *is stable. Moreover the induced map* $k: P \longrightarrow S(X)$ *is a morphism and, if* h *denotes the positive generator of* $H^2(P)$, *then* $k^*(\alpha) = h$; $k^*(\beta) = h^2$.

Proof. If M is a line subbundle of E with $\deg M \geq 1$, the induced homomorphism $M \longrightarrow L$ is an isomorphism and hence (2) splits. The fact that k is a morphism follows from the universal property of $S(X)$ [6, Chapter II, Theorem 5]. Moreover, for each p in P,

$$((k \times 1_X)^* U)_p \cong F_p.$$

It follows easily [5, Lemma 2.5] that there exists a line bundle M over P such that

$$F \cong (k \times 1_X)^* U \otimes \pi_1^* M.$$

A simple calculation now shows that $k^*(\alpha) = h$, $k^*(\beta) = h^2$.

Proposition 2.3. α *generates* $H^2(S(X))$.

Proof. We know [3, Theorem 1, Corollary 2 and Theorem 3] that $H^2(S(X)) \cong \mathbf{Z}$. Since by Riemann-Roch $\dim P = g - 1 > 0$, the result follows at once from Proposition 2.2.

We now recall [3, p. 243] that there is a principal $PU(2)$-fibration

$$p:\ S_0^{(g)} \longrightarrow S(X).$$

This fibration is closely related to U. In fact, let $x \in X$; then we have

Proposition 2.4. *The* $\mathbf{P}^1$*-fibration associated with* p *is* (*differentiably*) *isomorphic to* $\mathbf{P}(U)_x$.

Proof. We consider the bundle E' over $S_0^{(g)}$ constructed in [1, pp. 1206–1207]. We have clearly

$$\mathbf{P}(E') \cong (p \times 1_X)^*(\mathbf{P}(U)). \tag{3}$$

Note that the action of $PU(2)$ on $S_0^{(g)}$ extends in a natural way to actions on both sides of (3) and that these actions are compatible with (3). Moreover $(\mathbf{P}(E')/PU(2))_x$ is just the $\mathbf{P}^1$-fibration associated with p. Hence this fibration is isomorphic to $(\mathbf{P}(U))_x$ as required.

Proposition 2.5. *If* $g \geq 3$, β *generates the kernel of the homomorphism*

$$p^*: \ H^4(S(X)) \to H^4(S_0^{(g)}).$$

Proof. Note first that the bundle End U can be induced from the principal $PU(2)$-fibration associated with $\mathbf{P}(U)$ by means of a suitable representation of $PU(2)$. It follows at once from (1) and Proposition 2.4 that $\beta \in \operatorname{Ker} p^*$. Now an inspection of the spectral sequence of p shows that $\operatorname{Ker} p^* \cong \mathbf{Z}$. The result follows from Proposition 2.2 (note that $\dim P = g - 1 \geq 2$).

Proposition 2.6. *If* $g \geq 3$, *there is an exact sequence*

$$\cdots \to H^{r-1}(S_0^{(g)}; \mathbf{Q}) \to H^{r-4}(S(X); \mathbf{Q}) \xrightarrow{\cup\beta} H^r(S(X); \mathbf{Q}) \xrightarrow{p^*} H^r(S_0^{(g)}; \mathbf{Q}) \to \cdots.$$

Moreover the homomorphism

$$p^*: \ H^r(S(X); \mathbf{Q}) \to H^r(S_0^{(g)}; \mathbf{Q})$$

is an isomorphism for $r \leq 3$.

Proof. The exact sequence is just the Gysin sequence of the fibration p, which exists since $PU(2)$ has the same rational cohomology as S^3 and takes the stated form because of Proposition 2.5. The facts about p^* follow from the exact sequence and the fact that $\beta \neq 0$.

Finally, we note that these results enable us to deduce Theorem 1 from

Theorem 1′. *For* $r \leq 3g - 3$, $H^r(S_0^{(g)}; \mathbf{Q})$ *is contained in the subring of* $H^*(S_0^{(g)}; \mathbf{Q})$ *generated by classes of dimensions* 2 *and* 3.

In fact, since $S(X)$ is a projective variety of dimension $3g - 3$, it is sufficient to prove that, for $r \leq 3g - 3$, $H^r(S(X); \mathbf{Q})$ is contained in the subring of $H^*(S(X); \mathbf{Q})$ generated by $\alpha, \beta, \psi_1, \cdots, \psi_{2g}$. For $g = 2$, this follows at once from Propositions 2.1 and 2.3; while, for $g \geq 3$, it follows from Theorem 1′ and Propositions 2.1, 2.3 and 2.6.

We shall state and prove a more precise version of Theorem 1′ in the next section.

3. **Cohomology of** $S_0^{(g)}$. In this section we shall be using the methods of [3] and we shall adopt the notation of [3] with one exception: $H^*(\ )$ will continue to denote cohomology with integral coefficients. We recall in particular [3, pp. 242–243] that $SU(2)^{2g}$ has submanifolds with boundary $S_0^{(g)'}$ and $N^{(g)'}$ such that

$$S_0^{(g)'} \cup N^{(g)'} = SU(2)^{2g}, \qquad S_0^{(g)'} \cap N^{(g)'} = \partial S_0^{(g)'} = \partial N^{(g)'}; \tag{4}$$

moreover there is a diffeomorphism

$$w = w_g \colon S_0^{(g)} \times D^3 \to S_0^{(g)\prime}.$$

Note that the restriction to w to $S_0^{(g)} \times S^2$ is a diffeomorphism onto $\partial N^{(g)\prime}$; we denote by w' the composite of this map with the inclusion of $\partial N^{(g)\prime}$ in $N^{(g)\prime}$.

Next let z be a point of $S_0^{(1)}$; we define, for $g \geq 1$,

$h\colon N^{(g)\prime} \to S_0^{(g+1)}$ by $h(x) = (w(z, -f_g(x)^{-1}), x)$,

$i_N\colon N^{(g)\prime} \to N^{(g+1)\prime}$ by $i_N(x) = (x, I, I)$,

$i_S\colon S_0^{(g)} \to S_0^{(g+1)}$ by $i_S(x) = (x, I, I)$,

$j\colon S^2 \to S_0^{(g)} \times S^2$ by $j(x) = ((z, I, \cdots, I), x)$.

We have the diagram

$$(5)\qquad \begin{array}{ccccccc} S^2 & \xrightarrow{j} & S_0^{(g)} \times S^2 & \xrightarrow{w'} & N^{(g)\prime} & \xrightarrow{h} & S_0^{(g+1)} \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i_S \times 1} & & \downarrow{\scriptstyle i_N} & & \downarrow{\scriptstyle i_S} \\ S^2 & \xrightarrow{j} & S_0^{(g+1)} \times S^2 & \xrightarrow{w'} & N^{(g+1)\prime} & \xrightarrow{h} & S_0^{(g+2)} \end{array}$$

The left-hand square of (5) is clearly commutative; so also is the right-hand square, since $f_{g+1}(x, I, I) = f_g(x)$. The middle square is not commutative, but it is homotopy commutative. In fact, since i_N maps $\partial N^{(g)\prime}$ into $\partial N^{(g+1)\prime}$, we have a map

$$(w')^{-1} i_N w' \colon S_0^{(g)} \times S^2 \to S_0^{(g+1)} \times S^2$$

and it is sufficient to prove that this map is homotopic to $i_S \times 1$. But this was proved in [3, pp. 254–255] in the case $g = 1$ and the same proof works for all g.

Proposition 3.1. *The maps $w'j$, h and i_N induce isomorphisms on H^2; if $g \geq 2$, so also does i_S.*

Proof. We consider first the Mayer-Vietoris sequence corresponding to (4). In particular we have a homomorphism

$$(6)\qquad H^2(N^{(g)\prime}) \oplus H^2(S_0^{(g)\prime}) \to H^2(S_0^{(g)} \times S^2)$$

induced on the first factor by w' and on the second by w; in fact, by [3, Lemma 2], (6) is an isomorphism. Moreover, since w extends to a diffeomorphism of $S_0^{(g)\prime}$ with $S_0^{(g)} \times D^3$, the image of $H^2(S_0^{(g)\prime})$ under (6) is just the kernel of j^*. Hence $(w'j)^*$ is an isomorphism as required. The result for h follows easily from what we have just proved and the Mayer-Vietoris sequence of [3, §6]. Finally, for i_N and i_S, the result follows from the homotopy commutativity of (5).

Now let λ be a generator of $H^2(N^{(1)\prime})$; we denote also by λ the generators of $H^2(N^{(g)\prime})$ and $H^2(S_0^{(g)})$ ($g \geq 2$) induced from this generator by the isomorphisms of Proposition 3.1. We denote by $\mu_1, \cdots, \mu_{2g}$ the elements of $H^3(N^{(g)\prime})$ and $H^3(S_0^{(g)})$ induced by the inclusions of these spaces in $SU(2)^{2g}$ from the obvious generators of $H^3(SU(2)^{2g})$.

We need also one further observation. Note that we have a natural inclusion $N^{(g)'} \times N^{(1)'} \subset SU(2)^{2g} \times SU(2)^2 = SU(2)^{2g+2}$. A simple argument (similar to that at the top of p. 256 of [3]) shows that we can deform this map into a map

$$l\colon\ N^{(g)'} \times N^{(1)'} \longrightarrow N^{(g+1)'};$$

we need to observe that the deformation used to define l can be performed so that at every stage the images of the subspaces $N^{(g)'} \times \{(I, I)\}$ and $\{(I, \cdots, I)\} \times N^{(1)'}$ are contained in $N^{(g+1)'}$.

Proposition 3.2. (i) $l^*(\lambda^p) = \lambda^p \otimes 1 \pm p\lambda^{p-1} \otimes \lambda$;

(ii) $l^*(\mu_i) = \mu_i \otimes 1 \quad (i \le 2g), \quad = 1 \otimes \mu_{i-2g} \quad (i = 2g+1, 2g+2)$;

(iii) $b^*(\mu_1) = b^*(\mu_2) = 0; \quad b^*(\mu_i) = \mu_{i-2} \quad (3 \le i \le 2g+2)$.

Proof. All parts of the proposition follow easily from the definitions. For (i), we need the observation after the definition of l; also the fact that the homeomorphism $\rho\colon\ SU(2)^{2g+2} \to SU(2)^{2g+2}$ defined by

$$\rho(A_1, \cdots, A_{2g+2}) = (A_{2g+1}, A_{2g+2}, A_1, \cdots, A_{2g})$$

restricts to a homeomorphism of $N^{(g+1)'}$ and therefore induces an automorphism of $H^2(N^{(g+1)'})$, which must be multiplication by ± 1. This is the source of the sign ambiguity in (1).

Proposition 3.3. *The elements*

$$\lambda^p \mu_{q_1} \cdots \mu_{q_r} \qquad (p \ge 0,\ 1 \le q_1 < \cdots < q_r \le 2g,\ p + r \le g) \tag{7}$$

are linearly independent in $H^*(N^{(g)'})$.

Proof. For $g = 1$, this follows at once from [3, §5]. Now let $g \ge 2$ and suppose the proposition is true for $N^{(g-1)'}$. Suppose we have some linear combination Σ of the elements (7) in $H^*(N^{(g)'})$ such that $\Sigma = 0$. We look at the component of $l^*(\Sigma)$ in $H^*(N^{(g-1)'}) \otimes H^2(N^{(1)'})$. Using Proposition 3.2 and the induction hypothesis, we see that the only terms in Σ which can have nonzero coefficients are of one of the following two types:

(a) those for which $p = 0$,

(b) those involving μ_{2g-1} or μ_{2g}.

By composing l with powers of the map ρ used in the proof of Proposition 3.2, we see that (b) can be replaced by

(b)$'$ those involving at least one of μ_{2i-1} and μ_{2i} for all i, $1 \le i \le g$.

Now in view of the restriction on p and r in (7), terms satisfying (b)$'$ necessarily satisfy (a). Hence Σ is a linear combination of the $\mu_{q_1} \cdots \mu_{q_r}$ with $r \le g$ and the result follows from [3, Lemma 2].

Proposition 3.4. *The elements*

(8) $$\lambda^p \mu_{q_1} \cdots \mu_{q_r} \qquad (p \geq 0,\ 1 \leq q_1 < \cdots < q_r \leq 2g,\ p + r \leq g - 1)$$

are linearly independent in $H^*(S_0^{(g)})$.

Proof. This is obvious for $g = 1$. For $g \geq 2$, let Σ be a linear combination of the elements (8) such that $\Sigma = 0$. Using the maps $\rho^t h$ and Propositions 3.2 and 3.3, we see that the only terms which can have nonzero coefficients in Σ are those of the form (b)$'$. But in view of the restriction $p + r \leq g - 1$ there are no such terms.

Theorem 1$'$ now follows at once from Proposition 3.4 and Theorem 2 of [3] and this completes the proof of Theorem 1.

4. Chern and Pontrjagin classes of $S(X)$. Let T denote the tangent bundle of $S(X)$ and V the subbundle of End U consisting of the endomorphisms of trace zero. We have obviously

(9) $$\operatorname{End} U \cong I \oplus V.$$

Lemma. *Let π denote the projection of $S(X) \times X$ on the first factor. Then*
(i) $R^i\pi(V) = 0$ *for* $i \neq 1$,
(ii) $R^1\pi(V)$ *is the sheaf of sections of* T.

Proof. (i) follows from the fact that every stable bundle is simple [2, Corollary to Proposition 4.3], while (ii) is a simple consequence of (i) and the general theory of deformations (see [5, Lemma 2.6]).

Theorem 2. *The Chern character* $\operatorname{ch} T$ *is given by the formulae*

$$\operatorname{ch}_0 T = 3g - 3$$

$$(2n)!\operatorname{ch}_{2n} T = 2(g-1)\beta^n \qquad (n \geq 1)$$

$$(2n-1)!\operatorname{ch}_{2n-1} T = 2\beta^{n-2}[\alpha\beta - 4(n-1)\gamma] \qquad (n \geq 1)$$

where γ *is defined by* $\psi^2 = \gamma \otimes f$.

Proof. We apply the Grothendieck-Riemann-Roch Theorem to π and V. Using the lemma, we get $\operatorname{ch} T = -\pi_*(\operatorname{ch} V \cdot \tau(X))$ (here $\tau(X)$ denotes the total Todd class of X). The result now follows by a formal computation from (1), (9) and the fact that $\tau(X) = 1 - (g-1)f$.

Corollary 1. $c_1(T) = 2\alpha$.

Proof. Obvious.

Corollary 2. *The total rational Pontrjagin class* $p(T)$ *is given by* $p(T) = (1+\beta)^{2g-2}$.

Proof. We deduce at once from the theorem that, for $n \geq 1$,

$$(2n)!\,\mathrm{ch}_{2n}(T \oplus T^*) = 4(g-1)\beta^n.$$

A formal computation now shows that $c(T \oplus T^*) = (1-\beta)^{2g-2}$ and the result follows at once.

Remark. Those two corollaries enable us (at least theoretically) to express the Todd class of T in terms of α and β. At the moment this is of limited usefulness since we do not know the relations in $H^*(S(X); \mathbf{Q})$ (see §5).

To obtain explicit formulae for the Chern classes of T seems to be more difficult. A formal computation, whose details we shall omit, does however lead to the following recurrence relation:

$$(n+1)c_{n+1} - 2\alpha c_n - 2(n-g)\beta c_{n-1} + 2(\alpha\beta + 4\gamma)c_{n-2} + (n-2g-1)\beta^2 c_{n-3} = 0.$$

5. **Relations.** Theorem 1, though interesting, is of rather limited value in the absence of information about the relations among the classes $\alpha, \beta, \psi_1, \cdots, \psi_{2g}$. For many purposes, it would be sufficient to know the relations which involve only α, β and γ, or even just α and β.

Some information can be obtained by the methods of this paper. In fact, Proposition 3.4 allows us to write down an additive basis for each $H^r(S(X); \mathbf{Q})$. It follows in particular that there are no nontrivial relations in dimension $< 2g$, and (up to a scalar multiple) precisely one in dimension $2g$. It is possible to obtain all the relations in the cohomology ring of $S_0^{(g)}$ by a more careful argument on the same lines as that of §3. Unfortunately this does not tell us a great deal about $S(X)$.

Another method of obtaining information about relations is by using a refinement of the construction of §2. The nontrivial extensions

$$0 \longrightarrow M \longrightarrow E \longrightarrow L \otimes M^* \longrightarrow 0,$$

where M is a line bundle of degree 0 over X, can be made into a family whose base P is a bundle over the Jacobian of X with fibre $\mathbf{P}^{g-1}$. By using the known structure of the cohomology ring of P, we can obtain information about that of $S(X)$.

These two approaches are sufficient to enable us to obtain the relation in dimension $2g$ for $g \leq 7$ but seem to be insufficient for larger values of g. For $g = 2$ and $g = 3$, the complete structure of the cohomology ring is known ([4], [5]), while for $g = 4$ the second method outlined above allows us to obtain the relations involving α, β and γ. One or two interesting features emerge from these calculations. Thus, for $g \leq 4$, we have

(a) $\beta^g = 0$; hence (by Corollary 2 to Theorem 2) $p_r(T) = 0$ for $r \geq g$; also all Pontrjagin numbers of $S(X)$ are zero.

(b) $c_r(T) = 0$ for $r \geq 2g - 1$.

(c) $\chi(S(X), T) = -(3g - 3)$.

It seems reasonable, though perhaps optimistic, to conjecture that (a), (b) and (c) hold for all values of g. Another conjecture, related to (c), is

(d) $\dim H^1(S(X); T) = 3g - 3$.

This would mean that the number of moduli of the variety $S(X)$ is the same as that of the curve X. In this context, it may be noted that the complex structure of $S(X)$ determines that of X [1, p. 1201].

It is hoped to investigate some of these problems more fully in a future paper.

REFERENCES

1. D. Mumford and P. E. Newstead, *Periods of a moduli space of bundles on curves*, Amer. J. Math. 90 (1968), 1200–1208. MR 38 #3272.

2. M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) 82 (1965), 540–567. MR 32 #1725.

3. P. E. Newstead, *Topological properties of some spaces of stable bundles*, Topology 6 (1967), 241–262. MR 38 #341.

4. ———, *Stable bundles of rank* 2 *and odd degree over a curve of genus* 2, Topology 7 (1968), 205–215. MR 38 #5782.

5. S. Ramanan (to appear).

6. C. S. Seshadri, *Moduli of π-vector bundles over an algebraic curve*, Lectures at the Summer Institute, Varenna, 1969.

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