CHARACTERISTIC EQUATION METHOD FOR FRACTAL HEAT-TRANSFER PROBLEM VIA LOCAL FRACTIONAL CALCULUS

by

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In this paper the fractal heat-transfer problem described by the theory of local fractional calculus is considered. The non-differentiable-type solution of the heat-transfer equation is obtained. The characteristic equation method is proposed as a powerful technology to illustrate the analytical solution of the partial differential equation in fractal heat transfer.

Key words: *heat-transfer equation, analytical solution, local fractional calculus, characteristic equation method*

Introduction

The differential equations involving the local fractional calculus [1] were utilized to investigate the non-differentiable problems, *e. g.*, fractal diffusions [2-7], fractal oscillator [8], fractal wave [9], fractal Laplace [10, 11], fractal heat-conduction [12, 13], fractal Fokker-Planck [14], fractal Helmholtz [15] equations and others [16, 17]. Let us recall the local fractional derivative (LFD) of the function $\Pi(\zeta)$ of order θ $(0<\theta<1)$ at $\zeta=\zeta_0$, defined by [10-18]:

$$\mathrm{D}_{\zeta}^{(\theta)}\Pi(\zeta_0)=\frac{\mathrm{d}^{\theta}\Pi(\zeta)}{\mathrm{d}\zeta^{\theta}}\Big|_{\zeta=\zeta_0}=\lim_{\zeta\to\zeta_0}\frac{\Delta^{\theta}[\Pi(\zeta)-\Pi(\zeta_0)]}{(\zeta-\zeta_0)^{\theta}} \tag{1}$$

where

$$\Delta^{\theta}[\Pi(\zeta)-\Pi(\zeta_0)]\cong\Gamma(1+\theta)\Delta[\Pi(\zeta)-\Pi(\zeta_0)]$$

The LFD of the function $E_{\theta}(k\zeta^{\theta})(k\in\mathbb{R})$ was [1]:

$$\frac{\mathrm{d}^{\theta}E_{\theta}(k\zeta^{\theta})}{\mathrm{d}\zeta^{\theta}}=kE_{\theta}(k\zeta^{\theta}) \tag{2}$$

The heat-transfer equation involving the LFD in fractal media was written [18]:

$$\frac{\partial^{\theta}\Omega(\vartheta,\tau)}{\partial\tau^{\theta}}+\kappa\frac{\partial^{2\theta}\Omega(\vartheta,\tau)}{\partial\vartheta^{2\theta}}+\omega\Omega(\vartheta,\tau)=0 \tag{3}$$

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where κ is a heat-diffusive coefficient and ω – a constant related to the density and specific heat of fractal materials.

There are a lot of numerical and analytical methods for the local fractional partial differential equations, such as the decomposition method [2, 4, 15], differential transform [3], variational iteration method [5, 12, 14], homotopy perturbation method [6], similarity variable method [7], Laplace variational iteration method [9], series expansion method [10], function decomposition method [11], Fourier transform [13], exp-function method [16], Fourier transform [17], and characteristic equation method (CEM) [19]. The main aim of this paper is to present the CEM to solve the heat-transfer equation in fractal media.

Solve the heat-transfer equation in fractal media

By using the theory of CEM [19], we set the non-differentiable solution of eq. (3):

$$\Omega(\vartheta,\tau) = E_\theta(\rho\tau^\theta)E_\theta(\sigma\vartheta^\theta) \tag{4}$$

In view of eq. (4), we have:

$$\rho + \kappa\sigma^2 + \omega = 0 \tag{5}$$

such that

$$\Omega(\vartheta,\tau) = \varpi E_\theta[-(\kappa\sigma^2+\omega)\tau^\theta]E_\theta(\sigma\vartheta^\theta) \tag{6}$$

where κ is a heat-diffusive coefficient, ϖ – a constant, and the corresponding graph is represented in fig. 1.

By changing the dimension from $\theta = \upsilon\ (0<\upsilon<1)$ to 1, the conventional heat-transfer equation is written:

$$\frac{\partial\Omega(\vartheta,\tau)}{\partial\tau} + \kappa\frac{\partial^2\Omega(\vartheta,\tau)}{\partial\vartheta^2} + \omega\Omega(\vartheta,\tau) = 0 \tag{7}$$

Then, we obtain:

$$\Omega(\vartheta,\tau) = \varpi\exp[-(\kappa\sigma^2+\omega)\tau]\exp(\sigma\vartheta) \tag{8}$$

where κ is a heat-diffusive coefficient and ϖ – a constant.

Equation (8) represents the heat-transfer equation to account for the radiative loss of heat. The corresponding solutions are illustrated in fig. 2.

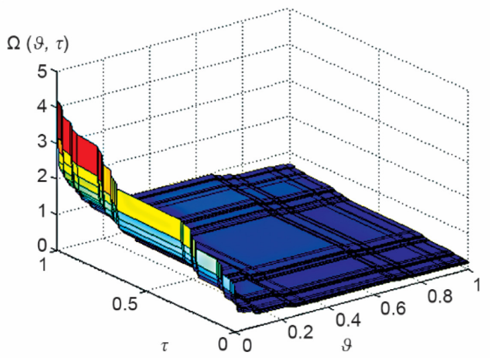

Figure 1. The solution of non-differentiable type of eq. (5) when $\varpi = 1$, $\kappa = 2$, $\omega = 1$, and $\sigma = 1$
(for color image see journal web-site)

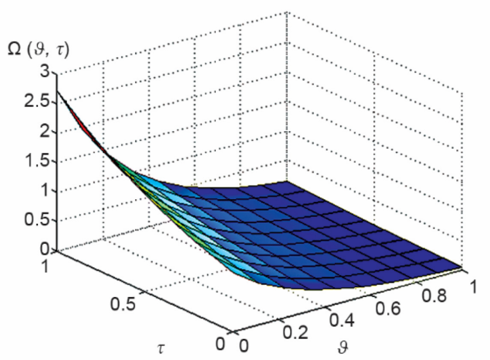

Figure 2. The differentiable solution for the conventional heat-transfer equation when $\varpi = 1$, $\kappa = 2$, $\omega = 1$, and $\sigma = 1$
(for color image see journal web-site)

Conclusion

The fractal heat-transfer problem involving the LFD has been investigated in the work. The non-differentiable solution for the heat-transfer equation in fractal media was obtained by using the CEM. The results for the fractal and conventional heat-transfer equations were compared. The obtained result is very efficient to show the fractal behaviour of heat transfer.

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Nomenclature

θ – fractal order, [–]
ϑ – space co-ordinate, [m]
$\Omega(\vartheta, \tau)$ –temperature, [Km^{-3}]
τ – time, [s]

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