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CHARACTERISTIC NUMBERS OF G-MANIFOLDS AND MULTIPLICATIVE INDUCTION

BY

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ABSTRACT. We determine those finite groups G for which characteristic numbers determine G-equivariant bordism in the unoriented and unitary cases.

It was shown in [6], [7] that suitable global characteristic numbers determine the bordism classes of unoriented G-manifolds if $G \cong (\mathbf{Z}_2)^k$ and of unitary G-manifolds if G is cyclic. We show in this paper that there are no other cases in which characteristic numbers determine bordism classes. The proof is based on an explicit computation of the equivariant characteristic numbers of certain manifolds and on the use of a new construction in bordism theory, which we call multiplicative induction, and which should have many more applications.

In §1 we review some well-known facts about characteristic numbers and explain the notation used in §2 to state the main results of this paper. §3 contains calculations. The definition of multiplicative induction and some applications are given in §4.

1. **Characteristic numbers.** Let G be a compact Lie group. We denote by $\mathfrak{N}_n^G(X, A)$ (respectively, $\mathfrak{U}_n^G(X, A)$) the geometric bordism group of n-dimensional (unitary) singular G-manifolds in (X, A), where X is any G-space and A is a G-subspace of X. If X is a point and A is the empty set, we write

$$\mathfrak{N}_n^G(\text{point}, \varnothing) = \mathfrak{N}_n^G \cong \mathfrak{N}_G^{-n} \quad \text{and} \quad \mathfrak{U}_n^G(\text{point}, \varnothing) = \mathfrak{U}_n^G \cong \mathfrak{U}_G^{-n}.$$

We suppress the G if G is the trivial group. Similarly, we use $N_n^G(X, A)$, $U_n^G(X, A)$, N_n^G, and U_n^G to denote the homotopical bordism groups, defined by means of equivariant Thom spectra [4]. We also use the corresponding cohomology groups $N_G^n(X, A)$ and $U_G^n(X, A)$. The Pontrjagin-Thom construction induces natural transformations of equivariant homology theories

$$i\colon \mathfrak{N}_n^G(X, A) \to N_n^G(X, A) \quad \text{and} \quad i\colon \mathfrak{U}_n^G(X, A) \to U_n^G(X, A).$$

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The bundling transformations

$$\alpha\colon N_G^n(X, A) \to N^n(EG \times_G X, EG \times_G A)$$

and

$$\alpha\colon U_G^n(X, A) \to U^n(EG \times_G X, EG \times_G A)$$

are as defined in [4], where EG is a universal free G-space with orbit space $EG/G = BG$.

The main ingredient in the definition of equivariant characteristic numbers is the Boardman map. In the case of unoriented bordism, this is a natural transformation of multiplicative equivariant cohomology theories

$$B\colon N^*(EG \times_G X) \to H^*(EG \times_G X)[[a_1, a_2, \dots]],$$

where cohomology is taken with $\mathbf{Z}_2$ coefficients and a_i has degree $-i$. The definition of B and its basic properties can be found in X. Kapitel of [2]. For the unitary case we consider characteristic numbers lying in equivariant K-theory. The Boardman map

$$B\colon U_G^*(X) \to K_G^*(X)[[a_1, a_2, \dots]],$$

where we use $\mathbf{Z}_2$-graded K_G-theory, is a natural transformation of multiplicative cohomology theories and is described in [7].

The characteristic number map for n-dimensional G-equivariant unoriented bordism, denoted χ_n^G, is defined to be the composition

$$B\alpha i\colon \mathfrak{N}_n^G \to N_n^G \cong N_G^{-n} \to N^{-n}(BG) \to H^*(BG)[[a_1, a_2, \dots]].$$

The characteristic number map χ_n^G for n-dimensional G-equivariant unitary bordism is the composition

$$Bi\colon \mathfrak{U}_n^G \to U_n^G \cong U_G^{-n} \to R(G)[[a_1, a_2, \dots]],$$

where $R(G) \cong K_G^0(\text{point})$ is the complex representation ring of G.

REMARKS. 1. The characteristic numbers of an n-manifold M, lying in $H^*(BG)$ or $R(G)$, are the coefficients in $\chi_n^G[M]$ of monomials in the a_i's.

2. We call the characteristic numbers defined above "global", because the definition does not involve the orbit structures of manifolds. One could, alternatively, define characteristic numbers using the normal bundles to various fixed point sets, as in the work of M. Rothenberg. Such numbers would combine the orbit structures with the global characteristic numbers.

3. Of course, one could define characteristic numbers with values in other equivariant cohomology rings by an analogous procedure.

2. **Characteristic numbers and bordism.** The notation in this section is as above. We only consider finite groups G.

THEOREM 1. (a) *The composition*

$$B\alpha\colon N_G^* \to N^*(BG) \to H^*(BG)[[a_1, a_2, \dots]]$$

is injective if and only if $G \cong (\mathbf{Z}_2)^k$.

(b) *The characteristic number map*

$$\chi_*^G\colon \mathfrak{N}_*^G \to H^*(BG)[[a_1, a_2, \dots]]$$

is injective if and only if $G \cong (\mathbf{Z}_2)^k$.

THEOREM 2. (a) *The Boardman map*

$$B\colon U_G^* \to R(G)[[a_1, a_2, \dots]]$$

is injective if and only if G is a cyclic group.

(b) *The characteristic number map*

$$\chi_*^G\colon \mathfrak{U}_*^G \to R(G)[[a_1, a_2, \dots]]$$

is injective if and only if G is a cyclic group.

REMARKS. 1. We are mainly interested in the second parts of the above theorems. But the algebraic reasons why the results hold are clearer for the homotopical theories than for the geometric ones.

2. We conjecture that the above theorems are true for compact Lie groups. The proofs below for Theorems 1(a) and 2(a) are valid for such a generalization.

The injectivity statements are already known. If $G \cong (\mathbf{Z}_2)^k$, it was shown in [6] that $B\alpha$ and χ_*^G in Theorem 1 are injective. If G is a finite cyclic group (more generally, the product of a torus and a finite cyclic group), it was proved in [7] that B and χ_*^G in Theorem 2 are injective.

PROOFS OF THEOREMS 1(a) AND 2(a). We begin with Theorem 1(a). If G is not isomorphic to $(\mathbf{Z}_2)^k$, G has an irreducible real representation of dimension greater than one. Let PV be the real projective space associated with such a G-module V, furnished with the G-action inherited from the linear G-action on V. Then the G-action on PV has no fixed points. Consider the commutative diagram

$$\begin{array}{ccc} N_G^* & \xrightarrow{B\alpha} & H^*(BG)[[a_1, a_2, \dots]], \\ \Big\downarrow{\scriptstyle i_1} & & \Big\downarrow{\scriptstyle i_2} \\ N_G^*(PV) & \xrightarrow{B\alpha(PV)} & H^*(EG \times_G PV)[[a_1, a_2, \dots]] \end{array}$$

where i_1 and i_2 map x to $x \cdot 1$. The map i_2 is injective, because $H^*(EG \times_G PV)$ is a free $H^*(BG)$-module. Assume that $B\alpha$ is injective. Then i_1 is injective, also. Now localize away from the multiplicatively closed subset S_G which contains 1 and the Euler classes of G-modules with no trivial summands. Since localization is an exact functor,

$$S_G^{-1}i_1\colon S_G^{-1}N_G^* \to S_G^{-1}N_G^*(PV)$$

is injective. But $S_G^{-1}N_G^* \neq 0$, by an analogue of Theorem 3.1 of [4], while

$$S_G^{-1}N_G^*(PV) \cong S_G^{-1}N_G^*(PV^G) = S_G^{-1}N_G^*(\varnothing) = 0,$$

by Satz 1 of [5]. This is a contradiction. Hence $B\alpha$ cannot be injective if G is not isomorphic to $(\mathbf{Z}_2)^k$.

Theorem 2(a) can be proved similarly. If B is injective, then $S_G^{-1}B$ is, too. But, by Lemma 1 of [7], $S_G^{-1}R(G)[[a_1, a_2, \ldots]]$ is nonzero if and only if G is cyclic.

Proof of Theorem 2(b). It remains to show that χ_*^G is not injective if G is not cyclic. As in [11], a set F of subgroups of G is called a family if it includes all of the subgroups and conjugates of each of its elements. Let EF denote the terminal object in the G-homotopy category of numerable G-spaces whose isotropy groups belong to F. Given a pair of families $F \supset F'$ and a G-homology theory h_*^G, let

$$h_*^G[F, F'](X, A) = h_*^G(EF \times X, (EF \times A) \cup (EF' \times X)).$$

For a subgroup H of G, let $h_*^H(X, A)^{G\text{-inv}}$ denote the "invariant elements", that is, the cokernel of the map

$$\begin{aligned} &h_*^G((G/H) \times (G/H) \times X, (G/H) \times (G/H) \times A) \\ &\qquad \to h_*^G((G/H) \times X, (G/H) \times A) \end{aligned}$$

defined by $(\mathrm{pr}_1 \times 1_X)_* - (\mathrm{pr}_2 \times 1_X)_*$, where $\mathrm{pr}_i\colon (G/H) \times (G/H) \to G/H$ for $i = 1, 2$ are the projections. Now if χ_*^G is injective, the rationalization $\chi_*^G \otimes \mathbf{Q}$ is injective, because the range of χ_*^G is torsion-free. There is a splitting of $\mathfrak{U}_*^G \otimes \mathbf{Q}$, described in more general terms in Theorem 1 of [8], which is given by

$$\mathfrak{U}_*^G \otimes \mathbf{Q} \cong \bigoplus_{(H)} \mathfrak{U}_*^H[\mathrm{All}, \mathrm{Prop}](\mathrm{point}, \varnothing)^{G\text{-inv}} \otimes \mathbf{Q},$$

where All is the family of all subgroups of G, Prop the family of proper subgroups of G, and H ranges over a complete set of conjugacy class representatives of the subgroups of G. Because the splitting is compatible with the natural transformations $i \otimes \mathbf{Q}$ and $B \otimes \mathbf{Q}$, there is a commutative diagram

$$\begin{array}{ccccc} \mathfrak{U}_*^G \otimes \mathbf{Q} & \xrightarrow{i \otimes \mathbf{Q}} & U_*^G \otimes \mathbf{Q} & \xrightarrow{B \otimes \mathbf{Q}} & R(G)[[a_1, a_2, \ldots]] \otimes \mathbf{Q} \\ \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_2} & & \downarrow{\scriptstyle s_3} \\ \bigoplus_{(H)} \mathfrak{U}_*^H[\mathrm{All}, \mathrm{Prop}]^{G\text{-inv}} \otimes \mathbf{Q} & \longrightarrow & \bigoplus_{(H)} S_H^{-1} U_*^H(X)^{G\text{-inv}} \otimes \mathbf{Q} & \longrightarrow & \bigoplus_{(H)} S_H^{-1} R(H)[[a_1, a_2, \ldots]] \otimes \mathbf{Q} \end{array}$$

where the vertical maps are the splitting maps. The group $U_*^H[\mathrm{All}, \mathrm{Prop}]$ may be thought of, as in [9], as $S_H^{-1}U_*^H$. The maps s_1 and s_2 are isomorphisms. The elements of $\mathfrak{U}_*^G \otimes \mathbf{Q}$ corresponding to summands $\mathfrak{U}_*^H[\mathrm{All}, \mathrm{Prop}]^{G\text{-inv}} \otimes \mathbf{Q}$ for which H is not cyclic are contained in the kernel of $\chi_*^G \otimes \mathbf{Q}$.

The remainder of this paper is chiefly concerned with a proof of the "only if" part of Theorem 1(b), that is, that χ_*^G is not injective when G does not have the form $(\mathbf{Z}_2)^k$. In fact, we exhibit explicit bordism classes which lie in the kernel of χ_*^G.

3. Computations of equivariant characteristic numbers. A convenient and conceptually simple method of calculating characteristic numbers uses exponential characteristic classes and the map given by evaluation on the fundamental class. In this section we describe that method.

If $\eta: E \to X$ is a real G-vector bundle, let η_G denote the bundle

$$1_{EG} \times_G \eta: EG \times_G E \to EG \times_G X.$$

If $A[[a_1, a_2, \dots]]$ is a power series ring, define conjugate generators $\bar{a}_1$, $\bar{a}_2, \dots$ by

$$1 + \bar{a}_1 t + \bar{a}_2 t^2 + \dots = \left(1 + a_1 t + a_2 t^2 + \dots\right)^{-1},$$

where t is an indeterminate. An exponential characteristic class for real G-vector bundles associates to each such bundle ξ an element $v(\xi) \in H^*(EG \times_G X)[[a_1, a_2, \dots]]$ of degree zero (recall that a_i has degree $-i$) with constant term 1. The characteristic class must be natural with respect to G-bundle maps and must satisfy the exponential law $v(\xi \oplus \eta) = v(\xi)v(\eta)$.

PROPOSITION 1. *There is exactly one exponential characteristic class v (respectively, $\bar{v}$) for real G-vector bundles which satisfies the condition that*

$$v(\eta) = 1 + (w_1(\eta_G))a_1 + (w_1(\eta_G))^2 a_2 + \dots \left(\bar{v}(\eta) = 1 + (w_1(\eta_G))\bar{a}_1 + (w_1(\eta_G))^2\bar{a}_2 + \dots\right)$$

for every real G-line bundle η, where w_1 denotes the first Stiefel-Whitney class. In general, $\bar{v}(\xi)$ can be obtained from $v(\xi)$ by applying the conjugation automorphism for a power series ring.

PROOF. It suffices to construct v for universal bundles. This can be done by using the splitting principle, as in VIII.X. Kapitel of [2].

REMARK. An analogous proposition is true for complex G-vector bundles, where now the characteristic classes lie in $K_G^*(X)[[a_1, a_2, \dots]]$, as in formula (4) on p. 35 of [7].

Now let M be a closed n-dimensional G-manifold. Let $x \mapsto x[M]$ be the $H^*(BG)$-linear map

$$H^*(EG \times_G M) \to H^{*-n}(BG)$$

known variously as the integration along the fibre map, evaluation on the fundamental class, or the Gysin homomorphism (recall we are using $\mathbf{Z}_2$ coefficients for cohomology). Extend this map linearly to power series rings, still denoting it $x \mapsto x[M]$. Let $\bar{v}$ be the exponential characteristic class defined in Proposition 1. We use τ_M to denote the tangent bundle of M, an n-dimensional real G-vector bundle.

PROPOSITION 2. $\chi_*^G[M] = (\bar{v}(\tau_M))[M]$.

PROOF. Observing that $v(\tau_M)v(\nu_M) = v(V)$, where ν_M is the normal bundle of a G-equivariant embedding of M in a real G-module V, the proof is formally the same as that of Satz 5.9 of [2].

Now let V be a real G-module and ε the trivial real G-line bundle over PV with trivial G-action on the fibre. Then

$$\tau_{PV} \oplus \varepsilon \cong V \otimes \eta, \tag{1}$$

where η is the Hopf line bundle over PV and V denotes the trivial vector bundle over PV with fibre V. To compute the characteristic numbers of PV, we use the following special case of the standard projective bundle theorem.

PROPOSITION 3. *$H^*(EG \times_G PV)$ is a free $H^*(BG)$-module on generators* 1, $b, b^2, \ldots, b^{n-1}$, *where $b = w_1(\eta_G) \in H^1(EG \times_G PV)$ and n is the dimension of V. Furthermore,*

$$b^n = \sum_{i=1}^{n} (w_i(V_G))b^{n-i},$$

where V_G is the real n-plane bundle $EG \times_G V \to BG$, the bundle map being projection on the first factor. The map $x \mapsto x[PV]$ sends $1, b, b^2, \ldots, b^{n-2}$ *to zero and b^{n-1} to* 1.

Now let ξ: $E \to M$ be a two-dimensional G-vector bundle with $w_1(\xi_G) = 0$ and $w_2(\xi_G) = x$. As usual, we can write Stiefel-Whitney classes formally as elementary symmetric polynomials $w_1 = y_1 + y_2$ and $w_2 = y_1 y_2$. Then

$$\begin{aligned} v(\xi) &= \left(1 + y_1 a_1 + y_1^2 a_2 + \ldots\right)\left(1 + y_2 a_1 + y_2^2 a_2 + \ldots\right) \\ &= 1 + y_1 y_2 a_1^2 + y_1^2 y_2^2 a_2^2 + \ldots \\ &= 1 + x a_1^2 + x^2 a_2^2 + \ldots, \end{aligned} \tag{2}$$

computing modulo 2 and using $y_1 + y_2 = 0$. We next apply this computation and Proposition 3 to the case in which

$$V = V(1) \oplus \cdots \oplus V(r) \oplus W(1) \oplus \cdots \oplus W(s),$$

where $V(i)$ and $W(j)$ are irreducible real G-modules, each $V(i)$ has dimension one, and each $W(j)$ has dimension two. Assume $w_1(V(i)_G) = x(i)$,

$w_1(W(j)_G) = 0$, and $w_2(W(j)_G) = y(j)$. Then using the notation of Proposition 3 we have

PROPOSITION 4.

$$v(\tau_{PV}) = \prod_{i=1}^{r}\left(1 + (x(i) + b)a_1 + (x(i) + b)^2 a_2 + \dots\right)$$
$$\times \prod_{j=1}^{s}\left(1 + (y(j) + b^2)a_1^2 + (y(j) + b^2)^2 a_2^2 + \dots\right).$$

PROOF. By Proposition 1 and (1),

$$\begin{aligned} v(\tau_{PV}) &= v(\tau_{PV})v(\varepsilon) = v(\tau_{PV} \oplus \varepsilon) = v(V \otimes \eta) \\ &= v\big((V(1) \oplus \cdots \oplus V(r) \oplus W(1) \oplus \cdots \oplus W(s)) \otimes \eta\big) \\ &= v\big((V(1) \otimes \eta) \oplus \cdots \oplus (V(r) \otimes \eta) \\ &\quad \oplus (W(1) \otimes \eta) \oplus \cdots \oplus (W(s) \otimes \eta)\big) \\ &= \prod_{i=1}^{r} v(V(i) \otimes \eta) \prod_{j=1}^{s} v(W(j) \otimes \eta). \end{aligned}$$

Since $V(i) \otimes \eta$ is a real G-line bundle, by Proposition 1 we have

$$\begin{aligned} v(V(i) \otimes \eta) &= 1 + \big(w_1((V(i) \otimes \eta)_G)\big)a_1 + \big(w_1((V(i) \otimes \eta)_G)\big)^2 a_2 + \dots \\ &= 1 + \big(w_1(V(i)_G) + w_1(\eta_G)\big)a_1 + \big(w_1(V(i)_G) + w_1(\eta_G)\big)^2 a_2 + \dots \\ &= 1 + (x(i) + b)a_1 + (x(i) + b)^2 a_2 + \dots . \end{aligned}$$

Similarly, $v(W(j) \otimes \eta)$ can be computed from (2) and the formula for the Stiefel-Whitney classes of a tensor product.

We apply Proposition 4 to the case in which $G \cong \mathbf{Z}_4$. Let 1 denote the trivial one-dimensional real representation of $\mathbf{Z}_4$, -1 its nontrivial one-dimensional real representation, and i the two-dimensional real representation which comes from the standard representation of $\mathbf{Z}_4 \subset S^1$ on $\mathbf{C}$. Let u denote the unique nonzero element of $H^1(B\mathbf{Z}_4)$, so that $u^2 = 0$, and d the unique nonzero element of $H^2(B\mathbf{Z}_4)$. Let $V = 1^{2n+1} \oplus i$ and $W = 1^{2n} \oplus (-1) \oplus i$.

By Proposition 3, $H^*(E\mathbf{Z}_4 \times_{\mathbf{Z}_4} PV)$ is a free $H^*(B\mathbf{Z}_4)$-module on generators $1, b, b^2, \dots, b^{2n+2}$, where $b^{2n+3} = db^{2n+1}$. The relation implies:

$$(3) \qquad \begin{aligned} b^{2m} &= d^{m-n-1}b^{2n+2}, \quad \text{if } m \geqslant n + 1, \\ b^{2m+1} &= d^{m-n}b^{2n+1}, \quad \text{if } m \geqslant n. \end{aligned}$$

Similarly, $H^*(E\mathbf{Z}_4 \times_{\mathbf{Z}_4} PW)$ has free $H^*(B\mathbf{Z}_4)$-generators $1, b, b^2, \dots, b^{2n+2}$, where $b^{2n+3} = udb^{2n} + db^{2n+1} + ub^{2n+2}$. Thus for $m \geqslant n + 1$ we have:

$$b^{2m} = d^{m-n-1}b^{2n+2} \quad \text{and} \tag{4}$$
$$b^{2m+1} = ud^{m-n}b^{2n} + d^{m-n}b^{2n+1} + ud^{m-n-1}b^{2n+2}.$$

Using the above notation and following [1], we have

PROPOSITION 5. $\chi^{\mathbf{Z}_4}_{2n+2}[PV] = \chi^{\mathbf{Z}_4}_{2n+2}[PW]$.

PROOF. Notice that

$$w_1((-1)_{\mathbf{Z}_4}) = u, \quad w_1(i_{\mathbf{Z}_4}) = 0, \quad \text{and} \quad w_2(i_{\mathbf{Z}_4}) = d.$$

Propositions 1, 2, and 4 imply that $\chi^{\mathbf{Z}_4}_{2n+2}[PV]$ is the conjugate of the coefficient of b^{2n+2} in $(\sum_{j=0}^{\infty} a_j^2(b^2+d)^j)(\sum_{j=0}^{\infty} a_j b^j)^{2n+1}$. And now using (3) we find that $\chi^{\mathbf{Z}_4}_{2n+2}[PV]$ has nonzero homogeneous components only in even dimensions. Similarly, $\chi^{\mathbf{Z}_4}_{2n+2}[PW]$ is the conjugate of the coefficient of b^{2n+2} in

$$\left(\sum_{j=0}^{\infty} a_j b^j\right)^{2n}\left(\sum_{j=0}^{\infty} a_j^2(b^2+d)^j\right)\left(\sum_{j=0}^{\infty} a_j(b+u)^j\right)$$

Because $u^2 = 0$, in even dimensions this product is

$$\left(\sum_{j=0}^{\infty} a_j b^j\right)^{2n+1}\left(\sum_{j=0}^{\infty} a_j^2(b^2+d)^j\right).$$

Since $b^{2m} = d^{m-n-1}b^{2n+2}$, if $m \geqslant n+1$, in both $H^*(E\mathbf{Z}_4 \times_{\mathbf{Z}_4} PV)$ and $H^*(E\mathbf{Z}_4 \times_{\mathbf{Z}_4} PW)$, we see that $\chi^{\mathbf{Z}_4}_{2n+2}[PV]$ and $\chi^{\mathbf{Z}_4}_{2n+2}[PW]$ are the same in even dimensions. Thus it only remains to show that b^{2n+2} has no nonzero odd-dimensional coefficients in

$$\left(\sum_{j=0}^{\infty} a_j b^j\right)^{2n}\left(\sum_{j=0}^{\infty} a_j^2(b^2+d)^j\right)\left(\sum_{j=0}^{\infty} a_j(b+u)^j\right).$$

But such a term would have to be of the form $a^i d^l (b+u)^{2j+1} b^{2k}$, where $k + j \geqslant n + 1$. In all such cases b^{2n+2} has a zero coefficient, since

$$b^{2m+1} + ub^{2m} = ud^{m-n}b^{2n} + d^{m-n}b^{2n+1}, \quad \text{if } m \geqslant n,$$

by (4).

As in [1], Proposition 5 implies

PROPOSITION 6. *$\chi^{\mathbf{Z}_4}_*$ is not injective.*

PROOF. The fixed point set of PV is $\mathbf{RP}^{2n}$. The fixed point set of PW is the disjoint union of a point and $\mathbf{RP}^{2n-1}$. Since, if $n \geqslant 1$, these two fixed point sets represent different bordism classes in the unoriented bordism ring $\mathfrak{N}_*$, PV and PW cannot represent the same equivariant bordism class in $\mathfrak{N}^{\mathbf{Z}_4}_*$. But their $\mathbf{Z}_4$-characteristic numbers are the same, by Proposition 5.

PROPOSITION 7. *If G is not a 2-group, χ^G_* is not injective.*

PROOF. Let H be a 2-Sylow subgroup of G. Since $|N_G(H) : H| \equiv 1 \pmod 2$, $\mathfrak{N}_0^G$ is a $\mathbf{Z}_2$-vector space having at least two generators, a point with trivial G-action and G/H with the G-action defined by left multiplication (Proposition 13.1 of [11]). But if M is any zero-dimensional G-manifold,

$$\chi_0^G[M] \in \left(H^*(BG)[[a_1, a_2, \dots]]\right)^0 \cong \mathbf{Z}_2$$

is nonzero if and only if M consists of an odd number of points. Since $|G : H| \equiv 1 \pmod 2$, the two generators of $\mathfrak{N}_0^G$ described above are both mapped into 1 by χ_0^G.

It follows from Proposition 7 that in order to prove Theorem 1(b) we need only consider 2-groups. We start with the examples used to prove Proposition 6 and construct new manifolds by a method which we call "multiplicative induction".

4. **Multiplicative induction.** Let G be a finite group and denote by $\underline{G}$ the category whose objects are G-homeomorphism classes of left G-spaces and whose morphisms are continuous G-maps. If H is a subgroup of G, the restriction functor r_H^G: $\underline{G} \to \underline{H}$ forgets the actions of the elements of G which are not in H. The functor r_H^G has a left adjoint which maps an H-space X to $G \times_H X$ with G-action given by $g_1(g_2, x) = (g_1 g_2, x)$. This left adjoint is additive, but is not, in general, multiplicative. A right adjoint m_H^G to r_H^G can be defined on objects of $\underline{H}$ by $m_H^G(X) = \mathrm{Hom}_H(G, X)$, where we consider G an H-space via left multiplication. The G-action on $m_H^G(X)$ is given by $(g_1 \cdot f)(g_2) = f(g_2 g_1)$. An H-map f: $X \to Y$ induces a G-map

$$m_H^G(f): m_H^G(X) \to m_H^G(Y)$$

by composition. The usual relationship between adjoint functors is, in our particular case, that

$$\mathrm{Hom}_{\underline{G}}\left(X, m_H^G(Y)\right) \cong \mathrm{Hom}_{\underline{H}}\left(r_H^G(X), Y\right)$$

for any G-space X and H-space Y. We call the functor m_H^G "multiplicative induction" because

$$m_H^G(X_1 \times X_2) \cong m_H^G(X_1) \times m_H^G(X_2)$$

for any H-spaces X_1 and X_2. Observe, also, that $m_H^G(X)$ is homeomorphic to the set of all continuous maps f: $G/H \to G \times_H X$ such that $\pi_1 \circ f = 1_{G/H}$, where π_1: $G \times_H X \to G/H$ is the projection on the first factor and the function space is given the compact-open topology. And now that space is, in turn, homeomorphic to the product of $|G : H|$ copies of X.

If M is an n-dimensional smooth H-manifold, then $m_H^G(M)$ is, in a natural way, a smooth G-manifold of dimension $n|G : H|$. It is surprising that the functor m_H^G is compatible with the bordism relation. That is, let f: $M \to X$ be a singular H-manifold in X. Then $m_H^G(f)$: $m_H^G(M) \to m_H^G(X)$ is a singular G-manifold in $m_H^G(X)$.

PROPOSITION 8. *The function $f \mapsto m_H^G(f)$ induces a well-defined map*

$$m_H^G\colon \mathfrak{N}_n^H(X) \to \mathfrak{N}_{n|G:H|}^G\big(m_H^G(X)\big)$$

which takes products to products but is, in general, not additive.

PROOF. A proof can be found in [10]. The idea is to apply a suitable Pontrjagin-Thom construction, as in [12], to convert the bordism relation into a homotopy relation, and then to use the fact that the image under m_H^G of an H-homotopy is a G-homotopy.

REMARKS. 1. It is, in general, extremely difficult to compute the effect of m_H^G on characteristic numbers. One approach would be to elaborate on the work of Evans.

2. The Steenrod power operation is a special case of multiplicative induction. For its effect on characteristic numbers, see 16.5 of [3].

3. Since m_H^G is not, in general, additive, a more general construction would start with nonhomogeneous bordism elements, that is, sums of manifolds of different dimensions.

To use m_H^G for our purposes, we study its effect on the Pontrjagin-Thom construction and the bundling map. Using a construction similar to m_H^G for H-spaces with base points, we construct induction maps for $N_H^*(X)$ and $N^*(EH \times_H X)$.

Let X_0 be a pointed H-space. Define an H-map $p\colon X = G \times_H X_0 \to G/H$ by $p(g, x) = gH$. Let

$$n_H^G(X) = \bigwedge_{a \in G/H} p^{-1}(a),$$

where $\bigwedge$ denotes the smash product, using the base point which $p^{-1}(a)$ inherits from X_0. Define a G-action on $n_H^G(X)$ by $g \cdot p^{-1}(a) = p^{-1}(ga)$. Recall that $m_H^G(X)$ is naturally G-homeomorphic to a similarly defined object, $\prod_{a \in G/H} p^{-1}(a)$. Clearly n_H^G extends to a functor from pointed H-spaces to pointed G-spaces and is compatible with equivariant pointed homotopies and smash products. That is, $n_H^G(X \wedge Y) \cong n_H^G(X) \wedge n_H^G(Y)$. Letting $M(\xi)$ denote the Thom space of the vector bundle ξ, there is a natural isomorphism between $n_H^G(M(\xi))$ and $M(m_H^G(\xi))$.

Recall that the group $\tilde{N}_H^n(X)$ is defined (at least for compact X) as a direct limit of pointed H-homotopy sets $[V^c \wedge X, M(\xi_H^{n+|V|})]_H^0$, where V^c is the one-point compactification of the G-module V of dimension $|V|$ and ξ_H^k is the universal k-dimensional H-vector bundle. Applying the functor n_H^G, we get maps

$$\begin{aligned}[V^c \wedge X, M(\xi_H^k)]_H^0 &\to \left[n_H^G(V^c \wedge X), n_H^G\left(M(\xi_H^k)\right)\right]_G^0 \\ &\xrightarrow{\cong} \left[\left(m_H^G(V)\right)^c \wedge n_H^G(X), M\left(m_H^G(\xi_H^k)\right)\right]_G^0 \\ &\to \left[\left(m_H^G(V)\right)^c \wedge n_H^G(X), M\left(\xi_G^{k|G:H|}\right)\right]_G^0,\end{aligned}$$

the last map being induced by the classifying map for $m_H^G(\xi_H^k)$. Passing to the limit, we obtain the multiplicative induction map for homotopical bordism,

$$n_H^G\colon \tilde{N}_H^n(X) \to \tilde{N}_G^{n|G:H|}\left(n_H^G(X)\right).$$

If X is a free G-space, there is a natural isomorphism $N_G^n(X) \cong N^n(X/G)$. Using this, the bundling map $\alpha\colon N_G^n(X) \to N^n(EG \times_G X)$ is induced by the projection $EG \times X \to X$. The multiplicative induction n_H^G for $N^*(EG \times_G X)$ is defined in these terms to be the composition

$$\begin{aligned}N_H^k(EH \times X) &\to N_G^{k|G:H|}\left(m_H^G(EH \times X)\right) \\ &\cong N_G^{k|G:H|}\left(m_H^G(EH) \times m_H^G(X)\right) \\ &\to N_G^{k|G:H|}\left(EG \times m_H^G(X)\right),\end{aligned}$$

where the last map is induced by the map $EG \to m_H^G(r_H^G(EG))$ which corresponds to the identity map of $r_H^G(EG)$ under the adjunction map mentioned above (we may take EH to be $r_H^G(EG)$). Now one only needs to apply the definitions to prove

PROPOSITION 9. *The following diagrams are commutative*:

$$\begin{array}{ccc} \mathfrak{N}_k^H & \xrightarrow{\quad i \quad} & N_k^H \cong N_H^{-k} \\ \Big\downarrow m_H^G & & \Big\downarrow n_H^G \\ \mathfrak{N}_{k|G:H|}^G & \xrightarrow{\quad i \quad} & N_{k|G:H|}^G \cong N_G^{-k|G:H|} \end{array}$$

$$\begin{array}{ccc} N_H^k(X) & \xrightarrow{\quad \alpha \quad} & N^k(EH \times_H X) \\ n_H^G \Big\downarrow & & \Big\downarrow n_H^G \\ N_G^{k|G:H|}(m_H^G(X)) & \xrightarrow{\quad \alpha \quad} & N^{k|G:H|}(EG \times_G m_H^G(X)) \end{array}$$

COROLLARY. *If M_1 and M_2 are H-manifolds with the same H-equivariant characteristic numbers, then $m_H^G(M_1)$ and $m_H^G(M_2)$ are G-manifolds with the same G-equivariant characteristic numbers.*

We can now prove Theorem 1(b), using the examples and notation of Proposition 5. Let G be a 2-group which is not of the form $(\mathbf{Z}_2)^k$. Then G has at least one subgroup H such that $H \cong \mathbf{Z}_4$. We fix one such H and consider the multiplicative induction m_H^G.

PROPOSITION 10. *For each positive integer n, the G-manifolds $m_H^G(PV)$ and $m_H^G(PW)$ represent different bordism classes in $\mathfrak{N}^G_{(2n+2)|G:H|}$.*

PROOF. It suffices to show that the H-fixed point sets of $m_H^G(PV)$ and $m_H^G(PW)$ represent different bordism classes in $\mathfrak{N}_*$. Let $G/H = \coprod_i M(i)$ be the decomposition of the H-space G/H into its orbits. Then $M(i) \cong H/H(i)$, where $H(i) \cong \{1\}$, $\mathbf{Z}_2$, or $\mathbf{Z}_4$. If M is any H-manifold,

$$\begin{aligned}(m_H^G M)^H &\cong \mathrm{Hom}_G(G/H, m_H^G(M)) \cong \mathrm{Hom}_H(G/H, M) \\ &\cong \prod_i \mathrm{Hom}_H(M(i), M) \cong \prod_i M^{H(i)}.\end{aligned}$$

Now suppose that $H(i) \cong \{1\}$ for b values of i, $\cong \mathbf{Z}_2$ for c values, and $\cong \mathbf{Z}_4$ for d values. Then

$$(m_H^G(PV))^H \cong (\mathbf{RP}^{2n+2})^b (\mathbf{RP}^{2n})^c (\mathbf{RP}^{2n})^d$$

and

$$(m_H^G(PW))^H \cong (\mathbf{RP}^{2n+2})^b (\mathbf{RP}^{2n})^c (\mathrm{point})^d.$$

And these manifolds are not bordant if $d > 0$ and $n > 0$.

Proposition 10 and the corollary to Proposition 9 complete the proof of Theorem 1(b).

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