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Characteristic polyhedra of singularities

By

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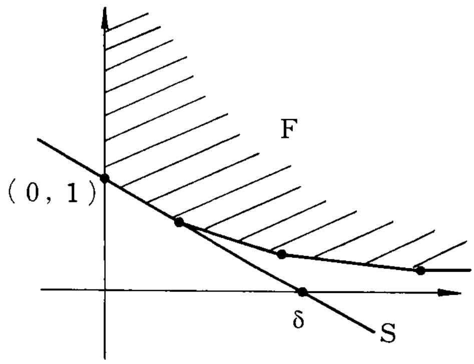
Introduction. The very basic idea of this work is that of Newton polygon. Just to illuminate this view-point, let us take an irreducible plane curve X through the origin defined by an equation $f(x, y)=0$, say over an algebraically closed field k. Let ν be the multiplicity of X at the origin 0, i. e., the order of f at 0. Then we choose the parameters in such a way that the coefficient of y^ν is not zero in $f(x, y)$. Let $f=\Sigma_{i,j}c_{i,j}x^iy^j$ be the power series expansion of f. We map each term $c_{ij}x^iy^j$ to the point $(i/\nu, j/\nu)$ of the Cartesian plane $\boldsymbol{R}^2$. Let E be the set of those points corresponding to nonzero terms of f, and F the convex closure of the union $\bigcup_{v\in E} v+\boldsymbol{R}_0^2$, where $\boldsymbol{R}_0$ denotes the set of nonnegative real numbers. The part of the boundary of F not contained in the two axes is called the *Newton polygon* of f with respect to the parameters x and y. As we have $c_{0\nu}\neq 0$ and $c_{ij}=0$ for all $i+j<\nu$, the first segment of the Newton polygon has $(0,1)$ as its left end point. Let S be the line containing the first segment, and d the intercept of S with the horizontal axis. This d, in general, depends upon the choice of parameters x and y. We can



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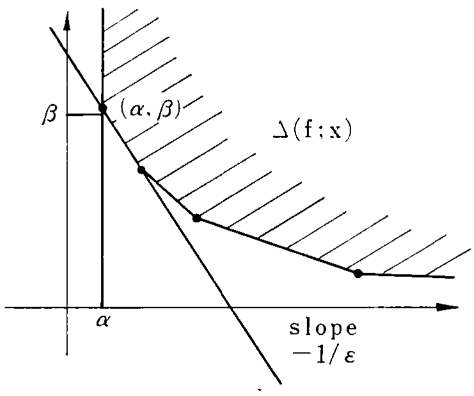
choose the parameters in such a way that d takes the largest possible value, which we denote by δ. It is not difficult to find a simple criterion for (x, y) to give $d=\delta$, as follows. Let f_S be the sum of those terms of f which are mapped to points on S. Then the necessary and sufficient condition for $d=\delta$ is that f_S is not a perfect power, i. e., it cannot be put into the form $c_{0\nu}(y-h)^{\nu}$ with $h\in k[x]$. (Here we exclude the case of a smooth point 0 of X.) The number δ is an intrinsic character of the singularity of X at 0, which is known as the *first characteristic exponent*. Rather, we put δ into the form m/n with relatively prime positive integers m and n, and call (m, n) the *first characteristic pair* of the singularity of X at 0. In [7], Ch. I, one can find a good short account of the characteristic pair (m, n) (and of the complete system of the characteristic pairs) from the algebraic and the topological point of view. Here we point out merely the fact that, $[\delta]$ being the integral part of δ, the exactly $[\delta]$ times applications of quadratic transformations to ν-fold points above 0 eliminate all such points. (In other words, following the classical terminology, we have exactly $[\delta]$ "infinitely near" ν-fold points of X at 0.) To be more precise, if $f: X'\to X$ is the quadratic transformation with center 0, then there exists at most one ν-fold point $0'$ of X' with $f(0')=0$, and if such $0'$ exists, then the new δ attached to $0'$ is exactly one less than the old.

In higher dimensions, the idea of Newton polygon becomes much less exact but certainly much deeper. What the number δ is generalized to in higher dimensions is not a number but a polyhedron with a finite number of vertices, which specializes itself to the half line $[\delta, \infty)$ in the case of a plane curve. To exhibit the essence of our generalization, let us take the next simplest (but enough complicated) case, i. e., that of a surface X in a 3-space over the base field k. Let us assume that X goes through 0 and is defined by $f(x_1, x_2, y)=0$. Let ν be again the multiplicity of X at 0. The most interesting and hardest case is when the leading form of

f is a ν-th power of a linear form, say y^{ν}. This is the case of cusp-like singularity. Let $f=\Sigma_{ijp}c_{ijp}x_1^i x_2^j y^p$ be the power series expansion. We map each term $c_{ijp}x_1^i x_2^j y^p$ to the point $(i/\nu, j/\nu, p/\nu)\in \boldsymbol{R}^3$. Let E be the set of those points corresponding to nonzero terms of f. We project the portion of E lying below the level of $(0,0,1)$, from the point $(0,0,1)$ into the horizontal plane $\boldsymbol{R}^2$. Let P be the projection, which lies in the first quadrant (including the positive half axes) of $\boldsymbol{R}^2$. Let $\{P\}$ be the convex closure of the union $\bigcup_{v\in P}v+\boldsymbol{R}_0^2$. This $\{P\}$ depends upon the choice of x_1, x_2 and y. With (x_1, x_2) fived but y varying, we can make the set $\{P\}$ the smallest in the sense of inclusions. (This fact is not obvious.) Let $\Delta(f; x)$ denote this smallest possible $\{P\}$, which depends only upon the singularity of X at 0 and the transversal parameters $x=(x_1, x_2)$. Define the numbers α, β and ε as indicated in the figure below, where $\Delta(f; x)$ is the shaded area. Note that ε is positive and can be $+\infty$ in an extreme case. Consider all the triples $(\beta, \varepsilon, \alpha)$ attached to various choices of the transversal parameters x, and let $(\tilde{\beta}, \tilde{\varepsilon}, \tilde{\alpha})$ be the smallest one among them in the sense of lexicographical ordering. The system of four numbers $(\nu, \tilde{\beta}, \tilde{\varepsilon}, \tilde{\alpha})$ is an intrinsic character of the singularity of X at 0, and it turns out to be a very useful measurement in describing the effects of certain monoidal and quadratic transformations to the singularity, especially in regards to the problem of reduction of singularities. To describe what happens in a little more details, let us localize the situation sufficiently near the singular point 0 of X, so that the ν-fold curves of X (if there are any) are all non-singular in themselves away from 0. Let us use the



term permissible transformation to mean either a quadratic transformation or a monoidal transformation whose center is a nonsingular ν-fold curve of X. What one can prove is that there exists a permissible transformation $f: X' \to X$, such that if $0'$ is any point of X' with $f(0')=0$, then the character $(\nu, \tilde{\beta}, \tilde{\varepsilon}, \tilde{\alpha})$ of X at 0 is *strictly* greater than the corresponding $(\nu', \tilde{\beta}', \tilde{\varepsilon}', \tilde{\alpha}')$ of X' at $0'$, where the ordering is lexicographical. In this way, one can obtain a new proof (without making any distinction with respect to the characteristic of the base field k) of the B. Levi-Zariski theorem of reduction of singularities of an algebraic surface, which was proven by Zariski in characteristic zero and by Abhyankar in positive characteristics. (What happens about $(\nu, \tilde{\beta}, \tilde{\varepsilon}, \tilde{\alpha})$ is this: $\nu \geq \nu'$; $\nu = \nu' \Rightarrow \tilde{\beta}' = \tilde{\beta} - d/(\nu!)$ with a nonnegative integer d; $(\nu, \tilde{\beta}) = (\nu', \tilde{\beta}')$ and $\tilde{\varepsilon} < \infty \Rightarrow \tilde{\varepsilon}' = \tilde{\varepsilon} - e/(\nu! \tilde{\beta})!$ with an integer $e \geq 0$; $(\nu, \tilde{\beta}) = (\nu', \tilde{\beta}')$and either $e=0$ or $\tilde{\varepsilon} = \infty \Rightarrow \tilde{\alpha}' = \tilde{\alpha} - c/(\nu!)$ with a positive integer c.)

The key substance of the above approach is the totality of those polyhedra $\Delta(f; x)$ with various transversal parameters x and the behavior of these polyhedra under suitable permissible monoidal transformations. Those numerical characters $(\tilde{\beta}, \tilde{\varepsilon}, \tilde{\alpha})$ are nothing more than an artificial and partial quantification of the behavior of those $\Delta(f; x)$, which happened to be sufficient for the mere purpose of reduction of singularities of surfaces.

In the case of dimension 3 or more, the behavior of $\Delta(f; x)$ (which is similarly defined if X is given as a hypersurface in a nonsingular algebraic variety) appears to be far more complicated and has not yet been fully investigated. At any rate, certain permissible transformations applied to the singularity are interpreted as certain transfigurations of the associated polyhedra in a real Cartesian space, and a little experiments lead us to an aphorism: Reduction of singularities is charpening of polyhedra.

So much for motivation, what is actually done in this paper is to generalize the definition of the characteristic polyhedra $\Delta(f; x)$ to the case of singularities of an arbitrary scheme X embedded in a regular scheme Z, and then to give a useful explicit description to them so that their behavior under permissible transformations can be analysed and, in some cases, numerically quantized. The large portion of this paper is devoted to overcome the difficulties in the case of large embedding codimensions, where there is no natural (or God-given) system of generators for the ideal of X in Z locally at a singular point in question. In the first section we will find an intrinsic definition of the characteristic polyhedra, and in the last an explicit description of them by means of suitably chosen ideal base and parameters. We do not know what this theory will develop into in the future, except that for sure the results of this paper will play an important role in the resolution of singularities of an arbitrary excellent surface at the very least.

§1. The characteristic polyhedron $\Delta(J; u)$

Throughout this paper, we fix the following notations:

$R=$ an arbitrary reqular local ring
$M=$ the maximal ideal of R
$k=$ the residue field of R, i.e., R/M
$J=$ an ideal in R, neither R nor (0)
$u=$ $(u_1, \cdots, u_p)=$a system of elements $u_i \in M$

such that

(1, 1) u can be extended to a regular sytem of parameters of R. We put $R'=R/(u)R, M'=M/(u)R$ and $J'=JR'$.

We shall work with various choices of a system $y=(y_1, \cdots, y_r)$ with $y_i \in M$ such that

(1.2) (u, y) is a regular system of parameters of R. Whenever such y is specified, we shall write $t=(t_1, \cdots, t_n)$ for (u, y), where $n=p+r$.

We shall write $\bar{u}=(\bar{u}_1, \cdots, \bar{u}_p)$ with the images $\bar{u}_i$ of u_i in $gr^1_M(R)$. Then we have a natural isomorphism

(1.3) $\quad gr_M(R)/(\bar{u})gr_M(R) \approx gr_{M'}(R')$

If y of (1.2) is chosen, then (1.3) induces an isomorphism

(1.4) $k[\bar{y}]\approx gr_{M'}(R')$, where $\bar{y}=(\bar{y}_1, \cdots, \bar{y})$ with the image $\bar{y}_i$ of y_i in $gr^1_M(R)$. By taking its inverse, we get a monomorphism of graded k-algebras

(1.5) e_y: $gr_{M'}(R')\rightarrow gr_M(R)$.

Let $\boldsymbol{R}^p$ be the p-dimensional vector space over the real number field $\boldsymbol{R}$. With the given (J, u) as above, we shall define a convex set $\Delta(J; u)$ in $\boldsymbol{R}^p$, which is contained in the first quadrant $\boldsymbol{R}^p_0$ where $\boldsymbol{R}_0$ denotes the set of nonnegative real numbers. A linear homogeneous function L on $\boldsymbol{R}^p$, real-valued, will be said to be *positive* if it takes only positive values in $\boldsymbol{R}^p_0-(0)$. Let $\boldsymbol{R}_+$ denote the set of positive real numbers. Let $\boldsymbol{L}_+$ denote the set of all positive linear homogenoeus functions on $\boldsymbol{R}^p$. Let us fix some notations as follows: Given y satisfying (1.2), $L\in\boldsymbol{L}_+$ and $b\in\boldsymbol{R}_+$, we define

(1.6) $I(L; b)_{u,y}=I(L; b)_t=$ the ideal in R generated by the monomials u^Ay^B with $A\in\boldsymbol{Z}^p_0$ and $B\in\boldsymbol{Z}^r_0$ such that $L(A)+|B|\geq b$, where $\boldsymbol{Z}_0$ denotes the set of nonnegative integers and $|B|$ the sum of the r components of B.

(1.7) $I^+(L; b)_{u,y}=I^+(L; b)_t=$ the ideal in R generated by those u^Ay^B such that $L(A)+|B|>b$.

For each pair (g, b) of $g\in$ J and $b\in\boldsymbol{R}_+$, such that $g\in I(L; b)_t$, we consider various expressions of g in the following from

(1.8) $g=\Sigma_{A,B}g(A, B)+h$, where $A\in\boldsymbol{Z}^p_0$, $B\in\boldsymbol{Z}^r_0$, $L(A)+|B|=b$, $g(A, B)\in u^Ay^BR$ and $h\in I^+(L; b)_t$. To each expression (1.8), we assign an element of $gr_M(R)$, denoted by $in(g; L; b)_{u,y}$ or simply $in(g; L: b)$, and defined by

(1.9) $in(g; L; b)=\Sigma_{A;B}G(A, B)$, where $G(A, B)$ is the image of $g(A, B)$ in $gr^*_M(R)$ with $*=|A|+|B|$ for *each* (A, B) of (1.8).

Now, given (J, u, y, L) as above, we define the following symbol

(1.10) $\{J, L\}_{u,y}=\{J; L\}_t=$ the ideal in $gr_M(R)$ which is generated by all those elements $in(g; L; b)$ obtained as above.

For $L\in\boldsymbol{L}_+$, we put $\Delta(L)=\{v\in\boldsymbol{R}^p_0|L(v)\geq 1\}$.

Defiinition (1.11) Given (J, u, y) as above, we define the symbol $\Delta(J;u;y)$ to be the intersection of $\Delta(L)$ for all those $L\in \boldsymbol{L}_{+}$ such that

(1.11.1) $$\{J;L\}_{u,y}=e_{y}(gr_{M'}(J',R'))gr_{M}(R)$$

where $gr_{M'}(J',R')$ denotes the associated graded ideal of J'. (cf. [3], Ch. II, §2.)

Deifinition (1.12) Given (J,u) as above, we define $\Delta(J;u)$ to be the intersection of $\Delta(J;u;y)$ for all those y satisfying (1.2). We call $\Delta(J;u)$ *the (first) characteristic polynedron of J with respect to u.*

Remark (1.13) Given $(J;u;y)$, it is possible that there exists no $L\in\boldsymbol{L}_{+}$ satisfying (1.11.1). In this case, $\Delta(J;u;y)=\boldsymbol{R}_{0}^{p}$. Moreover, if $\Delta(J;u;y)=\boldsymbol{R}_{0}^{p}$ for all y of (1.2), then $\Delta(J;u)=\boldsymbol{R}_{0}^{p}$. This is the case, for instance, if J is any non-zero ideal contained in $(u)R$. Such a case is certainly uninteresting. In the following section, we will find a condition under which such a case is excluded.

Remark (1.14) The most interesting is the following case:

(1.14.1) $\bar{u}$ is a free base of the k-module $gr_{M}^{1}(R)/T(J)$,

where

(1.14.2) $T(J)$ is the smallest k-submodule of $gr_{M}^{1}(R)$ such that $gr_{M}(J,R)=(gr_{M}(J,R)\cap k[T(J)])gr_{M}(R)$. (cf. [3], Ch. III, §4.)

In this case, $\Delta(J;u)$ is a nonempty convex subset of $\boldsymbol{R}_{0}^{p}$ which does not meet the *reference symplex* of $\boldsymbol{R}_{0}^{p}$, i.e., $\{(a_{1},\cdots,a_{p})\in\boldsymbol{R}_{0}^{p}\mid\sum_{i=1}^{p}a_{i}\leq 1\}$. We will find that, in this case, $\Delta(J;u)$ closely reflects the nature of singularities of the local scheme Spec(R/J).

The whole purpose of this study in this paper is to find an algorithm for computing, or at least a useful description of, the characteristic polyhedron $\Delta(J;u)$. In a later section, for instance, we shall prove the following

Theorem (1.15) Let the situation be the same as in (1.14).

Let us assume that R is complete. Then there exists a system y satisfying (1.2) and a standard base $f=(f_1, \cdots, f_m)$ of J, such that

(1.15.1) for each $L \in \boldsymbol{L}_+$, $L(v) \geq 1$ for all $v \in \Delta(J; u)$ if and only if $f_i \in I(L: \nu_i)_{u,y}$ with $\nu_i = \nu_M(f_i)$ for all i. (For the notion of *standard base*, see [3], Ch. III, §1.)

Our result is more precise and constructive than it is stated above. In a later section, we shall introduce the notion of *Δ-preparedness*. This notion is such that, in the situation of (1.14), if the pair $(f; y)$ is *Δ-prepared with respect to u*, then it has the property (1.15.1). When R is complete, starting from an arbitrary pair (f, y), we apply to it repeatedly what we call *vertex preparations* and finally obtain one which is Δ-prepared.

§2. The initial ideal $\{J; \Delta\}_{u,y}$ in $gr_M(R)$

Our primary aim in this section is to give a useful description of the ideal $\{J; L\}_{u,y}$ defined by (1.10), or more generally $\{J; \Delta\}_{u,y}$ which is to be defined below, in terms of a suitably chosen ideal base of J, such as a standard base.

An E-subset of $\boldsymbol{Z}_0^n$ is, by definition, a subset E such that $E+\boldsymbol{Z}_0^n=E$. (cf. [3], Ch. III, §7.) Every E-subset E of $\boldsymbol{Z}_0^n$ can be expressed as

(2.0) $E=\bigcup_{i=1}^s(A_i+\boldsymbol{Z}_0^n)$

where $\{A_1, \cdots, A_s\}$ is a *finite* subset of E. In other words, it is *bounded* by a finite subset. (loco cito) Such a subset $\{A_1, \cdots, A_s\}$ of E with the smallest possible s is uniquely determined by E and will be called the base of E. The intersection of E-subsets is an E-subset, and every subset F admits the smallest E-subset containing it, which will be denoted by [F]. Quite generally, if a system of n elements $t=(t_1, \cdots, t_n)$ is given in a ring R, then we associate with every subset F of $\boldsymbol{R}_0^n$, the ideal in R generated by all the monomials t^A with $A \in F \cap \boldsymbol{Z}_0^n$. This ideal will be denoted by $I(F)_t$ or simply $I(F)$ whenever the reference to t is clear. Clearly $I([F])=I(F)$ for a subset F of $\boldsymbol{Z}_0^p$. Moreover, if R is the polynomial (power series) ring of n indeterminates t over k, it is easy to show that $A\in[F]$ if and only if $t^A\in I(F)$. (One can prove (2.0) using this fact.)

Lemma (2.1) Let R be a noetherian local ring, and $t=(t_1, \cdots, t_n)$ a regular R-sequence in the maximal ideal of R. Let E_i be E-subsets of $\boldsymbol{Z}_0^n$. Then we have $\bigcap_i I(E_i)_t = I(\bigcap_i E_i)_t$.

Proof. Let us first consider the case of two E-subsets, in which the proof is by induction on n. We may assume $n>1$, and define $\beta(E_1, E_2)$ to be the sum $\Sigma_i|A_i|+\Sigma_j|B_j|$, where $(A_1, \cdots, A_a)$ (resp. $(B_1, \cdots, B_b)$) is the base of E_1(resp. E_2). We then take the second induction on $\beta(E_1, E_2)$. The assertion is trivial if $\min\{\Sigma_i|A_i|, \Sigma_j|B_j|\}=0$. It is immediate from the regularity assumption on t, if $\max\{\Sigma_i|A_i|, \Sigma_j|B_j|\}=1$. Pick any other case. We may assume that $\Sigma_i|A_i|>1$ and $A_a-e\in\boldsymbol{Z}_0^n$, where $e\in\boldsymbol{Z}_0^n$ is such that $t^e=t_n$. Then $I(E_1)\cap I([e])=I([A_a])+I([A_1, \cdots, A_{a-1}])\cap I([e])$. Apply the induction assumption to this last term, we get $I(E_1)\cap I([e])=I(E_1\cap[e])$. We need this result below. Let $R^*=R/t_nR$, $E_i^*\times(0)=E_i\cap\boldsymbol{Z}_0^{n-1}\times(0)$, and $t^*=(t_1^*, \cdots, t_{n-1}^*)$ with the image t_i^* of t_i in R^*. For any $F^*\subset\boldsymbol{Z}_n^{n-1}$, we define $I(F^*)_{t^*}$ in the same way as before, and write it as $I^*(F^*)$ for short. Now we have $\{I(E_1)\cap I(E_2)\}R^*\subset I(E_1)R^*\cap I(E_2)R^*=I^*(E_1^*)\cap I^*(E_2^*)$. By induction on n, this is equal to $I^*(E_1^*\cap E_2^*)=I(E_1\cap E_2)R^*$, as the regularity assumption on t implies the same on t^*. Thus, $I(E_1)\cap I(E_2)=I(E_1\cap E_2)+I(E_1)\cap I(E_2)\cap I([e])$. But this triple intersection is equal to $I(E_1\cap[e])\cap I(E_2\cap[e])$ by the above result and by the induction on β. Let E_i' be the E-subset of $\boldsymbol{Z}_0^n$ such that $E_i'+e=E_i\cap[e]$. Then $I(E_i\cap[e])=t_nI(E_i')$ and $t_nI(E_1')\cap t_nI(E_2')=t_n(I(E_1')\cap I(E_2'))=t_nI(E_1'\cap E_2')$ by induction on β. Hence $I(E_1)\cap I(E_2)=I(E_1\cap E_2)+t_nI(E_1'\cap E_2')=I(E_1\cap E_2)$. Now, for the general case, it is enough to prove: $I(\bigcap_i E_i)\equiv\bigcap_i I(E_i) \bmod I(E(m))$ for every m, where $E(m)=\{A\in\boldsymbol{Z}_0^n \mid |A|\geq m\}$. But this for each m is a case of finitely many E_i. Q. E. D.

If Δ is any subset of $\boldsymbol{R}_0^t$, then $\{\Delta\}$ will denote the smallest closed convex subset of $\boldsymbol{R}_0^p$ containing $v+\boldsymbol{R}_0^p$ for all $v\in\Delta$. If $\Delta=\{\Delta\}$, then we call Δ *an F-subset of* $\boldsymbol{R}_0^p$. For instance, $\Delta(L)$ for $L\in\boldsymbol{L}_+$ is an F-subset. We shall often identify $\Delta(L)$ with L itself, and in this sense we shall generalize the symbol $\{J; L\}_{u,y}$. For this purpose, we introduce the notion of the *essential boundary* of an F-subset Δ *of* $\boldsymbol{R}_0^p$, denoted by $\partial\Delta$ and defined by

(2.2) $\partial\Delta=$the subset of Δ consisting of those $v\in\Delta$ such that $v\notin v'+\boldsymbol{R}_0^p$ with $v'\in\Delta$ unless $v=v'$.

Every pair (A, B) with $A\in\boldsymbol{R}_0^p$ and $B\in\boldsymbol{R}_0^r$ will be viewed as an element of $\boldsymbol{R}_0^n$, where $n=p+r$. With every pair (Δ, b) of an F-subset Δ of $\boldsymbol{R}_0^p$ and $b\in\boldsymbol{R}_+$, we associate a subset of $\boldsymbol{R}_0^n$, $\{\Delta; b\}$ in symbol, which is defined by

(2.3) $\{\Delta; b\}=$the smallest convex subset of $\boldsymbol{R}_0^n$ containing $(b\Delta)\times\boldsymbol{R}_0^r$ and $\boldsymbol{R}_0^p\times W(b)$, where $W(b)=\{B\in\boldsymbol{R}_0^r \| B|\geq b\}$. Here $b\Delta=\{bv| v\in\Delta\}$ in the sense of scalar multiplication on vectors. By the convexity of Δ, $b\Delta=b_1\Delta+b_2\Delta$ if $b_i\in\boldsymbol{R}_+$ and $b=b_1+b_2$. Moreover, it is easy to see that $\{\Delta; b\}$ is an F-subset of $\boldsymbol{R}_0^n$.

Let us go back to the situation of §1. We choose y of (1.2) and put $t=(u, y)$. For every subset F of $\boldsymbol{R}_0^n$, we have defined the ideal $I(F)_{u,y}=I(F)_t=I(F\cap\boldsymbol{Z}_0^n)$. If F is an F-subset of $\boldsymbol{R}_0^n$, then we also define $I^+(F)_{u,y}(=I^+(F)_t)$ to be the ideal $I(F-\partial F)_t$. We now generalize the notations of (1.6) and (1.7) as follows: For an F-subset Δ of $\boldsymbol{R}_0^n$ and $b\in\boldsymbol{R}_+$,

(2.6) $I(\Delta; b)_{u,y}=I(\Delta; b)_t=I(\{\Delta; b\})_t$ and

(2.7) $I^+(\Delta; b)_{u,y}=I^+(\Delta; b)_t=I^+(\{\Delta; b\})_t$.

Lemma (2.8) Let F be any F-subset of $\boldsymbol{R}_0^n$. Then every $g\in I(F)_t$ can be expressed as

(2.8.1) $g=\Sigma_A g(A)+h$, where $g(A)\in t^A R$, $h\in I^+(F)_t$ and A ranges through the set $\partial F\cap\boldsymbol{Z}_0^n$. Moreover, if $G(A)$ denotes the image of $g(A)$ in $gr_M^*(R)$ with $*=|A|$, then

(2.8.2) $\Sigma_A G(A)$ for the same range of A as above, is independent of the expression (2.8.1).

Proof. The first assertion is trivial. To prove the second, let us pick any other expression of the same g, say $g=\Sigma_A g'(A)+h'$, which has the property of (2.8.1). Let $G'(A)$ be the image of $g'(A)$ in $gr^*_M(R)$ with $*=|A|$. We shall prove $G(A)=G'(A)$ for all A. Let $E=F\cap \boldsymbol{Z}^n_0$ and $A\in\partial F\cap \boldsymbol{Z}^n_0$. Then the complement of the element A in E is an E-subset of $\boldsymbol{Z}^n_0$, which will be denoted by E_0. Then we get $g(A)-g'(A)\in I(E_0)\cap I([A])=I(E_0\cap[A])$ by (2.1). Since $A\notin E_0$, $I(E_0\cap[A])\subset MI([A])\subset M^{|A|+1}$. Hence the image of $g(A)-g'(A)$ is zero in $gr^*_M(R)$ with $*=|A|$, which is $G(A)-G'(A)$. Q. E. D.

Definition (2.9) For $g\in I(F)_t$, the element of $gr_M(R)$ defined by (2.8.2) will be denoted by $in(g; F)_t$ and called *the initial polynomial of g in $gr_M(R)$ with respect to* $(F; t)$. We are particularly interested in the case of $F=\{\Delta; b\}$ with an F-subset Δ of $\boldsymbol{R}^0_p$ and $b\in \boldsymbol{R}_+$. In this case, $in(g; F)_t$ will be denoted by $in(g; \Delta; b)_{u,y}$ and called *the initial polynomial of g in $gr_M(R)$ with respect to* $(\Delta; b; u; y)$.

Remark (2.10) Whenever the following symbols make sense, we have

(2.10.1) $in(g\pm g'; \Delta; b)_{u,y}=in(g; \Delta; b)_{u,y}\pm in(g'; \Delta; b)_{u,y}$ and

(2.10.2) $in(g; \Delta; b)_{u,y}in(g'; \Delta; b')_{u,y}\equiv in(gg'; \Delta; b+b')_{u,y}$ mod $I^+(\Delta; b+b')_{\bar{t}}$, where the last ideal is defined in the same way as in (2.7) with the ring $gr_M(R)=k[\bar{t}]$ and the system $\bar{t}$. The first equality is immediate from (2.8) and (2.9). To prove the second congruence, it is enough to check: $\{\Delta; b\}+\{\Delta; b'\}=\{\Delta; b+b'\}$ and $(\{\Delta; b\}-\partial\{\Delta; b\})+\{\Delta; b'\}\subset\{\Delta; b+b'\}-\partial\{\Delta; b+b'\}$.

Remark (2.11) Suppose Δ is an F-subset of $\boldsymbol{R}^p_0$ such that its essential boundary $\partial\Delta$ is *convex*. Then we get the equality $j(\partial\Delta)+j'(\partial\Delta)=(j+j')(\partial\Delta)$ for all $j, j'\in\boldsymbol{R}_+$. It follows that $\partial\{\Delta; b\}$ is convex and has the same property. Then (2.10.2) implies that, in fact, the congruence can be replaced by the equality.

Definition (2.12) Δ being an F-subset of $\boldsymbol{R}^p_0$, we define the symbol $\{J;\, \Delta\}_{u,y}$ to be the ideal in $gr_M(R)$ which is generated by $in(g;\, \Delta;\, b)_{u,y}$ for all pairs (g, b) with $b \in \boldsymbol{R}_+$ and $g \in J \cap I(\Delta;\, b)_{u,y}$.

Remark (2.13) If $\Delta = \boldsymbol{R}_0^p$, then $in(g;\, \Delta;\, b)_{u,y}$ is nothing but the image of g in $gr_M^0(R) = k$. Hence we get $\{J;\, \Delta\}_{u,y} = (0)$, independent of J, u and y. On the other hand, if $\Delta = \Delta(L)$ with $L \in \boldsymbol{L}_+$, then Δ is neither $\boldsymbol{R}_0^p$ nor empty. In this case, we find that (1.9) (resp. (1.10)) is a special case of (2.9) (resp. (2.12)).

In general, an F-subset of $\boldsymbol{R}_0^p$ will be said to be *proper* if it is neither $\boldsymbol{R}_0^p$ itself nor empty.

Lemma (2.14) Let g be any non-zero element of M, and Δ a proper F-subset of $\boldsymbol{R}_0^p$. Let y be any system of (1.2). Let V be the set of those $b \in \boldsymbol{R}_+$ such that $g \in I(\Delta;\, b)_{u,y}$. Then V is an interval $(0, \nu]$ containing its least upper bound ν, provided V is not empty.

Proof. If $b > b' \in \boldsymbol{R}_+$, then $\{\Delta;\, b\} \subset \{\Delta;\, b'\}$ and $I(\Delta;\, b)_{u,y} \subset I(\Delta;\, b')_{u,y}$. Hence it is enough to prove that if V is not empty, then V contain its least upper bound. First of all V is bounded because for every integer $d > 0$, $I(\Delta;\, b)_{u,y} \subset M^d$ if b is sufficiently large. (For instance, choose b such that $b > d$ and d is less than the distance from the origin to $b\Delta$.) Let ν be the least upper bound for V. We want to prove that $g \in I(\Delta;\, \nu)_{u,y}$. It suffices that $g \in I(\Delta;\, \nu)_{u,y} + M^m$ for any given integer m. For a fixed m, there are only a finite number of $A \in \boldsymbol{Z}_0^n$ with $|A| < m$. It is therefore easy to find $b \in V$ (sufficiently close to ν) such that if $A \in \boldsymbol{Z}_0^n$ and $|A| < m$, then $A \in \{\Delta;\, b\}$ implies $A \in \{\Delta;\, \nu\}$. This implies $I(\Delta; b)_{u,y} \subset I(\Delta;\, \nu)_{u,y} + M^m$. But $g \in I(\Delta;\, b)_{u,y}$. Q. E. D.

Quite generally, we extend the notion of *power* I^b of an ideal I in a ring R to an arbitrary real number b as follows:

(2.15.1) For every real number b, let c (resp. c') be the smallest nonnegative integer $\geq b$ (resp. $> b$). Then we put $I^b = I^c$ and $I^{b+} = I^{c'}$.

(2.15.2) The graded R/I-algebra $gr_I(R)$ is viewed as the direct sum (as R/I-modules) of all I^b/I^{b+} for all real numbers b. In other words, the homogeneous parts of degrees b, $gr_I^b(R)$, are all zero except for nonnegative integers b.

Lemma (1.16) Let g, y and Δ be the same as in (2.14). Let g' be the image of g in R'. Pick $b\in \boldsymbol{R}_+$ such that $g\in I(\Delta; b)_{u,y}$. Then $g'\in M'^b$ and, if ψ' is the image of g' in $gr_{M'}^b(R')$, then

(2.16.1) $in(g;\Delta;b)_{u,y}-e_y(\psi')\in(\bar{u})gr_M(R)$ (cf.(1.3)–(1.5).)

Proof. It is clear from the definition that $I(\Delta; b)_tR'=M'^b=(y)^bR'$, where $t=(u,y)$. Hence we find $\mathrm{h}\in(y)^bR$ such that $g-\mathrm{h}\in I(\Delta; b)_t\cap(u)R$. By (2.10), $in(g; \Delta; b)_{u,y}-in(\mathrm{h}; \Delta; b)_{u,y}=in(g-\mathrm{h}; \Delta; b)_{u,y}$. Thanks to (2.1), we find this element in $(\bar{u})gr_M(R)$. It is easy to see that $in(\mathrm{h}; \Delta; b)_{u,y}=e_y(\psi')$. Q. E. D.

Corollary (2.16.2) Let $\varphi=in(g; \Delta; b)_{u,y}$. Let ψ be the initial form of g' in $gr_{M'}(R')$. Suppose $\varphi\in k[\bar{y}]$ and $\neq 0$. Then we have $b=\nu_{M'}(g')=$the largest number with $g\in I(\Delta; b)_{u,y}$, and $\varphi=e_y(\psi)$. In particular, b is a positive integer and φ is homogeneous of degree b.

Proof. Thanks to (2.16), it is enough that b is the largest number with $g\in I(\Delta: b)_{u,y}$. Suppose $g\in I(\Delta; b')_{u,y}$ with $b'>b$. Then by (2.16), $g'\in M'^{b'}$ so that by (2.16.1), $\varphi\equiv 0 \mod (\bar{u})gr_M(R)$. This is impossible, as $\varphi\neq 0$. Q. E. D.

We are interested in the following condition on a pair (y, Δ) of a system y of (1.2) and a proper F-subset Δ of $\boldsymbol{R}_0^p$:

(2.17) $\{J; \Delta\}_{u,y}=(\{J; \Delta\}_{u,y}\cap k[\bar{y}])gr_M(R)$

This condition implies $\{J; \Delta\}_{u,y}$ is homogeneous by (2.16.2). As will become clear later, the condition is very significant for a certain type of F-subsets Δ but not for some others at all. It is most significant, when Δ is effective in the following sense.

Definition (2.18) An F-subset Δ of $\boldsymbol{R}_0^p$ is said to be *effective* if it is a proper F-subset such that $\boldsymbol{R}_0^p-\Delta$ is bounded.

Remark (2.18.1) For $L\in \boldsymbol{L}_+$, the associated F-subset $\Delta(L)$ is effective. In general, if Δ is an effective F-subset of $\boldsymbol{R}_0^p$, the $\{\Delta;b\}$ with $b\in\boldsymbol{R}_+$ is an effective F-subset of $\boldsymbol{R}_0^n$. (cf. (2.3).)

Lemma (2.19) Let $g\in M$, $\neq 0$, and let Δ be an effective F-subset of $\boldsymbol{R}_0^p$. Then there exists the largest $\nu\in\boldsymbol{R}_+$ such that $g\in I(\Delta;\nu)_{u,y}$. Moreover $in(g;\Delta;b)_{u,y}$ for $b\leqq\nu$ is zero if and only if $b<\nu$.

Proof. In this case, the set V of (2.14) is not empty, because Δ being effective, every point of $\boldsymbol{R}_0^n$ other the origin is contained in some $\{\Delta;b\}$ with $b\in\boldsymbol{R}_+$. Hence ν exists by (2.14). For $b>b'\in\boldsymbol{R}_+$, $\{\Delta;b\}=(b/b')\{\Delta;b'\}$, so that $\{\Delta;b\}\subset\{\Delta;b'\}-\partial\{\Delta;b'\}$. Hence it is clear that $in(g;\Delta;b)_{u,y}=0$ for all $b<\nu$. Now, suppose $in(g;\Delta;\nu)_{u,y}=0$. If we take an expression (2.8.1) with $F=\{\Delta;\nu\}$, then all the $G(A)$ of (2.8.2) are zero. This implies that $g(A)\in I^+(F)_t$ for all A. Hence $g\in I^+(F)_t=I((F-\partial F)\cap\boldsymbol{Z}_0^n)_t$. Since F is effective, ∂F has a positive distance from $(F-\partial F)\cap\boldsymbol{Z}_0^n$. Let e be this distance. Let $m=\max\{|v|$ for $v\in\partial F\}$. Then for $c\in\boldsymbol{R}_+$ with $1<c<1+e/m$, $cF\supset(F-\partial F)\cap\boldsymbol{Z}_0^n$. But $cF=\{\Delta;c\nu\}$ and $g\in I^+(F)_t\subset I(cF)_t=I(\Delta;c\nu)_{u,y}$. This contradicts the maximality of ν. Q. E. D.

Recall that a *standard base* of a homogeneous ideal in a graded algebra is, by definition, a minimal base consisting of homogeneous elements which are arranged in the order of nondecreasing degrees.

Definition (2.20) A system of elements $f=(f_1,\cdots,f_m)$ with $f_i\in J$, $\neq 0$, will be called a *(u)-effective base* of J if there exists a system y satisfying (1.2) and an effective F-subset Δ of $\boldsymbol{R}_0^p$ such that if ν_i is the largest number with $f_i\in I(\Delta;\nu_i)$ for all i,

(2.20.1) $k[\bar{y}]$ contains the initial polynomials $\varphi_i=in(f_i;\Delta;\nu_i)_{u,y}$ for all i, $1\leq i\leq m$, and

(2.20.2) $\{J;\Delta\}_{u,y}=(\varphi_1,\cdots,\varphi_m)gr_M(R)$.

A (u)-effective base is called a *(u)-standard base* if $(\varphi_1,\cdots,\varphi_m)$ is a standard base of the ideal of (2.20.2). A pair (y,Δ) having the above property will be called a *reference datum* of the (u)-effective (or (u)-standard) base of J.

Remark (2.20.3) Let $f=(f_1, \cdots, f_m)$ be any standard base of J in the sense of [3], Ch. III, §1. Let T be any k-submodule of $gr^1_M(R)$ such that the initial forms φ_i of f_i in $gr_M(R)$ are contained in $k[T]$ for all i. If the system u of (1.1) is such that $\bar{u}$ is a free base of $gr^1_M(R)/T$, then f is a (u)-standard base of J in the sense of (2.20). In fact, let y be any system such that $\bar{y}$ is a free base of T, and let $\Delta=\{(a_1, \cdots, a_p)\in \boldsymbol{R}^p_0 \mid \sum_{i=1}^p a_i \geq 1\}$. Then (y, Δ) is a reference datum for that f.

We shall use the symbol $I(\Delta; b)_{u,y}$ of (2.6) in the following extended sense: $I(\Delta; b)_{u,y}=R$ if b is any nonpositive real number. Recall that (2.6) defined it only for positive b.

Theorem (2.21) Let $f=(f_1, \cdots, f_m)$ be a (u)-effective base of J. Let f'_i be the image of f_i in R' of (1.1), and let $\nu_i=\nu_{M'}(f'_i)$, $1\leq i\leq m$. Let Δ be a proper F-subset of $\boldsymbol{R}^p_0$ and y a system of (1.2). Suppose $f_i\in I(\Delta; \nu_i)_{u,y}$ for all i. Then, for every $b\in \boldsymbol{R}_+$ and E-subset N of $\boldsymbol{Z}^p_0$, we have

${}_NI(\Delta; b)_{u,y}\cap J=(\sum_{i=1}^m {}_NI(\Delta; b-\nu_i)_{u,y}f_i)+({}_NI^+(\Delta; b)_{u,y}\cap J)$ where ${}_NI$ substitutes for $I(N)_u\cap I$. (Note: ${}_NI=I$ if $N=\boldsymbol{Z}^p_0$.)

Proof. Let (z, Δ°) be a reference datum of the (u)-effective base f of J. Throughout this proof, we shall abreviate symbols as follows: $in(g; b)$ (resp. $in(g; b)^\circ$) for $in(g; \Delta; b)_{u,y}$ (resp. $in(g; \Delta^\circ; b)_{u,y}$). We shall first consider the case in which $z=y$. Until we discuss the case of $z\neq y$, we shall suppress the reference indices (u, y) attached to various symbols. Now, to prove the theorem, we shall consider an arbitrary pair (g, b) with $b\in \boldsymbol{R}_+$ and $g\in J\cap {}_NI(\Delta; b)$, $\neq 0$. We then want to prove that

(2.21.0) there exists $g^*\in \sum_{i=1}^m {}_NI(\Delta; b-\nu_i)f_i$ such that $g-g^*\in I^+(\Delta; b)$.

We will prove this for each fixed $b\in \boldsymbol{R}_+$, by the descending induction on the number $d=d(g)$ which is the largest real number such that $g\in I(\Delta^\circ; d)$. (Note that, for every real number e, say $\in \boldsymbol{R}_+$, the set $(\boldsymbol{R}_0^n - \{\Delta;\, e\}) \cap \boldsymbol{Z}_0^n$ is finite and hence there are only a finite number of possible values of $d(g)$ which are less than e.) If $d(g)$ is big enough (with respect to b), then $g \in I^+(\Delta;\, b)$ and the assertion (2.21.0) is trivial. Now, pick any (g, b) as above, so that $g \in_N I(\Delta;\, b) \cap I(\Delta^\circ;\, d) = I(\{\Delta;\, b\} \cap \{\Delta^\circ;\, d\} \cap N \times \boldsymbol{Z}_0^r)$ by (2.1). Then we have an expression

(2.21.1) $g = \Sigma_{A \in E}\, g(A) + h$, which has the property of (2.8.1) for $F = \{\Delta;\, b\}$, so that $in(g;\, b) = \Sigma_{A \in E}\, G(A)$ in the sense of (2.8.2), where the range E of A is equal to $\partial\{\Delta;\, b\} \cap \{\Delta^\circ;\, d\} \cap N \times \boldsymbol{Z}_0^r$. Note that this implies

(2.21.2) ${}_N I(\Delta^\circ;\, d)$ contains all the $g(A)$ and h.

If $G(A) = 0$, then $g(A) \in_N I^+(\Delta;\, b)$. Hence we may assume that

(2.21.3) $g(A) = 0$ if $G(A) = 0$ for each $A \in E$.

We have $g(A) \in t^A R$ with $t = (u, y)$. Hence $in(g(B);\, d)^\circ = in(g(B);\, b) = G(B)$ for all $B \in E \cap \partial\{\Delta^\circ;\, d\}$. Let us put $E_\circ = E \cap \partial\{\Delta^\circ;\, d\} \cap \{A \mid G(A) \neq 0\}$. Let $H = in(h;\, d)^\circ$. This is well defined by (2.21.2). Let P be the subset of $\boldsymbol{Z}_0^p$ such that

(2.21.4) $H = \Sigma_{a \in P}\, \bar{u}^a H_a$, where $H_a \in k[\bar{y}]$ and $\neq 0$ for each $a \in P$.

Let Q be the subset of $\boldsymbol{Z}_0^p$ such that

(2.21.5) $\Sigma_{A \in E_\circ}\, G(A) = \Sigma_{a \in Q}\, \bar{u}^a G_a$

where $G_a \in k[\bar{y}]$ and $\neq 0$ for each $a \in Q$. We then claim

(2.21.6) $P \cap Q = \phi$.

In fact, let $a \in P \cap Q$, so that there exist $A = (a, c) \in \boldsymbol{Z}_0^n$ and $A' = (a, c') \in \boldsymbol{Z}_0^n$ such that $\bar{t}^A$(resp. $\bar{t}^{A'}$) has non-zero coefficident in the polynomial H (resp. $\Sigma_{A \in E_\circ}\, G(A)$). Then, since $\partial\{\Delta^\circ;\, d\}$ contains both A and A', $a \in (d - |c|)\partial\Delta^\circ \cap (d - |c'|)\partial\Delta^\circ$. But this intersection is nonempty if and only if $|c| = |c'|$. Since $A' \in \partial\{\Delta;\, b\}$, this equality implies $A \in \partial\{\Delta;\, b\}$. But this contradicts $h \in I^+(\Delta;\, b)$. (2.21.6) is thus established. We know that

(2.21.7) $in(g;\, d)^\circ = \Sigma_{A \in E_\circ}\, G(A) + H$, which belongs to the ideal

$\{J;\Delta^{\circ}\}_{u,y}$. This ideal will be denote by $\{J\}^{\circ}$ in short. Let $\varphi_i = in(f_i, b_i)^{\circ}$ with the largest $b_i \in \boldsymbol{R}_+$ such that $f_i \in I(\Delta^{\circ}; b_i)$. By assumption, $\varphi_i \in k[\bar{y}]$ for all i and $\{J\}^{\circ} = (\varphi_1, \cdots, \varphi_m) gr_M(R)$. By (2.19) and (2.20), $\varphi_i \neq 0$ for any i. Hence by (2.16.2) we get $b_i = \nu_i$ for all i. Now, (2.21.4)-(2.21.7) readily imply that $\{J\}^{\circ}$ contains H_a for all $a \in P$ and G_a for all $a \in Q$. Hence we can write

(2.21.8) $H_a = \sum_{i=1}^{m} H_{a,i}\varphi_i$ with forms $H_{a,i} \in k[\bar{y}]$ of degrees $\deg H_a - \nu_i$, and

(2.21.9) $G_a = \sum_{i=1}^{m} G_{a,i}\varphi_i$ with forms $G_{a,i} \in k[\bar{y}]$ of degrees $\deg G_a - \nu_i$. Here it should be notes that all the H_a and G_a are homogeneous. Let us then choose an element $h_{a,i}$(resp. $g_{a,i}$) of the ideal $(y)^* R$ whose image in $gr_M^*(R)$ is equal to $H_{a,i}$(resp. $G_{a,i}$), where $* = \deg H_a - \nu_i$ (resp. $\deg G_a - \nu_i$). Let us then put

(2.21.10) $h_{\circ} = \sum_{a \in P} u^a \sum_{i=1}^{m} h_{a,i} f_i$, and

(2.21.11) $g_{\circ} = \sum_{a \in Q} u^a \sum_{i=1}^{m} g_{a,i} f_i$.

Let $g' = g - h_{\circ} - g_{\circ}$. Then we have

(2.21.12) $g - g' \in \sum_{i=1}^{m} {}_N I(\Delta; b - \nu_i) f_i$.

In fact, this ideal contains both $h_{\circ}$ and $g_{\circ}$. Take $h_{\circ}$ for instance. Due to our selection of E in (2.21.1), we have $h \in I(\{\Delta; b\} \cap \{\Delta^{\circ}; d\} \cap N \times \boldsymbol{Z}_0^r)$. By the assumptions in (2.21.4), $(a,b) \in \{\Delta; b\} \cap \{\Delta^{\circ}; d\} \cap N \times \boldsymbol{Z}_0^p$ for all $a \in P$ and all $c \in \boldsymbol{Z}_0^r$ with $|c| = \deg H_a$. Therefore every $H_{a,i} \neq 0$ in (2.21.8), $(a, c') \in \{\Delta; b - \nu_i\} \cap N \times \boldsymbol{Z}_0^p$ for all $c' \in \boldsymbol{Z}_0^r$ with $|c'| = \deg H_{a,i}$. This implies $u^a h_{a,i} \in {}_N I(\Delta; b - \nu_i)$ for all such (a, i). (The proof for $g_{\circ}$ is quite similar.) By the above induction, it is now suffices to prove that $d(g') > d$. In fact, we will then have g'^* having the property (2.21.0) for g', so that $g^* = g_{\circ} + h_{\circ} + g'^*$ has the same for g. Now, we have $f_i \in I(\Delta^{\circ}; \nu_i)$ for all i. (We proved $b_i = \nu_i$ for all i.) Therefore, it follows from (2.21.10) and (2.21.11) that $I(\Delta^{\circ}; d)$ contains both $h_{\circ}$ and $g_{\circ}$. Moreover, by (2.21.4)-(2.21.9), we get $in(h_{\circ}; d)^{\circ} = H = in(h; d)^{\circ}$ and $in(g^{\circ}; d)^{\circ} = \sum_{A \in E_{\circ}} G(A) = in(\sum_{A \in E_{\circ}} g(A); d)^{\circ}$. Hence, by (2.10), (2.21.7) shows $in(g'; d)^{\circ} = in(g; d)^{\circ} - in(g_{\circ}; d)^{\circ} - in(h_{\circ}; d)^{\circ} = 0$.

Hence, Δ° being effective, (2.19) implies that $g' \in I(\Delta^\circ; d')$ for some $d' > d$, i.e., $d(g') > d$. This completes the proof of the theorem for the case of $y = z$. In the general case, we can find a third system $x = (x_1, \cdots, x_r)$ satisfying (1.2), such that $y \equiv x \bmod (u)R$ and $(x)R = (z)R$. Pick any effective F-subset D of $\boldsymbol{R}_0^p$ which contains both Δ° and $\{(a_1, \cdots, a_p) \in \boldsymbol{R}_0^p \mid \sum_i a_i \geq 1/2\}$. Then we have $y_i - x_i \in I^+(D; 1)_{u,x} = I^+(D; 1)_{u,z}$ for all i, $1 \leq i \leq r$. It follows that $I(D; b)_{u,y} = I(D; b)_{u,x} = I(D; b)_{u,z}$ and $I^+(D; b)_{u,y} = I^+(D; b)_{u,z}$ for all $b \in \boldsymbol{R}_+$. Let us denote these ideals by $I(b)$ and $I^+(b)$ respectively. Pick any $g \in I(b)$ with $b \in \boldsymbol{R}_-$. On one hand, we have $u^a y^c \equiv u^a x^c \bmod I^+(b)$ for all $(a, c) \in \partial\{D; b\} \cap \boldsymbol{Z}_0^n$. Hence, $in(g; D; b)_{u,y}$ is gotten from $in(g; D; b)_{u,x}$ by replacing $\bar{x}$ by $\bar{y}$ (and fixing $\bar{u}$). On the other hand, if $E(j) = (b-j)(\partial D) \cap \boldsymbol{Z}_0^p$ for each integer j, $0 \leq j \leq b$, then $\partial\{D; b\} \cap \boldsymbol{Z}_0^n = \bigcup_j E(j) \times \{v \in \boldsymbol{Z}_0^r \mid |v| = j\}$. Therefore, if we write $g \equiv \sum_j \sum_{a \in E(j)} g^{(a)} \bmod I^+(b)$ with $g^{(a)} \in u^a(x)^j R = u^a(z)^j R$, then both $in(g; D; b)_{u,x}$ and $in(g; D; b)_{u,z}$ are equal to $\sum_j \sum_{a \in E(j)} G^{(a)}$ where $G^{(a)}$ denotes the image of $g^{(a)}$ in $gr_M^*(R)$ with $* = |a| + j$. Thus, if q is the $k[\bar{u}]$-automorphism of $gr_M(R)$ with $q(\bar{x}_i) = \bar{y}_i$, $q(in(g; D; b)_{u,z}) = in(g; D; b)_{u,y}$. This being so for every (g, b) as above, we get

(2.21.13) $\quad \{J; D\}_{u,y} = q(\{J; D\}_{u,z})$

Since $D \supset \Delta^\circ$, $\varphi_i = in(f_i; D; \nu_i)_{u,y} = q(in(f_i; D; \nu_i)_{u,z})$ for all i. By Theorem (2.21) for the case in which the proof is already done, we can associate with every pair (g, b) as above, an element $g^* \in \sum_{i=1}^m I(D; b-\nu_i)_{u,z} f_i$ such that $g - g^* \in I^+(b)$. Write $g^* = \sum_i h_i f_i$ accordingly. Then, since $f_i \in (z)^{\nu_i} R + I^+(\nu_i)$ for all i, it is easy to see that if $\varphi_i = q(\varphi_i^*)$, $in(g^*; D; b)_{u,z} = \sum_i in(h_i; D; b-\nu_i)_{u,z}\, \varphi_i^*$. Clearly this coincides with $in(g; D; b)_{u,z}$, and it follows that $(\varphi_1^*, \cdots, \varphi_m^*)$ generates the ideal $\{J; D\}_{u,z}$. Therefore, in view of (2.21.13), we find that (y, D) is also a reference datum for the given (u)-effective base f of J. In other words, we could choose (y, D) instead of (z, Δ°) from the very beginning. Q.E.D.

Theorem (2.21) has many useful consequences, some of which are stated below as corollaries.

Corollary (2. 21. a) Let the assumptions be the same as in (2. 21). Suppose $\partial\Delta$ is convex. Then the ideal $\{J;\Delta\}_{u,y}$ is generated by $in(f_i;\Delta;\nu_i)_{u,y}$ for $1\leq i\leq m$.

Proof. Pick any pair (g,b) with $b\in \boldsymbol{R}_+$ and $g\in I(\Delta;b)_{u,y}$. Then, by (2. 21), there exists $h_i\in I(\Delta;b-\nu_i)_{u,y}$, $1\leq i\leq m$, such that $g-\sum_i h_i f_i\in I^+(\Delta;b)_{u,y}$. Since $\partial\Delta$ is convex, (2. 11) is applicable to this case. Namely, by (2. 10) and (2. 11), we have the equality $in(g;\Delta;b)_{u,y}=in(\sum_i h_i f_i;\Delta;b)_{u,y}=\sum_i in(h_i;\Delta;b-\nu_i)_{u,y}in(f_i;\Delta;\nu_i)_{u,y}$, where $in(*;\Delta;d)=0$ if $d<0$ and $=$ (the residue class of $*$ mod M) if $d=0$. Q. E. D.

Corollary (2. 21. b) Let the assumptions be the same as in (2. 21). Suppose we have $f_i\in(y)^{\nu_i}R+I^+(\Delta;\nu_i)_{u,y}$ for all i. Let $\varphi_i=in(f_i;\Delta;\nu_i)_{u,y}$, $1\leq i\leq m$. Then we have $\{J;\Delta\}_{u,y}=(\varphi_1,\cdots,\varphi_m)gr_M(R)=e_y(gr_{M'}(J',R'))gr_M(R)$, where e_y is the homomorphism of (1. 5)

Proof. Pick (g,b) and put $g-\Sigma_i h_i f_i\in I^+(\Delta;b)_{u,y}$ in the same way as we did in the proof of (2. 21. a). By (2. 10), we have the congruence $in(g;\Delta;b)_{u,y}\equiv\Sigma_i in(h_i;\Delta;b-\nu_i)_{u,y}\varphi_i$ mod $I^+(\Delta;b)_{\bar{t}}$. But, by the assumption on f_i, φ_i are forms only in $\bar{y}$. Hence the right hand side of the congruence (as well as the left hand side) is a linear combination of only those monomials of the form $\bar{t}^A$ with $A\in\partial\{\Delta;b\}\cap \boldsymbol{Z}_0^n$. Since $I^+(\Delta;b)_{\bar{t}}$ is generated by those monomials $\bar{t}^A$ with $A\in(\{\Delta;b\}-\partial\{\Delta;b\})\cap\boldsymbol{Z}_0^n$, which is an E-subset of $\boldsymbol{Z}_0^n$, the above congruence can be replaced by the equality. We conclude that $\{J;\Delta\}_{u,y}=(\varphi_1,\cdots,\varphi_m)gr_M(R)$. Now, if ψ_i denotes the initial form of f'_i in $gr_{M'}(R')$, then $\varphi_i=e_y(\psi_i)$ for all i. (See (1. 16. 2).) We now have only to prove that $gr_{M'}(J',R')=(\psi_1,\cdots,\psi_m)gr_{M'}(R')$. Pick any form $q\in gr_{M'}(J',R')$, $\neq 0$. Then there exists an element g of J such that, g' being the image of g in R', q is the initial form of g' in $gr_{M'}(R')$. Then we can find an effective F-subset D of $\boldsymbol{R}_0^p$, so large that $D\supset\Delta$ and $g\in(y)^b+I^+(D;b)_{u,y}$ with the integer $b=$

deg q. Then, by what is already proven above, $\{J;D\}_{u,y}$ is generated by $in(f_i;D;\nu_i)_{u,y}=in(f_i;\Delta;\nu_i)_{u,y}=\varphi_i$, $1\leq i\leq m$. Hence, if $s=in(g;D;b)_{u,y}$, then $s\in(\varphi_1,\cdots,\varphi_m)gr_M(R)$. Since $k[\bar{y}]$ contains all the φ_i and s, we get $s\in(\varphi_1,\cdots,\varphi_m)k[\bar{y}]$. Since e_y is injective, $q\in(\psi_1,\cdots,\psi_m)gr_{M'}(R')$. Q.E.D.

Corollary (2.21.c) Let the assumptions be the same as in (2.21.b). Assume that Δ is effective. Then (y,Δ) is a reference datum for the (u)-effective base f of J.

Proof. By (2.16) and (2.19), ν_i is the largest number with $f_i\in I(\Delta;\nu_i)_{u,y}$ for each i. Then the assumption on f_i of (2.21.b) implies that $in(f_i;\Delta;\nu_i)_{u,y}\in k[\bar{y}]$ for all i, i, e., (2.20.1). On the other hand, we have (2.20.2) by (2.21.b). Q.E.D.

Corollary (2.21.d) Let the assumptions by the same as in (2.21). Assume that Δ is effective. Then, for every $b\in\boldsymbol{R}_+$, we have $I(\Delta;b)_{u,y}\cap J=\sum_{i=1}^m I(\Delta;b-\nu_i)_{u,y}f_i$. In particular, every (u)-effective base of J generates J.

Proof. Since $J\subset M$ and Δ is effective, we have $J\subset I(\Delta;b)_{u,y}$ for sufficiently small $b\in\boldsymbol{R}_+$. Thus the first assertion implies the second. Let H and H' denote the left and the right hand side of the claimed equality, respectively. Let $V=\{d\in\boldsymbol{R}_+\mid H\subset I(\Delta;d)_{u,y}+H'\}$. This set V is unbounded above. In fact, if otherwise, we have the least upper bound b of V. As is easily seen, if $\beta>0$ is small enough, $\{\Delta;b-\beta\}\cap\boldsymbol{Z}_0^n=\{\Delta;b\}\cap\boldsymbol{Z}_0^n$ and hence $I(\Delta;b-\beta)_{u,y}=I(\Delta;b)_{u,y}$. It follows that $b\in V$. Pick any element $g\in J\cap I(\Delta;b)_{u,y}$. Then by (2.21), there exists $g^*\in H'$ such that $g-g^*\in I^+(\Delta;b)_{u,y}$. Since Δ is effective, there exists $\alpha>0$, so small that $(\{\Delta;b\}-\partial\{\Delta;b\})\cap\boldsymbol{Z}_0^n=\{\Delta;b+\alpha\}\cap\boldsymbol{Z}_0^n$. This implies $I^+(\Delta;b)_{u,y}\subset I(\Delta;b+\alpha)_{u,y}$. It then follows that $b+\alpha\in V$, which contradicts the assumption on b. Q.E.D.

Let us recall the definiton (1.11) of the symbol $\Delta(J;u;y)$.

Corollary (2.21.e) Let the assumptions be the same as in (2.21). Then Δ contains $\Delta(J;u;y)$.

Proof. Take any $L\in \boldsymbol{L}_+$ which takes only values >1 at the points of $\varDelta$. We shall then prove that L takes only values ≥ 1 on $\varDelta(J; u; y)$. This certainly suffices. The assumption on L implies that $f_i\in(y)^{\nu_i}R+I^+(D; \nu_i)_{u,y}$ with $\nu_i=\nu_{M'}(f'_i)$ and $D=\varDelta(L)$. Hence by (2.21.b), this L satisfies the condition (1.11.1). Q.E.D.

Corollary (2.21.f) Let the assumptions be the same as in (2.21). Let N be any E-subset of $\boldsymbol{Z}_0^p$. Then we have $I(N)_u\cap J=\sum_{i=1}^m I(N)_u f_i$.

Proof. In (2.21), choose $\varDelta$ to be effective and b to be sufficiently small. Then we get $I(N)_u={}_N I(\varDelta; b)_{u,y}={}_N I(\varDelta; b-\nu_i)_{u,y}$ for all i. In the same way as we did in the proof of (2.21.d), we apply to our situation the theorem (2.21) for a fixed $\varDelta$ and various b, and prove that $I(N)_u\cap J=\sum_{i=1}^m I(N)_u f_i+I(N)_u\cap I(\varDelta; b)_{u,y}\cap J$ for all $b\in\boldsymbol{R}_+$. The equality of (2.21.f) follows from this by the closedness of every ideal in a local ring. Q.E.D.

If there exists a (u)-effective base of J, then there exists a (u)-standard base of J. In fact, as is easily seen from the definition (2.20), the latter can be obtained as a suitably reordered subsystem of the former. It should be noted, however, that there does not always exist any (u)-effective base of J. We thus look for some useful criteria for the existence of such a base.

Lemma (2.22) Let D be any proper F-subset of $\boldsymbol{R}_0^p$, and y a system of (1.2). Let $\widehat{R}$ be the completions of R, $\widehat{M}=M\widehat{R}$ and $\widehat{J}=J\widehat{R}$. Then for every $b\in\boldsymbol{R}_+$ and $g\in I(D; b)_{u,y}$, the symbol $in(g; D; b)_{u,y}$ defines the same element when R is replaced by $\widehat{R}$, in terms of the canonical isomorphism of $gr_M(R)$ and $gr_{\widehat{M}}(\widehat{R})$. Moreover, we have $\{\widehat{J}; D\}_{u,y}=\{J; D\}_{u,y}$.

Proof. The notations $I(D; b)_{u,y}$ and $I^+(D; b)_{u,y}$ will be meant with reference to R. The corresponding ones with reference to $\widehat{R}$ will be denoted by $\widehat{I}(D; b)_{u,y}$ and $\widehat{I}(D;b)_{u,y}$. Clearly, we have $\widehat{I}(D; b)_{u,y}=I(D; b)_{u,y}\widehat{R}$ and $\widehat{I}^+(D; b)_{u,y}=I^+(D; b)_{u,y}\widehat{R}$. Take any $b\in\boldsymbol{R}_+$.

Then $\{D; b\} \cap Z_0^n$ is an E-subset of Z_0^n, and hence as in (2.0), it has a finite base $\{A_1, \cdots, A_s\}$. *Let* $t = \max_i(|A_i|)$. Now take any $\hat{g} \in \hat{J} \cap \hat{I}(D; b)_{u,y}$. This ideal is equal to $(J \cap I(D; b)_{u,y})\hat{R}$, and hence there exists $g \in J \cap I(D; b)_{u,y}$ such that $\hat{g} \equiv g \mod \hat{M}^{t+1}$. Then by the selection of t, we can prove $in(\hat{g}; D: b)_{u,y} = in(g; D; b)_{u,y}$. This proves (2.22). Q. E. D.

Lemma (2.23) Quite generally, let R be a noetherian local ring with the maximal ideal M, and $u = (u_1, \cdots, u_p)$ a regular R-sequence with $u_i \in M$. Let J be an ideal in R which is contained in M. Then the following conditions are equivalent to one another:

(2.23.1) u is a regular R/J-sequence.

(2.23.2) $gr_{(u)}(R/J)$ is a polynomial ring of p indeterminates over $R/(J, u)R$ where (u) denotes the ideal $(u)R$.

(2.23.3) $(u)^q R \cap J = (u)^q J$ for all integers $q \geq 1$.

(2.23.4) $(u)R \cap J = (u)J$.

Proof. Let K_u denote the Koszul complex generated by u over R. Knowing that K_u is acyclic, (2.23.4) can be proven to be epuivalent to $H_1(K_u \otimes_R R/J) = (0)$. Hence Propcsition (2.8) of [2] proves the equivalence between (2.23.1) and (2.23.4). We have proven the equivalence of (2.23.1) and (2.23.2), by Lemma (1.9) of [4]. (2.23.3) $\Rightarrow$(2.23.4) is trivial. Therefore, it is enough to show: (2.23.2)+(2.23.4)$\Rightarrow$(2.23.3). This is done by induction on q. Let $q \geq 1$ and pick any $g \in (u)^{q+1} R \cap J$. Assume $g \in (u)^q J$. We can write $g = \Sigma_A u^A g_A$ with $g_A \in J$, where the range of A is $\{A \in Z_0^p \mid |A| = q\}$. Let $S = R/J$, and $\bar{g}_A$ (resp. $\bar{u}_i$) the image of g_A (resp. u_i) in $S/(u)S$ (resp. $gr^1_{(u)}(S)$). Then we get $\Sigma_A \bar{u}^A \bar{g}_A = 0$ because $g \in (u)^{q+1} R$. By (2.23.2), we get $\bar{g}_A = 0$ for all A, i.e., $g_A \in (u)R \cap J$. But this ideal is $(u)J$ by (2.23.4). Q. E. D.

Let us return to the situation of §1. Note that the above Lemma (2.23) is not only needed in the proof of the following theorem but also important as a supplement to the statement of the theorem itself.

Theorem (2.24) The following conditions ars equivalent to one another:

(i) There exists a (u)-standard base of J.

(ii) There exists a (u)-effective base of J.

(iii) u is a regular R/J-sequence.

(iv) There exists an effective F-subset Δ of $\boldsymbol{R}_0^e$ and a system y satisfying (1.2) such that (2.17) holds.

Remark (2.24.1) Under the assumption (iv), we find a (u)-standard base of J with reference datum (y, Δ).

Proof. (i)$\Leftrightarrow$(ii)$\Rightarrow$(iv) are trivial. Assume (ii). Then we get (2.23.4) as a special case of (2.21.f), and hence (iii) by (2.23). Thus (ii)$\Rightarrow$(iii). We shall next prove (iv)$\Rightarrow$(i). Let $G=\{J;\Delta\}_{u,y}$ and $\overline{G}=$ the $k[\overline{y}]$-part*) of G. The condition (2.17) says that $G=\overline{G}gr_M(R)$. Hence we can find a system of elements $f=(f_1,\cdots,f_m)$ with $f_i\in J\cap I(\Delta;\nu_i)_{u,y}$ for certain positive integers ν_i, such that if $h_i=in(f_i;\Delta;\nu_i)_{u,y}$ and $\overline{h}_i=$ the $k[\overline{y}]$-part*) of h_i, then $\overline{G}=(\overline{h})k[\overline{y}]$ with $\overline{h}=(\overline{h}_1,\cdots,\overline{h}_m)$. Replacing f by a suitably reordered subsystem, we may assume that $\overline{h}$ is a standard base of $\overline{G}$. Let i be any positive integer such that $\overline{h}_j=h_j$ for all $j<i$. Let P be the subset of $\boldsymbol{Z}_0^e$ such that $h_i-\overline{h}_i=\Sigma_{A\in P}\overline{u}^Ah_A$ with $h_A\in k[\overline{y}]$, $\neq 0$. Then h_A are all homogeneous and such that $(A,B)\in\partial\{\Delta;\nu_i\}$ for all $B\in\boldsymbol{Z}_0^r$ with $|B|=\deg h_A$. Moreover, $|A|>0$ for all $A\in P$ and hence $\deg h_A<\deg\overline{h}_i=\nu_i$. Therefore, as we have $h_i-\overline{h}_i\in G=\overline{G}gr_M(R)$, the ideal $(\overline{h}_1,\cdots,\overline{h}_{i-1})k[\overline{y}]$ must contain h_A for all $A\in P$. Let us write $h_A=\sum_{j=1}^{i-1}h_{Aj}\overline{h}_j$ with forms h_{Aj} of degrees $\deg h_A-\nu_j$. Then let us pick any $g_{Aj}\in(y)^*R$ such that h_{Aj} is the image of g_{Aj} in $gr^*(R)$, where $*=\deg h_A-\nu_j$. Let $f''=f_i-\Sigma_{A\in P}\sum_{j=1}^{i-1}u^Ag_{Aj}f_j$. Then the above remarks on $A\in P$ imply that $f''\in I(\Delta;\nu_i)_{u,y}$ and $in(f'';\Delta;\nu_i)_{u,y}=\overline{h}_i$. We can then replace f_i by f'' without affecting the assumptions on f. In other words, we could choose the above f in such a way that $h_i=\overline{h}_i$ for all i, so that f is a (u)-standard base of J with reference datum (y,Δ). Next, to

*) For the meaning of "part", refer to the paragraph preceeding Def. (3.8) below.

prove (iii)$\Rightarrow$(ii), we fix any system y of (1.2). For each integer $d>0$, let $D_d=\{v\in\boldsymbol{R}_0^p \mid |v|\geq 1/d\}$. Let $\{J\}_d$ denote the $k[\bar{y}]$-part of the ideal $\{J;\ D_d\}_{u,y}$. If $d>d'$, then $\{J\}_d\supset\{J\}_{d'}$. Hence, $gr_M(R)$ being noetherian, there exists a such that $\{J\}_a=\{J\}_d$ for all $d>a$. Replacing a by a still larger integer if necessary, we may assume that there exists a system of elements $f=(f_1,\cdots,f_m)$ with $f_i\in J\cap((y)^{\nu_i}+I^+(D_a;\ \nu_i)_{u,y})$ for certain positive integers ν_i, such that $\varphi_i=in(f_i;\ D_a;\ \nu_i)_{u,y}\in k[\bar{y}]$ for all i and $\{J\}_a=(\varphi)k[\bar{y}]$ with $\varphi=(\varphi_1,\cdots,\varphi_m)$. Now, pick any $b\in\boldsymbol{R}_+$ and $g\in J\cap I(D_a;\ b)_{u,y}$. We then want to prove that

(ii*) $\quad in(g;\ D_a;\ b)_{u,y}\in\{J\}_a gr_M(R)$.

If this is done, then it is clear that there exists a suitably reordered subsystem of f which is a (u)-standard base of J. Now, to prove (ii*), we may assume that R is complete, by virtue of (2.22). Let us assume $g\neq 0$ and define $q=q(g)$ to be the largest integer such that $g\in(u)^qR$. If $q>b$, then $g\in I^+(D_a;\ b)_{u,y}$ and (ii*) is then trivially true. We shall prove (ii*) in general by the descending induction on $q(g)$. Let $s=s(g)$ be the largest integer such that we have an expression $g=\sum_{A\in E}u^Ag_A$ with $g_A\in J\cap((y)^sR+(u)R)$ and $E=\{A\in\boldsymbol{Z}_0^p \mid |A|=q\}$. Such s exists for the following reasons. First, by (2.23), (iii) implies that $J\cap(u)^qR=(u)^qJ$ and hence we find such an expression of g for at least one nonnegative integer s. Secondly, such an expression is not possible for all $s>0$, because if it is, we get $g\in\bigcap_{s=1}^{\infty}\sum_{A\in E}u^A(J\cap((y)^sR+(u)R))\subset(u)^{q+1}R$. Now, an expression of g as above being given, we let g'_A (resp. h'_A) denote the image of g_A (resp. g'_A) in R' (resp. $gr^s_{M'}(R')$). Let $h_A=e_y(h'_A)$. Then as is easily seen, $h_A=in(g_A;\ D_d;\ s)_{u,y}$ for all sufficiently large $d>a$. Hence $h_A\in\{J\}_d=\{J\}_a$. Write $h_A=\sum_i h_{Ai}\varphi_i$ with forms h_{Ai} of degrees $s-\nu_i$ in $k[\bar{y}]$. Pick any $g_{Ai}\in(y)^*R$ whose image in $gr^*_M(R)$ is h_{Ai}, where $*=s-\nu_i$. Then let $g''=g-\sum_{A\in E}u^A\sum_{i=1}^m g_{Ai}f_i$. By the definition of $s(g)$, $h_A\neq 0$ for at least one $A\in E$. Therefore, since $g\in I(D_a;\ b)_{u,y}$, $\{D_a;\ b\}$ contains all (A,B) with $A\in E$ and

$B \in Z_0^r$ such that $|B| \geq s$. Hence, $g_{Ai} \in I(D_a;\ b-\nu_i)_{u,y}$ for all (A, i). It follows that

(ii**) $in(g'';\ D_a;\ b)_{u,y} \equiv in(g;\ D_a;\ b)_{u,y} \mod \{J\}_a gr_M(R)$.

Moreover, if s is sufficiently large in comparison with b and q, then the congruence of (ii**) becomes an equality. We repeat the modification from g to g'' as long as possible, and then, by the completeness of R, we find an element $\hat{g} \in J \cap (u)^{q+1}R$ such that (ii**) holds for $\hat{g}$ instead of g''. Then (ii**) implies (ii*) by induction hypothesis. Q. E. D.

Theorem (2. 24) gives us various interpretations of the number depth(R/J). For instance, we get

Corollary (2. 24. 2) $\tau(J) \geq \dim R - \mathrm{depth}(R/J)$.

Proof. Recall the definition of $\tau(J)$. ([3], Ch. III, §4, Def. 6.) It is the rank of the smallest k-submodule $T = T(J)$ of $gr_M^1(R)$ such that $(gr_M(J;\ R) \cap k[T])gr_M(R) = gr_M(J;\ R)$. (2. 24) follows (2. 24. 2) by (2. 20. 3). Q. E. D.

§3. Vertex-preparations

Definition (3. 1) Let $f = (f_1, \cdots, f_m)$ be a finite system of elements f_i of M such that the images f_i' of f_i in R' are not zero. Let $\nu_i = \nu_{M'}(f_i')$, $1 \leq i \leq m$. Then the symbol $\Delta(f;\ u;\ y)$ denotes the intersection of all those F-subsets D of $\mathbf{R}_0^r$ such that $f_i \in I(D;\ \nu_i)_{u,y}$ for all i.

If f is a (u)-effective (or (u)-standard) base of J, then we have $\Delta(f;\ u;\ y) \supset \Delta(J;\ u;\ y)$ by (2. 21. e), and hence

(3. 2) $\Delta(f;\ u;\ y) \supset \Delta(J;\ u)$.

Clearly, $\Delta(f;\ u;\ y)$ is more explicit and easier to deal with than $\Delta(J;\ u)$, while the latter is more intrinsic than the former as data to describe the singularity of Spec(R/J). Our aim in this section and in the following is to find a process of successive modifications, applied to f and y, such that in the end the inclusion of (3. 2) is replaced by an equality.

Lemma (3. 3) Let y by a system of (1. 2), and $t=(u, y)$. Then, for each $g\in M$, there exists the smallest F-subset F of $\boldsymbol{R}_0^n$ such that $g\in I(F)_t$.

Proof. An intersection of F-subsets are an F-subset. Let F_α be F-subsets of $\boldsymbol{R}_0^n$ such that $g\in I(F_\alpha)_t$ for all α. Then $g\in\bigcap_\alpha I(F_\alpha)_t$ $=I(\bigcap_\alpha F_\alpha)_t$ by (2. 1). Q. E. D.

Let $F(g; t)$ denote the smallest F-subset of (3. 3). Let g' be the image of g in R'. Then for every $b\in\boldsymbol{R}_+$ such that $b\leq\nu_{M'}(g')$, there exists the smallest F-subset Δ of $\boldsymbol{R}_0^p$ such that $\{\Delta; b\}\supset F(g; t)$. This existence is clear from the equality: $\bigcap_\alpha\{\Delta_\alpha; b\}=\{\bigcap_\alpha\Delta_\alpha; b\}$ for an arbitrary family of F-subsets Δ_α of $\boldsymbol{R}_0^p$.

Definition (3. 4) For a pair (g, b) with $g\in M$ and $b\in\boldsymbol{R}_+$, such that $b\leq\nu_{M'}(g')$ with the image g' of g in R', the symbol $\Delta(g; b; u; y)$ will denote the smallest F-subset Δ of $\boldsymbol{R}_0^p$, such that $\{\Delta; b\}\supset F(g; t)$. If $b=\nu_{M'}(g')\in\boldsymbol{R}_+$, then we write $\Delta(g; u; y)$ for $\Delta(g; b; u; y)$.

Remark (3. 4. 1) If $f=(f_1, \cdots, f_m)$ is a system of (3. 1), then $\Delta(f; u; y)=\{\bigcup_{i=1}^m\Delta(f_i; u; y)\}$, where $\{*\}$ denotes the F-subset spanned by * in the sense of the paragraph following (2. 1).

Remark (3. 4. 2) If D is an F-subset of $\boldsymbol{R}_0^n$ (or $\boldsymbol{R}_0^p$) then a point $v\in D$ is called a *vertex* of D if there exists a positive linear homogeneous function L such that $D\cap\{w\,|\,L(w)=1\}=v$. We note that $\Delta(g; b; u; y)$ of (3. 4) can be derived from $F(g; t)$ by the following procedure: Let V be the set of vertices of $F(g; t)$. Let W be the set of those $a\in\boldsymbol{R}_0^p$ for which there exists $c\in\boldsymbol{R}_0^r$ and $j\in\boldsymbol{R}_+$ such that $|c|=b-j$ and $(ja, c)\in V$. Then $\Delta(g; b; u; y)$ is the F-subset of $\boldsymbol{R}_0^p$ spanned by W.

Lemma (3. 5) For $g\in M$, every vertex of $F(g; t)$ is integral, i. e., it belongs to $\boldsymbol{Z}_0^n$.

Proof. Let v be any vertex of $F(g; t)$. We then have a positive linear homogeneous function L on $\boldsymbol{R}^n$ such that $F(g; t)\cap\{w| L(w)=1\}=v$. If v is not integral then there exists a neighborhood of v which contains no integral points. Therefore, as is easily seen, we then find $\alpha\in\boldsymbol{R}_+$ so small that $F(g; t)\cap\boldsymbol{Z}_0^n$ is contained in $\{w| L(w)\geq 1+\alpha\}$. Namely, if F' denotes the F-subset $F(g; t)\cap\{w|L(w)\geq 1+\alpha\}$, then $g\in I(F(g; t))_t=I(F')_t$, which is impossible by the minimality of $F(g; t)$. Q. E. D.

If $b\in\boldsymbol{R}_+$, then $b\boldsymbol{Z}_0^p$ denotes $\{v\in\boldsymbol{R}_0^p|(1/b)v\in\boldsymbol{Z}_0^p\}$.

Lemma (3. 6) Let the assumptions be the same as in (3. 1). Let e be any positive integer which is divisible by every positive integer not exceeding $\max_i\{\nu_i\}$. Then $\Delta(f; u; y)$ coincides with the convex set spanned by the intersection $\Delta(f; u; y)\cap(1/e)\boldsymbol{Z}_0^p$. In particular, every vertex of $\Delta(f; u; y)$ belongs to $(1/e)\boldsymbol{Z}_0^p$.

Proof. Let us recall the notation of (3. 4. 2), as we choose (f_i, ν_i) for (g, b) there, $1\leq i\leq m$. By (3. 5), the set V consists of integral points. Since $b=\nu_i$ is an integer, every $a\in W$ must be such that $ja\in\boldsymbol{Z}_0^p$ for some positive integer $j\leq\nu_i$. Let W^* denote the union of those W for various i. Then $W^*\subset(1/e)\boldsymbol{Z}_0^p$, and by (3. 4. 1), $\Delta(f; u; y)$ is the F-subset of $\boldsymbol{R}_0^p$ spanned by W^*. Hence $\Delta(f ; u; y)$ is the convex set spanned by $\bigcup_{w\in W^*}w+\boldsymbol{Z}_0^p$. Q. E. D.

If $b\in\boldsymbol{R}_+$, then an *E-subset of* $b\boldsymbol{Z}_0^p$ will mean a subset E of $b\boldsymbol{Z}_0^p$ such that $(1/b)E$ is an E-subset of $\boldsymbol{Z}_0^p$ in the sense of the early paragraph of §2.

Corollary (3. 6. 1) $\Delta(f; u; y)$ has only a finite number of vertices.

Proof. Since $\Delta(f; u; y)$ is an F-subset, $\Delta(f; u; y)\cap(1/e)\boldsymbol{Z}_0^p$ is an E-subset of $(1/e)\boldsymbol{Z}_0^p$. Call it E. Let $(A_1, \cdots, A_s)$ be the base of eE in the sense of (2. 0). Then by (3. 6), every vertex of $\Delta(f; u; y)$ is of the form $(1/e)A_j$, $1\leq j\leq s$. Q. E. D.

Let h be an element of $gr_M(R)=k[\overline{t}]$ with a certain $t=(u, y)$, and write it as: $h=\sum_{A\in E}c(A)\overline{t}^A$ with $c(A)\in k$ and a subset E of $\boldsymbol{Z}_0^n$. Let $v\in\boldsymbol{R}_0^p$ and $b\in\boldsymbol{R}_0$. Then *the* $(v; b)$*-part of* h will mean the partial sum $h^*=\sum_{A\in E^*}c(A)\overline{t}^A$ where $E^*=E\cap\partial\{v+\boldsymbol{R}_0^b;\ b\}$. If there exists a nonzero homogeneous element $\varphi\in k[\overline{y}]$ such that $h-\varphi\in(\overline{u})gr_M(R)$, then the $(v;\ \deg\varphi)$-part of h will be called simply *the v-part of h.*

Definition (3.7) Let the assumptions be the same as in (3.1). Let $\Delta=\Delta(f;\ u;\ y)$ and v a vertex of Δ. Then *the v-initial of f with respect to* $(u;\ y)$, denoted by $in_v(f)_{u,y}$, will mean the system $(h_1, \cdots, h_m)$ with $h_i=$ the v-part of $in(f_i;\ \Delta;\ \nu_i)_{u,y}$.

Remark (3.7.1) Let us pick $L\in\boldsymbol{L}_+$ such that $\Delta\cap\{a\,|\,L(a)=1\}=v$. Then the v-initial $in_v(f)_{u,y}$ coincides with the system of the initial polynomials $in(f_i;\ L;\ \nu_i)_{u,y}$, $1\leq i\leq m$.

Let U be a graded k-algebra with a field k, and U_+ a maximal ideal of U such that $k\approx U/U_+$ in a natural way. Let $w=(w_1, \cdots, w_r)$ be a system of indeterminates over U. (For instance, let $U=k[\overline{u}]$, $U_+=(\overline{u})U$ and $w=\overline{y}$.) Then, for every element $h\in U[w]$, there exists a unique $\overline{h}\in k[w]$ such that $h-\overline{h}\in U_+[w]$. We shall call $\overline{h}$ *the* $k[w]$*-part of h.* If G is an ideal in $U[w]$, we obtain an ideal $\overline{G}$ in $k[w]$, called *the* $k[w]$*-part of* G, which consists of the $k[w]$-parts of elements of G.

Definition (3.8) We say that an ideal G (resp. a system of elements $g=(g_1, \cdots, g_m)$) is *solvable* in U if there exists a U-automorphism q of $U[w]$ such that $q(w_i)-w_i\in U$ for all i and that $q(G)$ is generated by the $k[w]$-part $\overline{G}$ of G (resp. $q(g_i)$ is equal to the $k[w]$-part of g_i for all i). The system $s=(s_1, \cdots, s_r)$ with $s_i=q(w_i)-w_i$ will then be called a *solution for* G (resp. g) *in* U.

Definition (3.9) Let the assumptions be the same as in (3.7). We say that f is *v-solvable with respect to* $(u;\ y)$, if $in_v(f)_{u,y}$ is solvable in $k[\overline{u}]$ with a solution $s=(s_1, \cdots, s_r)$ of the form $s_i=c_i\overline{u}^v$ with $c_i\in k$ for all i. Here $\overline{u}^v=0$ if $v\notin\boldsymbol{Z}_0^b$. Such a solution s will be called a *v-solution for f with respect to* $(u;\ y)$.

Remark (3.9.1) We shall later find out that under a reasonable assumption (which is always satisfied in the case of our major interest) if $in_v(f)_{u,y}$ is solvable in $k[\bar{u}]$ in the sense of (3.8), then the solution s is unique and necessarily of the form described in (3.9), i.e., it is a v-solution for f with respect to $(u; y)$.

Remark (3.9.2) Following (3.7), let $in_v(f)_{u,y}=(h_1, \cdots, h_m)$. Then we have $h_i \notin k[\bar{y}]$ (i.e., h_i is different from its $k[\bar{y}]$-part) for at least one i. In fact, L being the same as in (3.7.1), if $\beta \in \boldsymbol{R}_+$ is small enough, then $h_i \in k[\bar{y}]$ implies $f_i \in I(D; \nu_i)_{u,y}$ for $D=\{a| L(a) \geq 1+\beta\}$. But $v \in \Delta = \Delta(f; u; y)$ and D cannot contain Δ. We thus get the assertion. It follows that a v-solution cannot be zero and it exists only if $v \in \boldsymbol{Z}_0^p$.

Lemma (3.10) Let the assumptions be the same as in (3.1). Let v be a vertex of $\Delta(f; u; y)$. Suppose we have a v-solution $s=(s_1, \cdots, s_r)$ for f with respect to $(u; y)$. Let $d=(d_1, \cdots, d_r)$ be any system of elements $d_i \in u^v R$ such that s_i is the image of d_i in $gr_M^{|v|}(R)$ for each i. Let $z=y-d$. Then z has the property (1.2) and

(3.10.1) $\Delta(f; u; z) \subset \Delta(f; u; y)$,

(3.10.2) $\Delta(f; u; z)$ does not contain v but does every other vertex v' of $\Delta(f; u; y)$, and

(3.10.3) for every v' of (3.10.2), we have $in_{v'}(f)_{u,z}=q_{y,z}(in_{v'}(f)_{u,y})$ where $q_{y,z}$ is the $k[\bar{u}]$-automorphism of $gr_M(R)$ such that $q_{y,z}(\bar{y}_i)=\bar{z}_i$ for all i.

Proof. Let $D=\Delta(f; u; y)$. By the assumptions of (3.1), $\Delta(f; u; y) \neq \boldsymbol{R}_0^p$ and hence $v \neq 0$. Hence $z_i \equiv y_i \mod (u)R$ for all i, so that clearly z satisfies (1.2) as y does. Since $v \in D$, if $(a, c) \in \{D; b\}$ (resp. $\{D; b\} - \partial\{D; b\}$) with $a \in \boldsymbol{Z}_0^p$ and $c \in \boldsymbol{Z}_0^r$, then $(a+(|c|-|c'|)v, c') \in \{D; b\}$ (resp. $\{D; b\} - \partial\{D; b\}$) with every $c' \in \boldsymbol{Z}_0^r$ such that $c \in c' + \boldsymbol{Z}_0^r$. It follows that $I(D; b)_{u,y}=I(D; b)_{u,z}$ and $I^+(D; b)_{u,y}=I^+(D; b)_{u,z}$. Hence $f_i \in I(D; \nu_i)_{u,z}$ for all i, and (3.10.1) follows. Now, write $in_v(f)_{u,y}=(h_1, \cdots, h_m)$ and fix any $L \in \boldsymbol{L}_+$ such that $D \cap \{a | L(a)=1\}=v$. We then have $h_i=in(f_i; L; \nu_i)_{u,y}$ for all i. Let φ_i be the $k[\bar{y}]$-part of h_i. Then, viewing each element of $gr_M(R)$ as a polynomial in $\bar{y}$ with coefficients in $k[\bar{u}]$, we have $h_i(\bar{y}+s)=\varphi_i(\bar{y})$ for all i. From the definition (2.9) of initial polynomial (and also (2.8)), it is easily seen that $h_i(\bar{z}+s)=in(f_i; L; \nu_i)_{u,z}$ where $\bar{z}=(\bar{z}_1, \cdots, \bar{z}_r)$ with the image $\bar{z}_j$ of z_j in $gr_M^1(R)$. (In fact, we can show $I(L; b)_{u,y}=I(L; b)_{u,z}$ and $I^+(D; b)_{u,y}=I^+(L; b)_{u,z}$ in the same way as above. It then is enough to check that $in(u^a(z+d)^c; L; \nu_i)_{u,z}=\bar{u}^a(\bar{z}+s)^c$ for all $(a, c)\in\partial\{D; \nu_i\}\cap Z_0^n$). This shows $v\notin\Delta(f; u; z)$ by (3.9.2). As the rest in (3.10.2) follows from (3.10.3), we have only to prove the last. Choose $L\in L_+$ such the $D\cap\{a|L(a)=1\}=v'$. Then $L(v)>1$. It follows that $u^a(z+d)^c\equiv u^a z^c$ mod $I^+(L; \nu_i)_{u,y}$ for all $(a, c)\in Z_0^n$ such that $L(a)+|c|=\nu_i$. Also $I^+(L; \nu_i)_{u,y}=I^+(L; \nu_i)_{u,z}$ as before. (3.10.3) is now clear from the definition (2.9).

Let U, U^+ and w be the same as in the paragraph of (3.8). We have the notion of *E-function* A^r: $k[w]\to Z_0^r$ and *E-set* $E^r(H)$ of a homogeneous ideal H in $k[w]$ with respect to $(k; w)$. (cf. [3], Ch. III, §7, p. 245.) Let $h=(h_1, \cdots, h_m)$ be a system of elements of $U[w]$ such that the $k[w]$-parts $\bar{h}_i$ of h_i are homogeneous for all $i<m$. We say that h is *normalized with respect to* $(U; w)$ if, for every i, the U-coefficient of the monomial w^A in h_i is zero for all $A\in E^r((\bar{h}_1, \cdots, \bar{h}_{i-1})k[w])$.

Definition (3.11) Let the assumptions be the same as in (3.7). We say that f is *v-normalized with respect to* $(u; y)$ if $in_v(f)_{u,y}$ is normalized with respect to $(k[\bar{u}]; \bar{y})$.

We shall avail ourselves of the following notation, supplementary to the definition (3.7): Under the assumptions of (3.1),

(3.12.1) $in(f)_{u,y}=(H_1, \cdots, H_m)$, where $H_i=in(f_i; \Delta; \nu_i)_{u,y}$

with $\Delta=\Delta(f; u; y)$. We call it the *total initial* of f with respect to $(u; y)$.

(3.12.2) $in_0(f)_{u,y}=(\varphi_1, \cdots, \varphi_m)$, where φ_i is the $k[\bar{y}]$-part of H_i as above. We call it the *basic initial* of f with respect to $(u; y)$. If $L\in L_+$ is any one which takes values >1 on the above Δ, then the φ_i can be obtained as $in(f_i; L; \nu_i)_{u,y}$ for $1\leq i\leq m$.

Definition (3.13) We say that f is *Δ-normalized* (resp. *0-normalized*) with respect to $(u; y)$, if $in(f)_{u,y}$ (resp. $in_0(f)_{u,y}$) is normalized with respect to $(k[\bar{u}]; \bar{y})$.

Lemma (3.14) Let the assumptions and the notation be the same as in (3.1) and (3.12.2). Suppose $\varphi=(\varphi_1, \cdots, \varphi_m)$ is a minimal base of the ideal which it generates. Then there exist $d_{ij}\in (y)^{\nu_i-\nu_j}R$ such that if $g_i=f_i-\sum_{j=1}^{i-1}d_{ij}f_j$, then $g=(g_1, \cdots, g_m)$ is 0-normalized with respect to $(u; y)$. Moreover, if g_i' is the image of g_i in R', then $\nu_{M'}(g_i')=\nu_i$ for all i and $\Delta(g; u; y)$ is contained in $\Delta(f; u; y)$. Furthermore, if f is a (u)-effective (respectively (u)-standard) base of J, so is g.

Proof. By Lemma (1.11) of [5], there exist forms c_{ij} of degrees $\nu_i-\nu_j$ in $k[\bar{y}]$ such that if $\psi_i=\varphi_i-\sum_{j=1}^{i-1}c_{ij}\varphi_j$, then $(\psi_1, \cdots, \psi_m)$ is normalized with respect to $(k; y)$ (and hence, with respect to $(k[\bar{u}]; \bar{y})$). Pick any $d_{ij}\in(y)^{\nu_i-\nu_j}R$ such that c_{ij} is the image of d_{ij} in $gr_M^*(R)$ with $*=\nu_i-\nu_j$. It is easy to check that $g_i\in I(\Delta; \nu_i)_{u,y}$ with $\Delta=\Delta(f; u; y)$ and $in(g_i; \Delta; \nu_i)_{u,y}=in(f_i; \Delta; \nu_i)_{u,y}-\sum_{j=1}^{i-1} c_{ij}in(f_j; \Delta; \nu_j)_{u,y}$ for all ν. (cf. *Remark* (2.10); a key point here is that $\partial\{\Delta; b\}+A\subset\partial\{\Delta; b+|A|\}$ for every $A\in(0)\times \boldsymbol{Z}_0^r\subset \boldsymbol{Z}_0^n$). It follows that ψ_i is the $k[\bar{y}]$-part of $in(g_i; \Delta; \nu_i)_{u,y}$ for all i. The assumption on φ implies that $\psi_i\neq 0$ for any i. Let $\psi=(\psi_1, \cdots, \psi_m)$. Then we have $(\varphi)gr_M(R)=(\psi)gr_M(R)$ and if φ is a standard base of this ideal, so is ψ. Moreover if $L\in \boldsymbol{L}_+$ is any one which takes values >1 on Δ, then $\varphi_i=in(f_i; L; \nu_i)_{u,y}$ and $\psi_i=in(g_i; L; \nu_i)_{u,y}$ for all i. Hence if f is a (u)-effective base of J, then by (2.21.b) the ideal $\{J; L\}_{u,y}$ is generated by $in(f_i; L; \nu_i)_{u,y}$ and hence by $in(g_i; L; \nu_i)_{u,y}$. According to (2.20), this shows that g is a (u)-effective base of J, too. The assertions of (3.13) are now immediate. Q.E.D.

Lemma (3.15) Let the assumptions be the same as in (3.1). Let us assume that f is 0-normalized with respect to $(u; y)$. Let v be a vertex of $\Delta(f; u; y)$. Then there exists $e_{ij}\in(u)R\cap I(v+\boldsymbol{R}_0^r;$

$\nu_i-\nu_j)_{u,y}$ such that if $h_i=f_i-\sum_{j=1}^{i-1} e_{ij} f_j$,

(3.15.1) $\Delta(h; u; y) \subset \Delta(f; u; y)$,

(3.15.2) if $v \in \Delta(h; u; y)$, then h is v-normalized with respect to $(u; y)$,

(3.15.3) if v' is any vertex of $\Delta(f; u; y)$ other than v, then v' is also a vertex of $\Delta(h; u; y)$ and $in_{v'}(f)_{u,y}=in_{v'}(h)_{u,y}$; in particular, $in_0(f)_{u,y}=in_0(h)_{u,y}$.

Proof. If $e_{ij} \in I(v+\boldsymbol{R}_0^p; \nu_i-\nu_j)_{u,y}$, then, since $v \in \Delta$, we have $h_i \in I(\Delta; \nu_i)_{u,y}$. This implies (3.15.1). Let $L \in \boldsymbol{L}_+$ be any such that $\Delta \cap \{a | L(a)=1\}=v'$ for any v' of (3.15.3). Then $L(v)>1$, and if the above $e_{ij} \in (u)R$ then $e_{ij} \in I^+(L; \nu_i-\nu_j)_{u,y}$ so that $h_i \equiv f_i$ mod $I^+(L; \nu_i)_{u,y}$ for all i. This implies (3.15.3). Thus we have only to prove the existence of such e_{ij} for which (3.15.2) holds. Let us write $in_v(f)_{u,y}=(\overline{f}_1, \cdots, \overline{f}_m)$. As before, we put $\overline{t}=(\overline{u}, \overline{y})$, so that $gr_M(R)=k[\overline{t}]$. An element $c \in k[\overline{t}]$ will be called a *(v, b)-form* (resp. *$(v, b)^*$-form*) if $c=\sum_A c(A)\overline{t}^A$ with $c(A) \in k$ and with $\partial\{v+\boldsymbol{R}_0^p; b\} \cap \boldsymbol{Z}_0^n$ (resp. $\partial\{v+\boldsymbol{R}_0^p; b\} \cap (\boldsymbol{Z}_0^n-(0)\times \boldsymbol{Z}_0^r)$) as the range of A. Note that $\overline{f}_i$ is a (v, ν_i)-form for each i. We shall prove that there exist $(v, \nu_i-\nu_j)^*$-forms $a_{ij} \in k[\overline{t}]$ such that if $H_i=\overline{f}_i-\sum_{j=1}^{i-1} a_{ij} \overline{f}_j$ then $(H_1, \cdots, H_m)$ is normalized with respect to $(k[\overline{u}]; \overline{y})$. If this is done, it follows without any difficulty that there exist $e_{ij} \in I(v+\boldsymbol{R}_0^p; \nu_i-\nu_j)_{u,y} \cap (u)R$ with $a_{ij}=in(e_{ij}; v+\boldsymbol{R}_0^p; \nu_i-\nu_j)_{u,y}$, and that if e_{ij} are such, then (3.15.2) holds. In fact, thanks to (2.10), if $in_v(h)_{u,y}=(H_1', \cdots, H_m')$, then $H_i' \equiv H_i$ mod $I^+(\Delta(f; u; y); \nu_i)_{\overline{t}}$. Since both H_i' and H_i are (v, ν_i)-forms with a vertex v of $\Delta(f; u; y)$, this congruence implies the equality $H_i'=H_i$. (3.15.2) is immediate from this. Now, suppose we have found a_{ij} for all (i, j) with $1 \leq j<i<\alpha$ for some α. Let $\varphi_i=$ the $k[\overline{y}]$-part of $\overline{f}_i$. Note that $(\varphi_1, \cdots, \varphi_m)$ is normalized by assumption. Let $G=\overline{f}_\alpha-\varphi_\alpha$. We want to find $(v, \nu_\alpha-\nu_j)^*$-forms b_j such that if $G^*=G-\sum_{j=1}^{\alpha-1} b_j H_j$, then $(H_1, \cdots, H_{\alpha-1}, G^*)$ is normalized with respect to $(k[\overline{u}]; \overline{y})$. The proof of this fact is similar to that of Lemma 17, Ch. III, [3], p. 246. In this case, we consider the following ordering in Z_0^r: $a>a'$ if either $|a|>|a'|$ or $|a|=|a'|$ and a' precedes a in the lexicographical ordering. The proof is then by induction on $a=a(G)$ which is the largest element in $E^r((\varphi_1, \cdots, \varphi_{\alpha-1})k[\bar{y}])$ such that the $k[\bar{u}]$-coefficient of $\bar{y}^a$ in G is not zero. If a is such, then there exists a form $C\in(\varphi_1, \cdots, \varphi_{\alpha-1})k[\bar{y}]$ with $A^r(C)=a$. Write $C=\sum_{j=1}^{\alpha-1} C_j \varphi_j$ with forms C_j of degrees $|a|-\nu_j$ in $k[\bar{y}]$. As G is a $(v, \nu_\alpha)^*$-form, the coefficient of $\bar{y}^a$ in G is of the form $d\bar{u}^{qv}$ with $d\in k$ and $q\in \boldsymbol{R}_+$ such that $|a|+q=\nu_\alpha$. Let d' be the coefficient of $\bar{y}^a$ in C. Then let $b'_j=(d/d')C_j\bar{u}^{qv}$, which are obviously $(v, \nu_\alpha-\nu_j)^*$-forms. Let $G'=G-\sum_{j=1}^{\alpha-1} b'_j H_j$. We know that H_j is *a* (v, ν_j)-form and φ_j is the $k[\bar{y}]$-part of H_j. With these facts, we can easily conclude that $a(G)>a(G')$ unless $(H_1, \cdots, H_{\alpha-1}, G')$ is normalized with respect to $(k[\bar{u}], \bar{y})$. Q. E. D.

Definition (3. 16) Let the assumptions be the same as in (3. 7). We say that $(f; y)$ is *v-prepared with respect to u*, if f is v-normalized and not v-solvable with respect to $(u; y)$. We say that $(f; y)$ is *totally prepared* (or *Δ-prepared*) *with respect to u*, if f is Δ-normalized and not w-solvable with respect to $(u; y)$ for any vertex w of $\Delta(f; u; y)$.

Remark (3. 16. 1) Note that the v-preparedness (resp. Δ-preparedness) of $(f; y)$ implies the 0-normalizedness of f. If $\Delta(f; u; y)$ is empty, then the Δ-preparedness of $(f; y)$ with respect to u is equivalent to the 0-normalizedness of f with respect to $(u; y)$. This is exactly the case in which $f_i\in(y)^{\nu_i}$ for all i.

Theorem (3. 17) Let the assumptions be the same as in (3. 1). Let $\Delta=\Delta(f; u; y)$. Let us assume that R is complete and that

(3. 17. 1) $in_0(f)_{u,y}$ is a minimal base of the ideal which it generates in $gr_M(R)$.

Then there exist $x_{ij}\in I(\Delta; \nu_i-\nu_j)_{u,y}$, $I\leq j<i\leq m$, and $d_\alpha\in I(\Delta; 1)_{u,y}\cap(u)R$, $1\leq\alpha\leq r$, such that

(3. 17. 2) $(g; z)$ is totally prepared with respect to u, where $g=(g_1, \cdots, g_m)$ with $g_i=f_i-\sum_{j=1}^{i-1} x_{ij}f_j$ and $z=(z_1, \cdots, z_r)$ with $z_\alpha=$

$y_\alpha - d_\alpha$.
Moreover, if f is 0-normalized with respect to $(u; y)$, then we can choose those x_{ij} in the ideal $I(\Delta; \nu_i - \nu_j)_{u,y} \cap (u)R$. Furthermore, if $\{v_\beta | \beta \in P\}$ is a nonempty set of vertices of $\Delta(f; u; y)$ and if $(f; y)$ is v_β-prepared with respect to u for all $\beta \in P$, then we can choose the above x_{ij} in $H(\nu_i - \nu_j)$ and d_α in $H(1)$ for all i, j, and α, where $H(b) = \bigcap_{\beta \in P} I^+(L_\beta; b)_{u,y} \cap I(\Delta; b)_{u,y} \cap (u)R$ with any $L_\beta \in \boldsymbol{L}_+$ such that $\Delta \cap \{a | L_\beta(a) = 1\} = v_\beta$.

Proof. By (3.14), if we put $g_i = f_i - \sum_{j=1}^{i-1} d_{ij} f_j$ with suitable $d_{ij} \in (y)^{\nu_i - \nu_j} R$, then g is 0-normalized with respect to $(u; y)$. Here clearly $(y)^{\nu_i - \nu_j} R \subset I(\Delta; \nu_i - \nu_j)_{u,y}$ and hence $\Delta(g; u; y) \subset \Delta(f; u; y)$. This shows that we have only to consider the case in which f is 0-normalized with respect to $(u; y)$ from the very beginning. From now on, we apply to f, repeatedly and alternately, the *vertex-normalization* in the sense of Lemma (3.15) and the *vertex-dissolution* in the sense of Lemma (3.10). To be precise, let us consider the following ordering in $\boldsymbol{R}_0^t$: $v > v'$ if either $|v| > |v'|$ or $|v| = |v'|$ and v' precedes v in the lexicographical ordering. Given (f, y) as above, let v be the smallest vertex v of $(f; u; y)$ such that either f is not v-normalized or f is v-solvable with respect to $(u; y)$. If f is not v-normalized with respect to $(u; y)$, then by (3.15) we have suitable $e_{ij} \in (u)R \cap I(v + \boldsymbol{R}_0^t; \nu_i - \nu_j)_{u,y}$ such that (3.15.1)-(3.15.3) hold. Then we replace each f_i by $f_i - \sum_{j=1}^{i-1} e_{ij} f_j$. Note that such e_{ij} belong to the ideals to which x_{ij} should, and that (3.15.1)-(3.15.3) assure us that the various starting assumptions are unaffected by the replacement. If f is v-normalized with respect to $(u; y)$, then by assumption, we have a v-solution for f with respect to $(u; y)$. By (3.10) we then have $d_i \in u^v R$ such that (3.10.1)-(3.10.3) hold. Then we replace each y_i by $y_i - d_i$. Again (3.10.1) assure us that the various starting assumptions remain unaffected by this replacement. We repeat this process as long as it is possible (i.e., as long as the goal of the theorem is not achieved). If an infinite repetition is possible, i.e., if we find an infinite sequence of vertices to which the above vertex-normalizations and vertex-dissolutions are applied successively, then this sequence must tend to ∞ in the space $\boldsymbol{R}^p$. This is due to the fact that vertices must belong to a certain lattice $(1/e)\boldsymbol{Z}_0^p$ by (3.6), and that each vertex can appear at most once in vertex-normalization and once in vertex-dissolution, thanks to (3.15.3), (3.10.2) and (3.10.3). The completeness of R now suffices for the proof. Q. E. D.

Corollary (3.17.3) Let us consider the case in which $(f; y)$ is v_β-prepared with respect to u for all $\beta \in P$. Then $(g; z)$ of (3.17) can be so chosen that v_β is also a vertex of $\Delta(g; u; y)$ for all $\beta \in P$ and $in_{v_\beta}(f)_{u,y} = q_{z,y}(in_{v_\beta}(g)_{u,z})$ for all $\beta \in P$, where $q_{z,y}$ is the $k[\bar{u}]$-automorphism of $gr_M(R)$ with $q_{z,y}(\bar{z}_i) = \bar{y}_i$ for all i.

Proof. The claimed equality is due to the condition that $x_{ij} \in H(\nu_i - \nu_j)$ and $d_\alpha \in H(1)$ for all i, j and α. Q. E. D.

Corollary (3.17.4) The assumptions being the same as in (3.17), if f is a (u)-standard base of J, so is g of (3.17.2).

Proof. To prove that g is a (u)-standard base of J, let $L \in \boldsymbol{L}_+$ by any such that L takes only values >1 on $\Delta(f; u; y)$. Let $\bar{x}_{ij} = in(x_{ij}; L; \nu_i - \nu_j)_{u,y}$, $\bar{f}_i = in(f_i; L; \nu_i)_{u,y}$ and $\bar{g}_i = in(g_i; L; \nu_i)_{u,y}$. Then these are forms of degrees $\nu_i - \nu_j$, ν_i and ν_i respectively, and $\bar{g}_i = \bar{f}_i - \sum_{j=1}^{i-1} \bar{x}_{ij} \bar{f}_j$. It is clear that $(\bar{g}_1, \cdots, \bar{g}_m)$ is a standard base of $\{J; L\}_{u,y}$. Q. E. D.

§4. Totally prepared (u)-standard base.

If a (u)-standard base of J is such that for some y of (1.2), $(f; y)$ is totally prepared with respect to u, then many of its properties turn out to be intrinsic of the ideal J itself. In this section, we shall prove that the inclusion of (3.2) becomes an equality in this case, i.e., $\Delta(f; u; y) = \Delta(J; u)$.

The following lemma will be found useful in regard to the uniqueness of a vertex-dissolution.

Lemma (4.1) Let $k \to K$ be a separable field extension, and $K[w]$ the polynomial ring of r indeterminates w_i over K. For each $s=(s_1, \cdots, s_r) \in K^r$, let us denote by $V(s)$ the graded k-subalgebra of $k[w]$ which is generated by those forms h such that $h(w+s)=h(w)$. Let $V(s)_1$ denote the homogeneous part of degree one of $V(s)$. Then we have $V(s)=k[V(s)_1]$.

Proof. Applying a suitable k-linear transformation to w (and accordingly to s), we may assume that $(w_{c+1}, \cdots, w_r)$ is a free base of $V(s)_1$ for a certain $c \leq r$. Put $w'=(w_1, \cdots, w_c)$ and $w''=(w_{c+1}, \cdots, w_r)$. If g is a form in $k[w]$, we can write $g=\sum g_E w''^E$ with $g_E \in k[w']$, where the range of E is a certain finite subset of Z_0^{r-c}. Then $g \in V(s)$ if and only if $g_E \in V(s)$ for all E. Therefore, replacing w by w' and accordingly the others, we can reduce the proof to the special case in which $V(s)_1=(0)$. In other words, it suffices to prove that, for any given s, if $V(s)$ contains a form g of positive degree then $V(s)_1 \neq (0)$. This statement with variable s will be proven by induction on deg g. Say deg $g=d>1$. Clearly $g(w+s)=g(w)$ implies $g_i(w+s)=g_i(w)$ for all i, where $g_i=\partial g/\partial w_i$. Hence by induction, either $V(s)_1 \neq (0)$ or $g_i=0$ for all i. If this last is the case, then the characteristic p of k must be positive and there exists a form h of degree d/p in $k[w]$ such that $g(w)=h(w^q)$ with $w^p=(w_1^p, \cdots, w_r^p)$. It follows that $h \in V(s^p)$ with $s^p=(s_1^p, \cdots, s_r^p)$. By induction, $V(s^p)_1 \neq (0)$. This implies that the r elements s_i^p are linearly dependent over k. Hence the s_i are linearly dependent over a certain purely inseparable extension of k. Since K is separable over k, the same must be true without any extension. This means that $V(s)_1 \neq (0)$.

Corollary (4.1.1) Let $h_1, \cdots, h_m$ be forms in $k[w]$. Suppose there is no k-submodule T of the linear homogeneous part $k[w]_1$, $\neq k[w]_1$, such that $h_i \in k[T]$ for all i. Let s and s' be two elements of K^r. If $h_i(w+s)=h_i(w+s')$ for all i, then $s=s'$.

Proof. We may assume $s'=(0)$, because $h_i(w+s)=h_i(w+s')$

implies $h_i(w+s-s')=h_i(w)$. Since $h_i\in k[V(s)_1]$ for all i by (4.1), we must have $V(s)_1=k[w]_1$ by assumption. Hence $s=(0)$. Q.E.D.

Let U, U_+ and w be the same as in the paragraph of the definition (3.8) of solvability in general. We shall follow the terminology introduced there. The polynomial ring $k[w]$ being graded in the usual manner, $k[w]_\alpha$ will denote the homogeneous part of degree α.

Proposition (4.2) Let G be an ideal in $U[w]$ and $\overline{G}$ its $k[w]$-part. Let us assume that

(4.2.1) U is an integral domain which is separable over k,

(4.2.2) $\overline{G}$ is homogeneous, and

(4.2.3) there is no k-submodule T of $k[w]_1$, $\neq k[w]_1$, such that $(\overline{G}\cap k[T])k[w]=\overline{G}$.

Then the solution for G in U is unique, if it exists at all.

Proof. Let s be any solution for G in U and let q be the U-automorphism of $U[w]$ associated with it. (See (3.8).) Then we have $q(G)=\overline{G}U[w]$. Therefore, to prove the uniqueness of s, we may assume $G=\overline{G}U[w]$ from the very beginning. The claim is then that the only solution is zero. Let us choose a standard base $(h_1, \cdots, h_m)$ of $\overline{G}$ with is normalized which respect to $(k; w)$. (See Lemma (1.11) of [5] for the existence of such a base.) We shall prove that $h_i(w+s)=h_i(w)$ for all i. By (4.1.1), it will then follow that $s=(0)$. The proof of $h_i(w+s)=h_i(w)$ is done by induction on $i\geq 0$. Now, suppose we have $h_j(w+s)=h_j(w)$ for all $j<i$ for a certain $i\geq 1$. Since $s\in U^r$, $h_i(w+s)-h_i(w)$ has degree $<\deg h_i$ in terms of w. Hence, this difference being in the ideal $\overline{G}U[w]$, it must be of the form $\sum_{j=1}^{i-1}c_jh_j$ with $c_j\in U[w]$. We know that the E-set $E^r((h_1, \cdots, h_{i-1})k[w])$ with respect to $(k; w)$ is egual to $E^r((h_1, \cdots, h_{i-1})U[w])$ with respect to $(U; w)$. (cf. Remark 1, §7, Ch. III, [3].) Call it E. If $h_i(w+s)\neq h_i(w)$, then E contains $A^r(\sum_{j=1}^{i-1}c_j h_j)=A^r(h_i(w+s)-h_i(w))$. Call this B. Then the rule of binomial expansion tells us that there exists at least one $B'\in B+\boldsymbol{Z}_0^r$ such that the coefficient of $w^{B'}$ in h_i is not zero. But $B\in E$ implies $B'\in E$, and this contradicts the normalizedness assumption on the h_i. Thus $h_i(w+s)=h_i(w)$. Q. E. D.

Proposition (4.3) Let G be an ideal in $U[w]$ and $g=(g_1, \cdots, g_m)$ a system of elements which generates G. Let $\bar{g}_i$ be the $k[w]$-part of g_i. Let us assume (4.2.2),

(4.3.0) $\deg \bar{g}_i > \deg(g_i - \bar{g}_i)$ (in w alone) for all i;

(4.3.1) $\bar{g}=(\bar{g}_1, \cdots, \bar{g}_m)$ is a standard base of the $k[w]$-part $\bar{G}$ of G, and

(4.3.2) g is normalized with respect to $(U; w)$.

Then g is solvable in U if and only if so is G with the same solution.

Proof. The only-if part is trivial. Let us now assume G is solvable in U. Let $s=(s_1, \cdots, s_r)$ be a solution for G in U, and q the U-automorphism of $U[w]$ associated with it. (cf. (3.8).) Let $g_i'=q(g_i)$. We shall prove $g_i'=\bar{g}_i$ for all i. Suppose the contrary, and let i be the smallest integer for which the equality fails. Then $g_i'-\bar{g}_i \in q(G)=\bar{G}U[w]$, and its degree in w is less than deg $\bar{g}_i$. Hence $g_i'-\bar{g}_i, \in (\bar{g}_1, \cdots, \bar{g}_{i-1})U[w]$. This implies that $(\bar{g}_1, \cdots, \bar{g}_{i-1}, g_i'-\bar{g}_i)$ is not normalized with respect to $(U; w)$. But this is impossible. In fact, first of all, (4.3.2) implies that $\bar{g}$ (and hence $(\bar{g}_1, \cdots, \bar{g}_i)$) is normalized with respect to $(U; w)$. Secondly, by the binomial expansion theorem, (4.3.2) also implies that $(\bar{g}_1, \cdots, \bar{g}_{i-1}, g_i')$ is normalized with respect to $(U; w)$. Q. E. D.

Let us consider the situation in which U is a multi-graded k-algebra with the homogeneous parts U_a of multi-degrees $a \in \boldsymbol{Z}_0^p$. As before, we set $U_v=(0)$ for every $v \in \boldsymbol{R}^p - \boldsymbol{Z}_0^p$.

Proposition (4.4) Let $g=(g_1, \cdots, g_m)$ be a system of elements in $U[w]$, and $\bar{g}_i$ the $k[w]$-part of g_i. Let us assume (4.2.1),

(4.4.1) $\bar{g}_i$ is homogeneous of degree $n_i \in \boldsymbol{Z}_+$, and there is no k-submodule T of $k[w]_1$, $\neq k[w]_1$, such that $\bar{g}_i \in k[T]$ for all i,

(4.4.2) there exists $v \in \boldsymbol{R}_0^p$, $\neq (0)$, such that $g_i \in \sum_{j \in R_0} U_{jv} k[w]_{n_i - j}$ for all i.

Then the solution $s=(s_1, \cdots, s_r)$ for g in U is unique and $s_i \in U_v$

for all i, if it exists at all.

Proof. Using the U-automorphism of $U[w]$ associated with a solution s, we can reduce the proof to the case of $g_i=\bar{g}_i$ for all i. Then the uniqueness follows by (4.1.1). We may assume $s\neq(0)$. Let v' be the smallest element of $\boldsymbol{R}_0^p$ in the lexicographical ordering such that either $v'=v$ or s_i has a nonzero homogeneous part of degree v' for at least one i. Suppose $v'\neq v$. Then the equality $g_i(w+s)=\bar{g}_i(w)$ implies that $\bar{g}_i(w+\bar{s})=\bar{g}_i(w)$ where $\bar{s}=(\bar{s}_1, \cdots, \bar{s}_r)$ with the homogeneous parts $\bar{s}_i$ of degree v' of s_i. (Here (4.4.2) is needed.) But this implies $\bar{s}=(0)$ by (4.1.1). Thus we must have $v'=v$. In view of (4.4.2), we then get $g_i(w+\bar{s})=\bar{g}_i(w)$ for all i. By the uniqueness of solution, we must have $\bar{s}=s$. Q.E.D.

Let $U=k[\bar{u}]$, $U_+=(\bar{u})U$ and $w=\bar{y}$. Then U satisfies (4.2.1) and has the multi-grading in which the k-module generated by $\bar{u}^a$ is the homogeneous part of multi-degree $a\in\boldsymbol{Z}_0^p$. Thus we get the following special case of (4.4).

Corollary (4.4.3) Let assumptions be the same as in (3.7). Let us assume that there exists no k-submodule T of $k[\bar{y}]_1$, $\neq k[\bar{y}]_1$, such that $in_0(f)_{u,y}\in k[T]^m$. Suppose $in_v(f)_{u,y}$ is solvable in $k[\bar{u}]$ in the sense of (3.8). Then the solution is unique and of the form $s=(s_1, \cdots, s_r)$ with $s_i\in k\bar{u}^v$ for all i. In other words, f is then v-solvable with respect to $(u; y)$ in the sense of (3.9).

From now on, the basic situation will be the same as in §1. Let $\hat{R}$ be the completion of R, $\hat{M}=M\hat{R}$ and $\hat{J}=J\hat{R}$. Every element of R will be identified with its image in $\hat{R}$ by the canonical homomorphism $R\rightarrow\hat{R}$. Let y be a system of (1.2), and $f=(f_1, \cdots, f_m)$ a system of (3.1). Then the symbol $\Delta(f; u; y)$ is given two meanings, the one with reference to R and the other with reference to $\hat{R}$ instead of R. The fact is that these two coincide. In fact, using the symbols $\hat{I}(\Delta; b)_{u,y}$ in the same sense as in the proof of (2.22), we see that $f_i\in\hat{I}(D; \nu_i)_{u,y}\Leftrightarrow f_i\in I(D; \nu_i)_{u,y}$ for all i and all F-subsets D of $\boldsymbol{R}_0^p$.

Lemma (4. 5) $\Delta(J; u)$ defined by (1. 12) with reference to R is equal to the coresponding $\Delta(\hat{J}; u)$ defined with reference to $\hat{R}$.

Proof. If $\hat{R}'=\hat{R}/(u)\hat{R}$ and $\hat{J}'=J\hat{R}'$, then $\hat{R}'$ is the completion of R' and $\hat{J}'=J'\hat{R}'$. Hence we get $gr_{M'}(J', R')=gr_{\hat{M}'}(\hat{J}', \hat{R}')$ with $\hat{M}'=M'\hat{R}'$. We also have $\{J; L\}_{u,y}=\{\hat{J}; L\}_{u,y}$ for every $L\in \boldsymbol{L}_{+}$, by (2. 22). Therefore, clearly $\Delta(J; u; y)=\Delta(\hat{J}; u; y)$ by (1. 11). Hence $\Delta(J; u)\supset\Delta(\hat{J}; u)$. Pick $v\in \boldsymbol{R}_0^p-\Delta(\hat{J}; u)$. We then have a system y^* of elements in $\hat{M}$, which satisfies (1. 2) with reference to $\hat{R}$, such that $v\notin\Delta(\hat{J}; u; y^*)$. We then have $L\in \boldsymbol{L}_{+}$ such that $L(v)<1$ and

$$(4.5.1)\quad \{\hat{J}; L\}_{u,y^*}=e_{y^*}(gr_{\hat{M}'}(\hat{J}', \hat{R}'))gr_{\hat{M}}(\hat{R}).$$

Clearly $\hat{I}^{+}(L; 1)_{u,y^*}$ (according to the notation used in the proof of (2. 22)) contains some power of $\hat{M}$, say $\hat{M}^s$. Assume $s\geq 2$. Let us then choose a system y of (1. 2) such that $y_i\in R$ and $y_i^*\equiv y_i$ mod $\hat{M}^s$ for all i. Note that, since $s\geq 2$, y_i and y_i^* have the same image in $gr_{\hat{M}}^1(\hat{R})$. Then, as is easily seen, $in(g^*; L; b)_{u,y^*}=in(g^*; L; b)_{u,y}$ for all $g^*\in\hat{I}(L;b)_{u,y^*}=\hat{I}(L;b)_{u,y}$. Moreover, $e_{y^*}=e_y$. Thus it follows that (4. 5. 1) remains true if y^* is replaced by y. Since $L(v)<1$, this proves $v\notin\Delta(\hat{J}; u; y)=\Delta(J; u; y)$, and hence $v\notin\Delta(J; u)$.

Q. E. D.

Lemma (4. 6) Let $f=(f_1, \cdots, f_m)$ be a system of elements of J, y a system of (1. 2) and Δ an effective F-subset of $\boldsymbol{R}_0^p$. Then f is a (u)-effective (resp. (u)-standard) base of J with reference datum (y, Δ) if it is the same of $\hat{J}$ with the same reference datum.

Proof. Let ν_i be the same as in (2. 20). Then $f_i\in\hat{I}^{+}(\Delta; \nu_i)_{u,y}+(y)^{\nu_i}\hat{R}$ if and only if $f_i\in I^{+}(\Delta; \nu_i)_{u,y}+(y)^{\nu_i}R$. Moreover, by (2. 22), $\{\hat{J}; \Delta\}_{u,y}=\{J;\Delta\}_{u,y}$. (4. 6) then follows from these by (2. 21. c).

Q. E. D.

In many important cases, the following condition is satisfied:

(4. 7) There exists no k-submodule T of $gr_{M'}^1(R')$, $\neq gr_{M'}^1(R')$, such that $(gr_{M'}(J', R')\cap k[T])\, gr_{M'}(R')=gr_{M'}(J', R')$.

For instance, let y be a regular system of τ-parameters for J in the sense of [3], Ch. III, and let u any system for which (2.1) holds. Then J with this u fulfills (4.7).

Theorem (4.8) Let f be a (u)-standard base of J, and y a system satisfying (1.2). Assume that (J, u) satisfies (4.7). If v is any vertex of $\Delta(f; u; y)$ such that $(f; y)$ is v-prepared with respect to u, then v is also a vertex of $\Delta(J; u)$. In particular, if $(f; y)$ is totally prepared with respect to u, then $\Delta(f; u; y)=\Delta(J; u)$.

Proof. By (4.5) and (4.6), we may assume that R is complete. Then, by (3.17) and (3.17.4), we can find a (u)-standard base g of J and a system z satisfying (1.2), such that $(g; z)$ is totally prepared with respect to u and that every vertex v of $\Delta(f; u; y)$ having the property in (4.8) is also a vertex of $\Delta(g; u; z)$. Therefore, we have only to prove the last statement of (4.8). We know $\Delta(f; u; y)\supset\Delta(J; u)$ by (3.2). Suppose these two are not the same. Then there exists a system $x=(x_1, \cdots, x_r)$ satisfying (1.2) and $L\in\boldsymbol{L}_+$ such that

(4.8.1) $\{J; L\}_{u,x}=e_x(gr_{M'}(J', R'))gr_M(R)$, and

(4.8.2) $L(w)<1$ for some $w\in\Delta(f; u; y)$.

By (2.24.1), there exists a (u)-standard base $f^*=(f_1^*, \cdots, f_m^*)$ of J with reference datum $(x, \Delta(L))$. Hence, ν_i being the largest number with $f_i^*\in I(L; \nu_i)_{u,x}$, we have $f_i^*\in(x)^{\nu_i}+I^+(L; \nu_i)_{u,x}$ for all i. By virtue of (3.5), this condition remains unaffected if L is replaced by $L+L'$ for any sufficiently small linear homogeneous function L' on $\boldsymbol{R}^p$. By (2.21.c), $(x, \Delta(L+L'))$ with such L' is also a reference datum for the (u)-standard base f^* of J. Therefore, by replacing L by $L+L'$ with a suitable L', we may assume (4.8.1), (4.8.2) and

(4.8.3) there exists a vertex v of $\Delta(f; u; y)$ such that $\Delta(f; u; y)\cap\{a\in\boldsymbol{R}_0^p | L(a)=s\}=v$, where $s=L(v)$.

We may replace x by any minimal base of the ideal $(x)R$ without affecting (4.8.1). Therefore, in view of (1.2), we may assume

$y_i \equiv x_i \bmod (u)R$ for all i. Let us take the smallest $t \in \boldsymbol{R}_0$ such that $y_i \in I(tL;1)_{u,x}$ for all i. Here, if $t=0$, tL should be identified with the empty F-subset of $\boldsymbol{R}_0^b$ and we can then choose $x=y$. In general, let $t'=\max(t, s^{-1})$. Since $t' \geq t$, $y_i \in I(t'L;1)_{u,x}$ and $in(y_i; t'L; 1)_{u,x} = \bar{x}_i + d_i$, where $\bar{x}_i$ is the initial form of x_i in $gr_M(R)$ and d_i is a k-linear combination of those monomials $\bar{u}^A$ with $t'L(A)=1$. Since $t' \geq s^{-1}$, $\Delta(t'L) \supset \Delta(f;u;y)$. Let $h=(h_1, \cdots, h_m)$ with $h_i = in(f_i; t'L; \nu_i)_{u,y}$ and $h^* = (h_1^*, \cdots, h_m^*)$ with $h_i^* = in(f_i; t'L; \nu_i)_{u,x}$. Let q' (resp. q'') be the $k[\bar{u}]$-automorphism of $gr_M(R)$ such that $q'(\bar{y}_i) = \bar{x}_i$ (resp. $q''(\bar{x}_i) = \bar{x}_i + d_i$) for all i. We have $q''q'(h_i) = h_i^*$ for all i. Let $h'=(h_1', \cdots, h_m')$ with $h_i' = q'(h_i)$. Since $t' \geq s^{-1} > 1$, by (4.8.1)

(4.8.4) $\{J; t'L\}_{u,x}$ is generated by its $k[\bar{x}]$-part.

This last ideal in $k[\bar{x}]$ will be denoted by $\bar{G}$. If $t' > s^{-1}$, then $t'L(v) > 1$ for v of (4.8.3), and $h = in_0(f)_{u,y}$. Therefore $h' = in_0(f)_{u,x}$. By (2.21.a), $\{J; t'L\}_{u,x}$ is generated by h^* and $\bar{G}$ is by h'. By the preparedness of $(f;y)$, h (resp. h') is a standard base of $(h)k[\bar{y}]$ (resp. $\bar{G}$), which is normalized with respect to $(k;\bar{y})$ (resp. $(k;\bar{x})$). Hence, by (2.21.a), $\{J; t'L\}_{u,y}$ is generated by h and, by $q''q'(h) = h^*$, $d=(d_1, \cdots, d_r)$ is a solution for $\{J; t'L\}_{u,y}$ in $k[\bar{u}]$. Clearly (0) is also a solution for the same, and by (4.2), $d=(0)$. This shows that $t' > t$, which contradicts $t' > s^{-1}$. Next we assume $t' = s^{-1}$. By (4.8.3), we then have $h = in_v(f)_{u,y}$ which is, by assumption, normalized with respect to $(k[\bar{u}], \bar{y})$. Hence h' is normalized with respect to $(k[\bar{u}], \bar{x})$. But $h^* = q''(h')$ and $\{J; t'L\}_{u,x} = (h^*)gr_M(R)$. Hence, by (4.3), (4.8.4) implies that d is a solution for h' in $k[\bar{u}]$. This then is also true for h in $k[\bar{u}]$, which contradicts the v-preparedness of $(f;y)$ by (4.4.3). Q.E.D.

Corollary (4.8.5) Let us assume that (J, u) satisfies the condition of (2.24) and (4.7). Let $(\nu_1, \cdots, \nu_m)$ be the degrees of a standard base of $gr_{M'}(J', R')$, and e any positive integer which is divisible by every positive integer not exceeding $\max_i\{\nu_i\}$. Then $\Delta(J;u)$ has only a finite number of vertices, all of which belong to the lattice $(1/e)\boldsymbol{Z}_0^b$.

Proof. Immediate from (4.8), (3.6) and (3.6.1). Q. E. D.

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