Characterization of 2-dimensional normalMather-Jacobian log canonical singularities

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Introduction

In birational geometry, the notion of discrepancy plays an important role. By using this discrepancy, we can define canonical, log canonical, terminal and log terminal singularities. Those singularities are all normal Q-Gorenstein singularities and these conditions are essential in birational geometry. But these conditions seem to be an unnecessary restriction for a singularities to be considered as a good singularities.

By using the jet schemes and the Nash blow-up of a variety, Ishii [3], de Fernex and Docampo [1] independently introduced the notion of Mather-Jacobian log discrepancy, which is a modification of the classical definition of discrepancy without the restriction of normal Q-Gorenstein property.

In [2], Ein and Ishii determined Mather-Jacobian canonical singularities of dimension 2. They also determined a complete intersection Mather-Jacobian log canonical singularities of dimension 2 and showed that a non-complete intersection Mather-Jacobian log canonical singularities of dimension 2 is embedded into an Mather-Jacobian log canonical complete intersection surface.

We determine a normal Mather-Jacobian log canonical surface singularity of non-complete intersection. More precisely, we prove the following theorem:

Main Theorem

Let (X, p) be a singularity on a normal surface X. We assume that (X, p) is not locally a complete intersection. Then the following are equivalent:
i) (X, p) is Mather-Jacobian log canonical singularity,
ii) (X, p) is a toric singularity with multiplicity 3,
iii) (X, p) is a toric singularity with embedding dimension 4.

Preliminaries on Mather-Jacobian log discrepancy

Definition 1. Mather discrepancy divisor

Let $\varphi : Y \to X$ be a resolution of singularities of X that factors through the Nash blow-up of X. The image of the canonical homomorphism

$$\varphi^*(\Omega_X^d) \to \Omega_Y^d$$

is an invertible sheaf of the form $Jac_f\Omega_Y^d$, where Jac_f is the relative Jacobian which is an invertible ideal on Y and defines an effective divisor supported on the exceptional locus of φ which is called the Mather discrepancy divisor and denoted by $\widehat{K}_{Y/X}$.

Definition 2. Jacobian discrepancy divisor

Let $\varphi : Y \to X$ is a log resolution of Jac_X, where Jac_X is the Jacobian ideal of a variety X. We denote by $J_{Y/X}$ the effective divisor on Y such that $Jac_X\mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$. This divisor is called the Jacobian discrepancy divisor.

Definition 3. Mather-Jacobian log discrepancy

Let E be a prime divisor over X and $\varphi : Y \to X$ be a log resolution of Jac_X, on which E appears. We define the Mather-Jacobian log discrepancy at E as

$$a_{\mathrm{MJ}}(E; X) := \mathrm{ord}_E(\widehat{K}_{Y/X} - J_{Y/X}) + 1.$$

Definition 4. Mather-Jacobian minimal log discrepancy

Let W be a closed subset of X such that it does not contain an irreducible component of X. The Mather-Jacobian minimal log discrepancy of X along W is defined as

$$\mathrm{mld}_{\mathrm{MJ}}(W; X) = \inf\{a_{\mathrm{MJ}}(E; X) \mid E \text{ prime divisor over } X \text{ with center in } W\}$$

Definition 5. Mather-Jacobian log canonical singularity

We call that a point p of X is Mather-Jacobian log canonical (Mather-Jacobian log canonical for short) singularity of X if the inequality $\mathrm{mld}_{\mathrm{MJ}}(p; X) \geq 0$ holds, i.e., for every exceptional prime divisor E over X with center $\{p\}$, we have inequality $a_{\mathrm{MJ}}(E; X) \geq 0$.

Sketch proof of the Main Theorem

The equivalence (ii)⇔(iii) in the Main Theorem follows immediately from Artin's formula $\mathrm{emb}(X, p) = \mathrm{mult}_p X + 1$, because a toric singularity is rational and we can apply the formula. The program of the rest of the proof is as follows:

First we prove that a 2-dimensional Mather-Jacobian log canonical singularity which is not a complete intersection is a rational triple point:

Proposition 1

Let (X, p) be a normal Mather-Jacobian log canonical surface singularity. Then (X, p) is a log terminal singularity with multiplicity 3 or a complete intersection log canonical singularity.

The rational triple singularities are dclassified into 9 types by Tyurina in [4]:

(1) $A_{k-1,l-1,m-1}$, $k \geq l \geq m \geq 1$

$$(x(y + w^k) - yw^m, yz - (y + w^k)w^l, xz - w^{l+m})$$

(2) $B_{m,n}$, $m \geq 0$

$n = 2k - 1$, $k \geq 2$

$$(xz - y^{m+1}(y^k + yw), (y^k + yw)w - z^2, xw - y^{m+1}z)$$

$n = 2k$, $k \geq 2$

$$(x(z + y^k) - y^{m+2}w, yw^2 - z(z + y^k), xw - y^{m+1}z)$$

(3) $C_{m,n}$, $m \geq 3$, $n \geq 0$

$$(xz - y^{n+1}(y^2 + w^{m-1}), (y^2 + w^{m-1})w - z^2, xw - y^{n+1}z)$$

(4) D_n, $n \geq 0$

$$(x(y^2 + z) - y^{n+1}w^2, w^3 - z(y^2 + z), xw - y^{n+1}z)$$

(5) $E_{6,0}$

$$(x^2z - (y + w^2)y, yw - z^2, x^2w - (y + w^2)z)$$

(6) $E_{0,7}$

$$((x^2 + w^3)z - y^2, yw - z^2, (x^2 + w^3)w - yz)$$

(7) $E_{7,0}$

$$(w^2z - (y + x^2)y, yw - z^2, w^3 - (y + x^2)z)$$

(8) F_n, $n \geq 0$

$$(xz - y^{n+1}(y^3 + w^3), (y^3 + w^3)w - z^2, xw - y^{n+1}z)$$

(9) H_n

$n = 3k - 1$, $k \geq 2$

$$((xw + x^k)z - y^2, yw - z^2, (xw + x^k)w - yz)$$

$n = 3k$, $k \geq 2$

$$(xwz - (y + x^k)y, yw - z^2, xw^2 - (y + x^k)z)$$

$n = 3k + 1$, $k \geq 2$

$$(xw(z + x^k) - y^2, yw - (z + x^k)z, xw^2 - yz)$$

Then by checking the list, we pick up possible rational triple points for being Mather-Jacobian log canonical:

Proposition 2

Let (X, p) be a normal Mather-Jacobian log canonical surface singularity. If p is log terminal singularity with multiplicity 3, then it is a toric singularity.

This completes the proof of (i)⇒(ii).
Finally, we prove (iii)⇒(i) by determining the defining ideal of a toric variety with embedding dimension 4 and apply Theorem 5.6 in [2].

Proposition 3

Let (X, p) be a toric singularity of embedding dimension 4, then X is Mather-Jacobian log canonical at p.

References

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