Characterization of $\alpha 1$ and $\alpha 2$-matrices*

Rafael Bru[1] Ljiljana Cvetković[2] Vladimir Kostić[2]
Francisco Pedroche[1]

1 Institut de Matemàtica Multidisciplinar, Universitat Politècnica de València. Camí de Vera s/n. 46022 València. Spain. {rbru,pedroche@mat.upv.es}

2 Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad. Serbia, 21000 Novi Sad. {lila,vkostic@im.ns.ac.yu}

Keywords. H-matrices, $\alpha 1$-matrices, Subdirect sum, Hadamard product.
AMS Subject Classification. 15A48, 15A18, 65F15

Abstract

This paper deals with some properties of α-matrices, which are subclasses of invertible H-matrices. In particular, new characterizations of $\alpha 1$ and of $\alpha 2$-matrices are given. Considering these characterizations some algebraic properties of these matrices such as the subdirect sum and the Hadamard product are studied.

1 Introduction

In this paper we give characterizations of subclasses of H-matrices which are being studied by different authors, see [7], [6], [16], [5]. In particular we deal with $\alpha 1$ and with $\alpha 2$-matrices. H-matrices may appear in many practical applications, e.g., in the numerical solution of Euler equations in fluid dynamics [8], in nonlinear boundary problems and in the Lyapounov stability analysis for large scale evolution systems; see [15] and the references therein. They were introduced by Ostrowsky in [14] as a generalization of M-Matrices. H-matrices are called this way in homage to Hadamard, while M-matrices homage to Minkowsky [16]. We recall that a nonsingular matrix A having all non-positive off-diagonal entries is called an *M-matrix* if the

*This version dated 22-9- 2008.

inverse is (entry-wise) nonnegative, i.e., $A^{-1} \geq O$; see, e.g., [1] for more characterizations. For any matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, its *comparison* matrix $\langle A \rangle = (\alpha_{ij})$ can be defined by

$$\alpha_{ii} = |a_{ii}|, \quad \alpha_{ij} = -|a_{ij}|, \quad i \neq j.$$

A matrix A is said to be an *H-matrix* if $\langle A \rangle$ is a nonsingular M-matrix. In particular, A is an H-matrix if and only if it is (strictly) generalized (row) diagonally dominant, i.e.,

$$|a_{ii}|w_i > \sum_{i\neq j} |a_{ij}|w_j, \quad i = 1, \dots, n,$$

for some positive vector $w = (w_1, \dots, w_n)^T$. This is equivalent to say that A is an H-matrix if and only if there exists a positive diagonal matrix $W = diag(w_1, w_2, \dots, w_n)$ such that AW is a strictly (row) diagonally dominant (SDD) matrix. Some attempts have made to give practical characterizations of H-matrices (see, e.g., [12], [10], [15], [11], [6]) mainly based on finding suitable scaling matrices W. Another approach to the problem of finding classes of H-matrices resides in describing subclasses which are easily characterizable. According to this approach, some new subclasses of H-matrices were recently introduced in [5]. In this paper we focus on the subclass of α-matrices, and we give characterizations for the subclasses called $\alpha 1$-matrices and $\alpha 2$-matrices, which are defined below. Recently a new generalized subclass of α-matrices has been introduced in [5]. All of these classes contain the subclass of SDD matrices, as shown in figure 1. Properties related with the Schur complement of these matrices can be found in [13]. Note that all H-matrices considered here belong to the invertible class of H-matrices, i.e., their comparison matrix is an invertible M-matrix, which is one of the three classes introduced in [2].

In this paper, we will also consider the concept of subdirect sum. This concept contains the usual sum of matrices as a particular case, and it has been studied in [9], [3], [4] and [17] for different kind of matrices.

The paper is structured as follows. First, we give the definitions of $\alpha 1$, $\alpha 2$. In section 2, we give a characterization of $\alpha 1$-matrices followed by some remarks. In section 3, we extend this characterization to $\alpha 2$-matrices. In section 4, these new characterizations of $\alpha 1$ and $\alpha 2$-matrices are used for studying some algebraic properties as subdirect sums and Hadamard products of those matrices. We end this work in section 5 with some comments about the results so far presented.

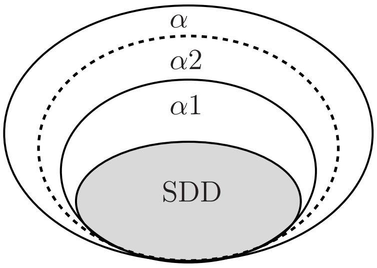

Figure 1: Some subclasses of H-matrices

We recall some definitions, see [5], [7], [16]. Throughtout the paper we deal with square matrices of order greater or equal to two and with nonempty subsets S of $N := \{1, 2, \ldots, n\}$.

Definition 1. *Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, and a nonempty subset S of N, let us define the ith deleted absolute row sum as*

$$r_i(A) = \sum_{j\neq i,\, j=1}^{n} |a_{ij}|, \quad \textit{for all } \; i = 1, 2, \ldots, n,$$

and the ith deleted absolute row sum over the indices of $S \subseteq N$ as

$$r_i^S(A) = \sum_{j\neq i,\, j\in S} |a_{ij}|, \quad \textit{for all } \; i \in N.$$

Given any nonempty set of indices $S \subseteq N$ we denote its complement in N as $\bar{S} := N\backslash S$. Note that for any $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ we have that $r_i(A) = r_i^S(A) + r_i^{\bar{S}}(A)$.

We are interested in $\alpha 1$ and $\alpha 2$-matrices. It is well known that α-matrices are nonsingular H-matrices; see [5], which are the invertible class of H-matrices given in [2].

2 Characterization of $\alpha 1$-matrices

We first recall the definition of $\alpha 1$-matrices from [5].

Definition 2. *A matrix* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, *is called an* $\alpha 1$ *matrix if there exists* $\alpha \in [0,1]$, *such that*

$$|a_{ii}| > \alpha r_i(A) + (1-\alpha) r_i(A^T), \quad \textit{for all } i \in N = \{1, 2, \ldots, n\}. \tag{1}$$

Note that the above condition can be rewritten as

$$|a_{ii}| > \alpha(r_i(A) - r_i(A^T)) + r_i(A^T), \quad \text{for all } i \in N.$$

According to this definition it is obvious that A can be an $\alpha 1$-matrix for different values of the parameter α.

Note that if $r_i(A) = r_i(A^T)$, then, for the index i the relation (1) is $|a_{ii}| > r_i(A^T) = r_i(A)$. In general, two more cases can occur: $r_i(A) > r_i(A^T)$ or $r_i(A) < r_i(A^T)$. Then, let us define the corresponding sets of indices.

Definition 3. *Given* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, *let us define the sets of indices* $\mathcal{L}(A)$, $\mathcal{H}(A)$ *and* $\mathcal{C}(A)$ *in the following way*

$$\begin{array}{rcll} \mathcal{L}(A) & = & \{i \in N : & r_i(A) - r_i(A^T) > 0\} \\ \mathcal{H}(A) & = & \{i \in N : & r_i(A) - r_i(A^T) < 0\} \\ \mathcal{C}(A) & = & \{i \in N : & r_i(A) - r_i(A^T) = 0\}. \end{array}$$

For any matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, and each $i \in N$, such that $i \notin \mathcal{C}(A)$, we define the quantity

$$\phi_i(A) = \frac{|a_{ii}| - r_i(A^T)}{r_i(A) - r_i(A^T)} \in \mathbb{R} \tag{2}$$

and now, using the quantity $\phi_i(A)$, we shall define the set of feasible values of a parameter α in order to obtain $\alpha 1$-matrices. First, let us see the following example.

Example 1. *Given the matrix*

$$A = \begin{bmatrix} 1 & 0 & 0.1 & 0.3 \\ 0.9 & 1.6 & 0.2 & 0.2 \\ 0.4 & 0.8 & 1.8 & 0 \\ 0.9 & 0 & 0.6 & 1.3 \end{bmatrix},$$

we have that $\phi_1(A) = \frac{2}{3}$, $\phi_2(A) = 1.6$, $\phi_3(A) = 3.0$ *and* $\phi_4(A) = 0.8$. *It is easy to see that* A *is an* $\alpha 1$*-matrix for any* $\alpha \in]\phi_1(A), \phi_4(A)[=]2/3, 0.8[$.

Note that the interval, of the above example, in which the parameter α can take values is a subset of the interval $[0,1]$ considered in Definition 2. In order to fix that interval, we give the following definition.

Definition 4. *Given $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, let us define the set $U(A)$ as follows*

$$U(A) =]-\infty, \min_{i\in\mathcal{L}(A)} \phi_i(A)[\quad \bigcap \quad]\max_{i\in\mathcal{H}(A)} \phi_i(A), +\infty[, \tag{3}$$

where by convention we take

$$\min_{i\in\mathcal{L}(A)} \phi_i(A) = +\infty \quad \textit{if } \mathcal{L}(A) = \emptyset, \quad \textit{and} \quad \max_{i\in\mathcal{H}(A)} \phi_i(A) = -\infty \quad \textit{if } \mathcal{H}(A) = \emptyset.$$

Bearing in mind this set, we characterize the class of $\alpha 1$-matrices.

Theorem 1. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then A is an $\alpha 1$-matrix if and only if the following conditions hold*

(i) $U(A) \cap [\,0,1\,] \neq \emptyset$,

(ii) $|a_{ii}| > r_i(A)$, *for all* $i \in \mathcal{C}(A)$.

Proof: First, let us assume that A is an $\alpha 1$-matrix. Consider $i \in \mathcal{L}(A)$. From equation (2) we have

$$|a_{ii}| = \phi_i(A)(r_i(A) - r_i(A^T)) + r_i(A^T), \tag{4}$$

where $r_i(A) - r_i(A^T) > 0$. Since A is an $\alpha 1$-matrix, there exists $\alpha \in [0,1]$ such that

$$|a_{ii}| > \alpha(r_i(A) - r_i(A^T)) + r_i(A^T), \quad \text{for all } \ i \in N. \tag{5}$$

Therefore, from (4) and (5), we conclude that $\phi_i(A) > \alpha$ for all $i \in \mathcal{L}(A)$ and thus we have

$$\min_{i\in\mathcal{L}(A)} \phi_i(A) > \alpha. \tag{6}$$

In an analogous way, it is easy to show that

$$\alpha > \max_{i\in\mathcal{H}(A)} \phi_i(A). \tag{7}$$

Note that (6) and (7) still hold if $\mathcal{L}(A)$ or $\mathcal{H}(A)$ are empty sets. Then,

$$\max_{i\in\mathcal{H}(A)} \phi_i(A) < \alpha < \min_{i\in\mathcal{L}(A)} \phi_i(A), \tag{8}$$

By definition 4, we have

$$U(A) =]\max_{i\in\mathcal{H}(A)} \phi_i(A), \ \min_{i\in\mathcal{L}(A)} \phi_i(A)[, \tag{9}$$

and therefore, we conclude that $\alpha \in U(A) \cap [0,1]$.

If $i \in \mathcal{C}(A)$ then $r_i(A) = r_i(A^T)$ and since A is an $\alpha 1$-matrix, the condition (ii) follows.

Conversely, assume that the conditions (i) and (ii) hold. From expresion (3), we have $U(A) =]\underset{i\in\mathcal{H}(A)}{max}\phi_i(A), \underset{i\in\mathcal{L}(A)}{min}\phi_i(A)[$.

Let us now show that A is an $\alpha 1$-matrix. More precisely, let us prove that (1) holds for each $i \in N$ and for some $\alpha \in U(A) \cap [0,1]$, which is not empty by the first condition (i). If $i \in \mathcal{L}(A)$, from the equation (2), we have

$$|a_{ii}| = \phi_i(A)(r_i(A) - r_i(A^T)) + r_i(A^T).$$

Since $\alpha < \underset{i\in\mathcal{L}(A)}{min}\phi_i(A)$, we obtain

$$|a_{ii}| > \alpha(r_i(A) - r_i(A^T)) + r_i(A^T). \tag{10}$$

Therefore the expresion (10) holds for all $\alpha < \underset{i\in\mathcal{L}(A)}{min}\phi_i(A)$. In analogous way we have that if $i \in \mathcal{H}(A)$, then, we have

$$|a_{ii}| > \alpha(r_i(A) - r_i(A^T)) + r_i(A^T), \tag{11}$$

for all $\alpha > \phi_i(A)$, and therefore the expresion (11) holds for all $\alpha > \underset{i\in\mathcal{H}(A)}{max}\phi_i(A)$. If $i \in \mathcal{C}(A)$, we have, by the condition (ii), that $|a_{ii}| > r_i(A^T)$. So, A is an $\alpha 1$-matrix for all $\alpha \in U(A) \cap [0,1]$, which is nonempty by the condition (i). $\square$

Note that the proof of Theorem 1 yields the interval where the parameter α can belong when we are dealing with $\alpha 1$-matrices. Then we can establish the following corollary.

Corollary 1. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be an $\alpha 1$-matrix, for some parameter α. Then $\alpha \in U(A) \cap [0,1]$, where $U(A)$ is given by the equation* (9).

Note that, in general, the set $U(A)$ and the interval $[0,1]$ may not be intersected. This opens the possibility to define a new class of α-matrices: those matrices $A \in \mathbb{C}^{n\times n}$ such that

$$|a_{ii}| > \alpha r_i(A) + (1-\alpha) r_i(A^T),$$

with $\alpha \in U(A)$. Unfortunately, this class is not a subclass of H-matrices, as the following example shows.

Example 2. *Given the matrix*

$$A = \begin{bmatrix} 4.7 & 0.8 & 0.8 & 0.8 \\ 0.9 & 0.7 & 0 & 0.5 \\ 0.4 & 0.2 & 1.9 & 0.7 \\ 0.4 & 0.7 & 0.3 & 0.1 \end{bmatrix},$$

we have $U(A) =]\phi_2(A), \phi_3(A)[=]10/3, 4.0[$. Since $U(A) \cap [0,1] = \emptyset$, by Theorem 1, A is not an $\alpha 1$-matrix. Moreover, a simple computation shows that $\langle A \rangle^{-1}$ is not a nonnegative matrix. Therefore A is not an H-matrix.

3 Characterization of α2-matrices

We recall the definition of α2-matrices from [5].

Definition 5. *A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, is said to be an $\alpha 2$-matrix if there exists $\alpha \in [0,1]$, such that*

$$|a_{ii}| > r_i(A)^{\alpha} \cdot r_i(A^T)^{1-\alpha},$$

for all $i \in N$.

To characterize this subclass of H-matrices of type I, we can use a similar construction as in the case of α1-matrices. In fact, due to the monotonicity of the log function, we can use the sets $\mathcal{L}(A)$, $\mathcal{C}(A)$ and $\mathcal{H}(A)$ given by Definition 3.

Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, for each $i \in N$ such that $i \notin \mathcal{C}(A)$, we define the quantity

$$\overline{\phi_i}(A) = \frac{\log |a_{ii}| - \log r_i(A^T)}{\log r_i(A) - \log r_i(A^T)} \quad \in \mathbb{R}, \tag{12}$$

and, similarly to Definition 4, we define the set $\overline{U}(A)$.

Definition 6. *Given $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, let us define the set*

$$\overline{U}(A) =]-\infty, \min_{i \in \mathcal{L}(A)} \overline{\phi_i}(A)[\quad \bigcap \quad]\max_{i \in \mathcal{H}(A)} \overline{\phi_i}(A), +\infty[, \tag{13}$$

where, by convention,

$$\min_{i \in \mathcal{L}(A)} \overline{\phi_i}(A) = +\infty \quad \text{if } \mathcal{L}(A) = \emptyset \quad \text{and} \quad \max_{i \in \mathcal{H}(A)} \overline{\phi_i}(A) = -\infty \quad \text{if } \mathcal{H}(A) = \emptyset.$$

The following result characterizes the $\alpha 2$-matrices. The proof is analogous to that of the Theorem 1, just working with $\overline{U}(A)$ and $\overline{\phi_i}(A)$ instead of $U(A)$ and $\phi_i(A)$, respectively.

Theorem 2. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then A is an $\alpha 2$-matrix if and only if the following conditions hold*

(i) $\overline{U}(A) \cap [\,0, 1\,] \neq \emptyset$,

(ii) $|a_{ii}| > r_i(A)$, *for all* $i \in \mathcal{C}(A)$.

Note that in the proof of Theorem 2 we obtain that the set $\overline{U}(A)$ is given by

$$\overline{U}(A) =]\max_{i\in\mathcal{H}(A)} \overline{\phi_i}(A),\ \min_{i\in\mathcal{L}(A)} \overline{\phi_i}(A)[. \tag{14}$$

The following example illustrates the characterization of $\alpha 2$-matrices.

Example 3. *Given the matrix A from Example 1, we have $\mathcal{H}(A) = \{1\}$, $\mathcal{L}(A) = \{2, 3, 4\}$ and $\max_{i\in\mathcal{H}(A)} \overline{\phi_i}(A) = \overline{\phi_1}(A) \approx 0.46$, and $\min_{i\in\mathcal{L}(A)} \overline{\phi_i}(A) = \overline{\phi_4(A)} \approx 0.87$. Therefore, $\overline{U}(A) \cap [0, 1] \approx]0.46, 0.87[$. In accordance with Theorem 2, matrix A is an $\alpha 2$-matrix for $\alpha \in \overline{U}(A) \cap [0, 1]$.*

Again, we have the interval in which the parameter α takes the values when the matrix is an $\alpha 2$-matrix. More precisely, we have the following result.

Corollary 2. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be an $\alpha 2$-matrix, for some parameter α. Then $\alpha \in \overline{U}(A) \cap [0, 1]$, where $\overline{U}(A)$ is given by the equation (14).*

Remark 1. *Since $\alpha 1$-matrices form a subclass of $\alpha 2$-matrices, for any $\alpha 1$-matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, then it is clear that $U(A) \cap \overline{U}(A) = U(A)$. Notice this inclusion in Examples 1 and 3.*

4 Algebraic properties of $\alpha 1$ and $\alpha 2$-matrices

In this section we study some algebraic properties of $\alpha 1$ and $\alpha 2$ matrices using the characterizations given in section 3. More precisely, we study the sum, in particular the subdirect sum of two matrices of the same class, the Hadamard product of two matrices of the same class and some other properties.

4.1 Subdirect sums

Let us recall some definitions and notation about the subdirect sum of two matrices. Let A and B be two square matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned into 2×2 blocks as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{15}$$

where A_{22} and B_{11} are square matrices of order k. According with [9], the following square matrix of order $n = n_1 + n_2 - k$

$$C = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \tag{16}$$

is called the k-subdirect sum of A and B, and denoted by $C = A \oplus_k B$.

It is easy to express each element of C in terms of those of A and B. To that end, let us define the following set of indices

$$\begin{aligned} S_1 &= \{1, 2, \ldots, n_1 - k\}, \\ S_2 &= \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\}, \\ S_3 &= \{n_1 + 1, n_1 + 2, \ldots, n\}. \end{aligned} \tag{17}$$

Denoting $C = [c_{ij}]$ and setting $t = n_1 - k$, we can write

$$c_{ij} = \begin{cases} a_{ij} & i \in S_1, \quad j \in S_1 \cup S_2 \\ 0 & i \in S_1, \quad j \in S_3 \\ a_{ij} & i \in S_2, \quad j \in S_1 \\ a_{ij} + b_{i-t,j-t} & i \in S_2, \quad j \in S_2 \\ b_{i-t,j-t} & i \in S_2, \quad j \in S_3 \\ 0 & i \in S_3, \quad j \in S_1 \\ b_{i-t,j-t} & i \in S_3, \quad j \in S_2 \cup S_3, \end{cases} \tag{18}$$

Note that $S_1 \cup S_2 \cup S_3 = \{1, 2, \ldots, n\}$ and $n = t + n_2$.

Note that when both matrices A and B are of the same size n and taking $k = n$ the k-subdirect sum becomes the standard sum of two matrices.

Now we give sufficient conditions so that the subdirect sum of two $\alpha 1$-matrices remains in the class.

Theorem 3. *Let A and B be $\alpha 1$-matrices of order n_1 and n_2, respectively. Let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned*

$$\begin{array}{cc}
 & \begin{array}{ccc} \quad S_1 \quad & \qquad\qquad S_2 \qquad\qquad & \qquad S_3 \end{array} \\
C = & \begin{pmatrix}
a_{11} & \cdots & a_{1p} & \cdots & a_{1,n_1} & \cdots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{p,1} & \cdots & a_{p,p}+b_{1,1} & \cdots & a_{p,n_1}+b_{1,n_1-t} & \cdots & b_{1,n-t} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{n_1,1} & \cdots & a_{n_1,p}+b_{n_1-t,1} & \cdots & a_{n_1,n_1}+b_{n_1-t,n_1-t} & \cdots & b_{n_1-t,n-t} \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \cdots & b_{n-t,1} & \cdots & b_{n-t,n_1-t} & \cdots & b_{n-t,n-t}
\end{pmatrix}
\begin{array}{c} S_1 \\ \\ S_2 \\ \\ S_3 \end{array}
\end{array}$$

Figure 2: Sets for the subdirect sum $C = A \oplus_k B$, with $t = n_1 - k$ and $p = t + 1$

as in (15) *and let* $U(A)$ *and* $U(B)$ *be given by Definition 4. If* $U(A) \cap U(B) \neq \emptyset$ *and all diagonal entries of* A_{22} *and* B_{11} *are positive (or all negative), then the k-subdirect sum* $C = A \oplus_k B$ *is an* α*1-matrix for all* $\alpha \in U(A) \cap U(B) \cap [0, 1]$.

Proof We want to show that

$$|c_{ii}| > \alpha r_i(C) + (1-\alpha) r_i(C^T) \tag{19}$$

for all $i \in S_1 \cup S_2 \cup S_3$ (see equation (18) and figure 2) and for $\alpha \in U(A) \cap U(B) \cap [0, 1]$. When $i \in S_1$ we have from (18) that $c_{ii} = a_{ii}$ and since A is an α1-matrix we obtain, for all $\alpha \in U(A) \cap [0, 1]$:

$$|c_{ii}| = |a_{ii}| > \alpha r_i^{S_1 \cup S_2}(A) + (1-\alpha) r_i^{S_1 \cup S_2}(A^T) = \alpha r_i(C) + (1-\alpha) r_i(C^T), \tag{20}$$

where we have used that $r_i(C) = r_i^{S_1 \cup S_2}(A)$ and $r_i(C^T) = r_i^{S_1 \cup S_2}(A^T)$ when $i \in S_1$. It is obvious that (20) still holds for all $\alpha \in U(A) \cap U(B) \cap [0, 1] \subseteq U(A) \cap [0, 1]$. In a similar way, when $i \in S_3$ we have from (18) that $c_{ii} = b_{i-t,i-t}$ and since B is an α1-matrix we get, for all $\alpha \in U(B) \cap [0, 1]$:

$$|c_{ii}| = |b_{i-t,i-t}| > \alpha r_{i-t}^{S_2 \cup S_3}(B) + (1-\alpha) r_{i-t}^{S_2 \cup S_3}(B^T) = \alpha r_i(C) + (1-\alpha) r_i(C^T), \tag{21}$$

where we have used that $r_i(C) = r_i^{S_2 \cup S_3}(B)$ and $r_i(C^T) = r_i^{S_2 \cup S_3}(B^T)$ when $i \in S_3$. It is obvious that (21) still holds for all $\alpha \in U(A) \cap U(B) \cap [0, 1] \subseteq U(B) \cap [0, 1]$.

Finally we have to deal with the case $i \in S_2$. We first note that since all diagonal entries of A_{22} and B_{11} are positive (or all negative) we have $|c_{ii}| = |a_{ii} + b_{i-t,i-t}| = |a_{ii}| + |b_{i-t,i-t}|$, $\forall i \in S_2$, with $t = n_1 - k$; see figure 2. We analyze the following subcases

(a) $i \in \mathcal{L}(A) \cup \mathcal{H}(A)$, $i - t \in \mathcal{L}(B) \cup \mathcal{H}(B)$.

(b) $i \in \mathcal{C}(A)$, $i - t \in \mathcal{L}(B) \cup \mathcal{H}(B)$.

(c) $i \in \mathcal{L}(A) \cup \mathcal{H}(A)$, $i - t \in \mathcal{C}(B)$.

(d) $i \in \mathcal{C}(A)$, $i - t \in \mathcal{C}(B)$.

In the subcase (a) we have, for all $\alpha \in U(A) \cap U(B)$:

$$\begin{aligned}
|c_{ii}| &= |a_{i,i} + b_{i-t,i-t}| = |a_{i,i}| + |b_{i-t,i-t}| \\
&> \alpha r_i^{S_1 \cup S_2}(A) + (1-\alpha) r_i^{S_1 \cup S_2}(A^T) + \alpha r_{i-t}^{S_2 \cup S_3}(B) + (1-\alpha) r_{i-t}^{S_2 \cup S_3}(B^T) \\
&= \alpha (r_i^{S_1 \cup S_2}(A) + r_{i-t}^{S_2 \cup S_3}(B)) + (1-\alpha)(r_i^{S_1 \cup S_2}(A^T) + r_{i-t}^{S_2 \cup S_3}(B^T)) \\
&\geq \alpha r_i(C) + (1-\alpha) r_i(C^T) \qquad (22)
\end{aligned}$$

where we have used the triangle inequality, i.e., $r_i^{S_1 \cup S_2}(A) + r_{i-t}^{S_2 \cup S_3}(B) \geq r_i(C)$, and the same for the corresponding term with A^T, B^T and C^T. It is clear that equation (22) still holds for $\alpha \in U(A) \cap U(B) \cap [0,1] \subseteq U(A) \cap U(B)$.

In the subcase (b) we have, for all $\alpha \in U(A) \cap U(B)$:

$$\begin{aligned}
|c_{ii}| &= |a_{i,i} + b_{i-t,i-t}| = |a_{i,i}| + |b_{i-t,i-t}| \\
&> r_i^{S_1 \cup S_2}(A^T) + \alpha r_{i-t}^{S_2 \cup S_3}(B) + (1-\alpha) r_{i-t}^{S_2 \cup S_3}(B^T) \\
&= \alpha (r_i^{S_1 \cup S_2}(A) + r_{i-t}^{S_2 \cup S_3}(B)) + (1-\alpha)(r_i^{S_1 \cup S_2}(A^T) + r_{i-t}^{S_2 \cup S_3}(B^T)) \\
&\geq \alpha r_i(C) + (1-\alpha) r_i(C^T) \qquad (23)
\end{aligned}$$

where we have used that $r_i^{S_1 \cup S_2}(A^T) = r_i^{S_1 \cup S_2}(A)$, since $i \in \mathcal{C}(A)$, and as before, the triangle inequality, i.e., $r_i^{S_1 \cup S_2}(A) + r_{i-t}^{S_2 \cup S_3}(B) \geq r_i(C)$, and the same for the corresponding term with A^T, B^T and C^T. It is clear that equation (23) still holds for $\alpha \in U(A) \cap U(B) \cap [0,1] \subseteq U(A) \cap U(B)$.

The subcase (c) is analogous to the subcase (b) and we omit the details.

In the subcase (d) we have, for all $\alpha \in U(A) \cap U(B)$:

$$\begin{aligned}
|c_{ii}| &= |a_{i,i} + b_{i-t,i-t}| = |a_{i,i}| + |b_{i-t,i-t}| \\
&> r_i^{S_1 \cup S_2}(A^T) + r_{i-t}^{S_2 \cup S_3}(B^T) \\
&= \alpha (r_i^{S_1 \cup S_2}(A) + r_{i-t}^{S_2 \cup S_3}(B)) + (1-\alpha)(r_i^{S_1 \cup S_2}(A^T) + r_{i-t}^{S_2 \cup S_3}(B^T)) \\
&\geq \alpha r_i(C) + (1-\alpha) r_i(C^T) \qquad (24)
\end{aligned}$$

where we have used that $r_i^{S_1\cup S_2}(A^T) = r_i^{S_1\cup S_2}(A)$, $r_{i-t}^{S_2\cup S_3}(B^T) = r_{i-t}^{S_2\cup S_3}(B)$, and as before, the triangle inequality, i.e., $r_i^{S_1\cup S_2}(A) + r_{i-t}^{S_2\cup S_3}(B) \geq r_i(C)$, and the same for the corresponding term with A^T, B^T and C^T. It is clear that equation (24) still holds for $\alpha \in U(A) \cap U(B) \cap [0,1] \subseteq U(A) \cap U(B)$. The proof is now completed. □

Example 4. *Given the matrix*

$$B = \begin{bmatrix} 2.4 & -0.4 & -0.6 & 0.2 \\ -0.4 & 2.2 & -0.2 & -0.7 \\ -0.8 & 0.9 & 2.3 & 1.0 \\ -0.8 & 0 & -0.5 & 2.6 \end{bmatrix},$$

we have $\phi_1(B) = -0.5$, $\phi_3(B) = 5/7$, $\phi_4(B) = -7/6$, and the sets of indices $\mathcal{H}(B) = \{1,4\}$ and $\mathcal{L}(B) = \{3\}$, and then $\max_{i\in\mathcal{H}(B)} \phi_i(B) = \phi_1(B) = -0.5$, and $\min_{i\in\mathcal{L}(B)} \phi_i(B) = \phi_3(B) = 5/7$ and therefore $U(B) =]-0.5, 5/7[$ and according to Theorem 1 B is an α1-matrix for $\alpha \in U(B) \cap [0,1] = [0, 5/7[$. Let A be given by Example 1. If we construct the 3-subdirect sum $C = A \oplus_3 B$ which results to be

$$C = \begin{bmatrix} 1.0 & 0.0 & -0.1 & -0.3 & 0.0 \\ -0.9 & 4.0 & -0.6 & -0.8 & 0.2 \\ -0.4 & -1.2 & 4.0 & -0.2 & -0.7 \\ 0.9 & -0.8 & 0.3 & 3.6 & 1.0 \\ 0.0 & -0.8 & 0.0 & -0.5 & 2.6 \end{bmatrix},$$

we have, in accordance with Theorem 3 that C is an α1-matrix for $\alpha \in U(A) \cap U(B) \cap [0,1] =]2/3, 5/7[\approx]0.67, 0.71[$. Indeed, the interval for which C is an α1-matrix is bigger in this case, since we have $\phi_1(C) = 2/3$, $\phi_2(C) = -4$, $\phi_3(C) = 2.0$,$\phi_4(C) = 1.5$ $\phi_5(C) = -7/6$ and the sets of indices $\mathcal{H}(C) = \{1,2,5\}$ and $\mathcal{L}(C) = \{3,4\}$, and then $\max_{i\in\mathcal{H}(C)} \phi_i(C) = 2/3$, and $\min_{i\in\mathcal{L}(C)} \phi_i(C) = 1.5$. Then $U(C) =]2/3, 1.5[$ and according to Theorem 1 C is an α1-matrix for $\alpha \in U(C) \cap [0,1] =]2/3, 1]$.

Theorem 3 cannot be extended to the subdirect sum of α2-matrices as the following example shows.

Example 5. *The matrix*

$$A = \begin{bmatrix} 1.9 & 0.8 & 0.4 & 0.8 \\ 0.7 & 1.7 & 0.3 & 1 \\ 0.5 & 0.6 & 1.2 & 1 \\ 0.6 & 0 & 0 & 1.6 \end{bmatrix}$$

is an $\alpha 2$-matrix with $\overline{U}(A) =]0.36, 0.49[$ but the 2-subdirect sum $C = A \oplus_2 A$ is not an $\alpha 2$-matrix, since the set $\overline{U}(C)$ is empty.

The following result gives sufficient condition for the subdirect sum of two $\alpha 2$-matrices be in the same class.

Theorem 4. *Let A and B be $\alpha 2$-matrices of order n_1 and n_2, respectively. Let $n_1 \geq 2$, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$, which defines the sets S_1, S_2, S_3 as in (17). Let A and B be partitioned as in (15). If $\overline{U}(A) \cap \overline{U}(B) \neq \emptyset$, the diagonals entries of A_{22} and B_{11} have the same sign pattern, and for any $i \in S_2$ the following conditions hold*

1. $i \in \mathcal{C}(A)$,
2. $i - t \in \mathcal{C}(B)$,
3. $r_i(C) = r_i(C^T)$,

then the k-subdirect sum $C = A \oplus_k B$ is an $\alpha 2$-matrix for all $\alpha \in \overline{U}(A) \cap \overline{U}(B) \cap [0, 1]$.

Proof: Consider first $i \in S_1$, we have $|c_{ii}| = |a_{ii}| > r_i(A)^\alpha \cdot r_i(A^T)^{1-\alpha} = r_i(C)^\alpha \cdot r_i(C^T)^{1-\alpha}$, for $\alpha \in \overline{U}(A)$. In a similar way, it is clear that for $i \in S_3$, we also have $|c_{ii}| > r_i(C)^\alpha \cdot r_i(C^T)^{1-\alpha}$ for $\alpha \in \overline{U}(B)$. In the case $i \in S_2$ we obtain

$$|c_{ii}| = |a_{ii}| + |b_{i-t,i-t}| > r_i(A^T) + r_{i-t}(B^T) > r_i(C^T) \tag{25}$$

where we have used conditions *1* and *2* combined with condition *3*, and the triangle inequality. Note that the equation (25) implies that C satisfies the condition of $\alpha 2$-matrix for $i \in S_2$. Then, C is an $\alpha 2$-matrix for all $\alpha \in \overline{U}(A) \cap \overline{U}(B) \cap [0, 1]$. $\square$

Note that condition (3) of the above theorem holds when A and B are doubly stochastic [1]. More precisely, if matrices A and B are of the type doubly stochastic for the rows and columns corresponding to the overlapping region (set S_2), we have $r_i(A) = r_i(A^T) = 1$ and $r_{i-t}(B) = r_{i-t}(B^T) = 1$ in these rows and therefore condition (3) of theorem 4 is fulfilled.

4.2 Hadamard product

In this section we show other application of the characterizations introduced for α-matrices. We focus on sufficient conditions for the Hadamard (entry-wise) product of $\alpha 2$-matrices to be in the class.

We first show that in general the Hadamard product of $\alpha 2$-matrices is not in the class.

Example 6. *Given*

$$A = \begin{bmatrix} 0.3 & 2.0 & 3.0 \\ 0.1 & 2.2 & 0.15 \\ 0.15 & 0.1 & 3 \end{bmatrix}$$

which is an $\alpha 2$-matrix for $\alpha \in \overline{U}(A) \approx]0.02, 0.06[$, we have that the Hadamard product

$$C = A \circ A^T = \begin{bmatrix} 0.09 & 0.2 & 0.45 \\ 0.2 & 4.84 & 0.015 \\ 0.45 & 0.015 & 9 \end{bmatrix}$$

is not an $\alpha 2$ matrix, since

$$c_{11} = 0.09 < r_1(C)^\alpha r_1(C^T)^{1-\alpha} = r_1(C) = 0.65$$

Note that A^T is an $\alpha 2$-matrix for $\overline{U}(A^T) \approx]0.94, 0.98[$, and therefore $\overline{U}(A) \cap \overline{U}(A^T) = \emptyset$.

In the following result we give a sufficient condition for the Hadamard product of $\alpha 2$-matrices to be in the class.

Theorem 5. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $B = [b_{ij}] \in \mathbb{C}^{n\times n}$ be $\alpha 2$-matrices such that $\overline{U}(A) \cap \overline{U}(B) \neq \emptyset$ then the Hadamard product $C = [c_{ij} = a_{ij}b_{ij}] \in \mathbb{C}^{n\times n}$ is an $\alpha 2$-matrix for any α in the set $\overline{U}(A) \cap \overline{U}(B) \cap [0, 1]$.*

Proof. Let $\alpha \in \overline{U}(A) \cap \overline{U}(B) \cap [0, 1]$. Then it is clear that

$$\begin{aligned} |c_{ii}| &= |a_{ii}||b_{ii}| \\ &> r_i(A)^\alpha \; r_i(A^T)^{1-\alpha} \; r_i(B)^\alpha \; r_i(B^T)^{1-\alpha} \\ &= [r_i(A)r_i(B)]^\alpha \; [r_i(A^T)r_i(B^T)]^{1-\alpha} \\ &\geq r_i(A \circ B)^\alpha \; r_i(A^T \circ B^T)^{1-\alpha} \end{aligned} \tag{26}$$

and the proof follows. □

Example 7. *Given the matrices* $A = \begin{bmatrix} 8.6 & 5.0 \\ 10 & 6.9 \end{bmatrix}$ *and* $B = \begin{bmatrix} 11.85 & 6 \\ 13 & 8.15 \end{bmatrix}$ *which are $\alpha 1$-matrices, and therefore they are also $\alpha 2$-matrices, we have $\overline{U}(A) \approx]0.22, 0.46[$ and $\overline{U}(B) \approx]0.12, 0.40[$, and hence, $A \circ B$ is an $\alpha 2$-matrix, according to theorem 5. It is worhtwhile to remark that $U(A) \approx]0.28, 0.38[$ and $U(B) \approx]0.16, 0.31[$, but, $A \circ B$ is not an $\alpha 1$-matrix. Therefore, theorem 5 does not hold for $\alpha 1$-matrices.*

In the following example we show that the conditions of theorem 5 are not necessary ones.

Example 8. *Given the $\alpha 2$-matrices*

$$A = \begin{bmatrix} 0.6 & 0.7 & 0.7 \\ 0.1 & 0.5 & 0.0 \\ 0.1 & 0.2 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1.6 & 0.7 & 0.8 \\ 0.3 & 0.9 & 0.5 \\ 0.9 & 0.8 & 1.8 \end{bmatrix}$$

we have $\overline{U}(A) \approx]0.27, 0.56[$ and $\overline{U}(B) \approx]0.81, 1.0]$ and therefore $\overline{U}(A) \cap \overline{U}(B) = \emptyset$. Nevertheless, the Hadamard product $C = A \circ B$ is an $\alpha 2$-matrix with $\overline{U}(C) \approx]0.12, 0.96[$.

4.3 Other algebraic properties

Given $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, let us denote by $|A|$ the matrix whose elements are $[|a_{ij}|]$. Since $r_i(A) = r_i(|A|)$ it is a routine to obtain the following results.

- Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then A is an $\alpha 1$-matrix if and only if $|A|$ is an $\alpha 1$-matrix. Moreover, $U(A) = U(|A|)$.
- Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then A is an $\alpha 2$-matrix if and only if $|A|$ is an $\alpha 2$-matrix. Moreover, $\overline{U}(A) = \overline{U}(|A|)$.

Given a real number $q \neq 0$, we have $r_i(qA) = |q| r_i(A)$ for all $i \in N$. Then, it is easy to show the following results.

- Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $q \in \mathbb{R}$. Then A is an $\alpha 1$-matrix if and only if qA is an $\alpha 1$-matrix. Moreover, $U(A) = U(qA)$.
- Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $q \in \mathbb{R}$. Then A is an $\alpha 2$-matrix if and only if qA is an $\alpha 2$-matrix. Moreover, $\overline{U}(A) = \overline{U}(qA)$.

Regarding the transposition it is easy to establish the following results for which we omit the proof.

- Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then A is an $\alpha 1$-matrix if and only if A^T is an $\alpha 1$-matrix. Moreover, $U(A) =]a, b[$ if and only if $U(A^T) =]1-b, 1-a[$.
- Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then A is an $\alpha 2$-matrix if and only if A^T is an $\alpha 2$-matrix. Moreover, $\overline{U}(A) =]a, b[$ if and only if $\overline{U}(A^T) =]1-b, 1-a[$.

Other algebraic operations such as matrix powering do not preserve, in general, the property of being an $\alpha 1$ or $\alpha 2$-matrix.

To end this section we note that given a permutation matrix $P \in \mathbb{R}^{n\times n}$ we have that A and $P^T AP$ have the same set of diagonal entries and the same set of row (and column) sums. As a consequence we have the following.

- Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be an $\alpha 1$-matrix and P a permutation matrix in $\mathbb{R}^{n\times n}$. Then $U(A)$ is invariant under a symmetric permutation of A. That is, $U(P^T AP) = U(A)$.
- Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be an $\alpha 2$-matrix and P a permutation matrix in $\mathbb{R}^{n\times n}$. Then $\overline{U}(A)$ is invariant under a symmetric permutation of A. That is, $\overline{U}(P^T AP) = \overline{U}(A)$.

5 Concluding remarks

We have shown practical characterizations for the classes of $\alpha 1$-matrices and of $\alpha 2$-matrices, which are subclasses of (nonsingular) H-matrices. The characterizations introduced allow a better comprehension of these kind of matrices in the sense that they bound the parameter α that actually defines each class. These characterizations using a bound for a parameter are analogous, in some way, to the characterizations of the S-SDD matrices given in [7]. In addition, we have studied some algebraic properties regarding the subdirect sum, the Hadamard product and some other operations such as the product by a scalar and the transposition.

Acknowledgments

This work is supported by an agreement of research cooperation between University of Novi Sad (Serbia) and Universitat Politècnica de València (Spain), by Spanish DGI grant MTM2007-64477, by the Provincial Secretariat of Science and Technological Development of Vojvodina, Serbia, grant 0708 and by the Ministry of Science and Environmental Protection of Serbia, grant 144025.

References

[1] A. Berman, and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Academic Press, New York. Reprinted and updated, SIAM, Philadelphia, 1994.

[2] R. Bru, C. Corral, I. Gimenez and J. Mas. Classes of H-matrices, *Linear Algebra and its Applications*, to appear, 2008, doi:10.1016/j.laa.2007.10.030.

[3] R. Bru, F. Pedroche, and D. B. Szyld. Subdirect sums of S-Strictly Diagonally Dominant matrices, *Electron. J. Linear Algebra*, 15:201–209, 2006.

[4] R. Bru, F. Pedroche, and D. B. Szyld. Subdirect sums of nonsingular M-matrices and of their inverses, *Electron. J. Linear Algebra*, 13:162–174, 2005.

[5] L. Cvetkovic. H-matrix theory vs. eigenvalue localization *Numerical Algorithms*, 42: 229-245, 2006.

[6] L. Cvetkovic and V. Kostic. New criteria for identifying H-matrices, *J. Comput. Appl. Math.*, 180:265–278, 2005.

[7] L. Cvetkovic, V. Kostic and R. S. Varga. A new Geršgoring-type eigenvalue inclusion set, *Electron. Trans. Numer. Anal.*, 18:73–80, 2004.

[8] L. Elsner and V. Mehrmann. Convergence of Block-Iterative Methods for Linear Systems Arising in the Numerical Solution of Euler Equations. *Numerische Mathematik*, Vol. 59, pp. 541-560, 1991.

[9] S. M. Fallat and C. R. Johnson. Sub-direct sums and positivity classes of matrices, *Linear Algebra Appl.*, 288:149–173, 1999.

[10] T. B. Gan, and T. Z. Huang. Simple criteria for nonsingular H-matrices, *Linear Algebra Appl.*, 374:317–326, 2003.

[11] T-Z Huang, J-S Leng, E.L. Wachspress and Y. Y. Tang. Characterization of H-matrices. *Computers & Mathematics with applications*, 48 (10-11): 1587-1601, 2004.

[12] B. Li, L. Li, M. Harada, H. Niki and M. J. Tsatsomeros, An iterative criterion for H-matrices, *Linear Algebra Appl.*, 271:179–190, 1998.

[13] J. Liu and Y. Huang. Some properties on Schur complements of H-matrices and diagonally dominant matrices, *Linear Algebra and its Applications*, 389:365–380, 2004.

[14] A. M. Ostrowski, (1937), Über die Determinanten mit überwiegender Hauptdiagonale, *Comentarii Mathematici Helvetici* **10** pp. 69–96.

[15] P. Spiteri. A new characterization of M-matrices and H-matrices. *BIT Numerical Mathematics*, 43:1019-1032, 2003.

[16] R. S. Varga. *Geršgorin and his circles*. Springer Series in Computational Mathematics, vol. 36. Springer, Berlin, Heidelberg, 2004.

[17] Y. Zhu and T-Z. Huang. Subdirect sums of Doubly Diagonally Dominant Matrices, *Electron. J. Linear Algebra*, 16:171–182, 2007.