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Characterization of feedback Nash equilibria for multi-channel systems via a set of non-fragile stabilizing state-feedback solutions and dissipativity inequalities

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Abstract We consider the problem of state-feedback stabilization for a multi-channel system in a game-theoretic framework, where the class of admissible strategies for the players is induced from a solution set of the individual objective functions that are associated with certain dissipativity properties of the system. In such a framework, we characterize the feedback Nash equilibria via a set of non-fragile stabilizing state-feedback solutions corresponding to a family of perturbed multi-channel systems. Moreover, we show that the existence of a weak-optimal solution to a set of constrained dissipativity problems is a sufficient condition for the existence of a feedback Nash equilibrium, whereas the set of non-fragile stabilizing state-feedbacks solutions is described in terms of a set of dilated linear matrix inequalities.

Keywords Dissipativity properties · Game theory · Multi-channel system · Nash strategy · Dilated linear matrix inequality · Robust stabilization

1 Introduction

In this paper, we consider a multi-channel system governed by several players (or *decision makers*) where the stability of the overall closed-loop system is a common objective while each player aims to maximize a different type of objective function. In such a scenario,

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Nash strategy offers a suitable framework to study inherent robustness or non-fragility of the strategies under a family of information structures, since no player can improve his payoff by deviating unilaterally from the Nash strategy once the equilibrium is attained (e.g., see references [17], [24], [25], [18], [33], [4], [5], [12]).

In the past, several theoretical results have been established to characterize control related problems in the context of Nash equilibria via a game theoretic interpretation [31], [26], [28], [34], [1], [35], [4] and [36]. For example, the existence of open-loop Nash strategies for linear-quadratic games over a finite time-horizon, assuming that all strategies lie in compact subsets of an admissible strategy space, has been addressed in [37], [24] and [32], and the existence of Nash strategies for linear-quadratic differential games over an infinite-horizon has been studied in detail in [31], [28], [1], [4] and [35]. We also note that some of these works have discussed the uniqueness of the optimal strategies for linear-quadratic games with structured uncertainties, where the bound for the objective function is based on the existence of a set of solutions for appropriately parameterized Riccati equations. Moreover, in the area of multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ control theory, the concept of differential games has been applied by interpreting uncertainty (or neglected dynamics) as a fictitious player while the model of the system is supposed to be well known; where the fictitious player is usually introduced in the criteria through a weighting matrix (e.g., see references [11], [22], [3], [34], [7] and [5]).

On the other hand, the use of different simplified models of the same system has been employed for capturing certain information structures, models or objective functions that individual players may hold about the overall system. Thus, the resulting problem can be best described by nonzero-sum differential games where the individual players are allowed to minimize different types of objective functions (e.g., see references [31], [8], [20], [30]). An extensive survey on the area of noncooperative dynamic games is provided in the book by Başar and Olsder [4].

Our main focus in this paper is to take this line of approach, where individual players have different objective functions that are associated with certain information structures, i.e., the dissipativity property of the multi-channel system and where the optimality concept is that of Nash equilibrium. We characterize the feedback Nash equilibria via a set of stabilizing state-feedback solutions corresponding to a family of perturbed multi-channel systems with dissipativity properties (see [38], [39], [41] and references therein for a review of systems with dissipativity properties). Specifically, we consider two fundamental problems: (i) We first isolate a condition that guaranteeing the control/strategy space is sufficiently *decentralized* to make the game-theoretic interpretation sensible, and (ii) We then provide a sufficient condition for the existence of *non-fragile* feedback Nash equilibrium, where the individual players have different objective functions that are associated with certain information structures, i.e., the dissipativity properties, of the system. We further show that the existence of a weak-optimal solution to the constrained dissipativity problem is a sufficient condition for the existence of a feedback Nash equilibrium with an additional desirable property of strong time consistency.

The rest of the paper is organized as follows. In Section 2, we present a verifiable stability condition for a multi-channel system via a set of dilated linear matrix inequalities (LMIs), with a certain dissipativity property being used to extend the stability condition when there is a model perturbation in the system. Section 3 presents the main results, where we provide a sufficient condition for the existence of Nash equilibria via weak-optimal solutions of the game corresponding to the dissipativity property of the system. In Section 4, we present a simple numerical example. Finally, Section 5 provides some concluding remarks.

Throughout the paper, we use the following notations. For a matrix $A \in \mathbb{R}^{n\times n}$, $\mathrm{He}\,(A)$ denotes a hermitian matrix defined by $\mathrm{He}\,(A) \triangleq (A + A^T)$, where A^T is the transpose of A. For a matrix $B \in \mathbb{R}^{n\times p}$ with $r = \mathrm{rank}\, B$, $B^{\perp} \in \mathbb{R}^{(n-r)\times n}$ denotes an orthogonal complement of B, which is a matrix that satisfies $B^{\perp}B = 0$ and $B^{\perp}B^{\perp T} \succ 0$. $\mathbb{S}^n_+$ denotes the set of strictly positive definite $n \times n$ real matrices and $\mathbb{C}^-$ denotes the set of complex numbers with negative real parts, that is $\mathbb{C}^- \triangleq \{s \in \mathbb{C} \,|\, \mathrm{Re}\{s\} < 0\}$. $\mathrm{Sp}(A)$ denotes the spectrum of a matrix $A \in \mathbb{R}^{n\times n}$, i.e., $\mathrm{Sp}(A) \triangleq \{\lambda \in \mathbb{C} \,|\, \mathrm{rank}(A - \lambda I) < n\}$ and $\mathrm{GL}_n(\mathbb{R})$ denotes the general linear group consisting of all $n \times n$ real nonsingular matrices.

2 Preliminaries

Consider a continuous-time N-channel system

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{N} B_j u_j(t), \quad x(0) = x_0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times r_j}$, $x(t) \in \mathbb{R}^n$ is the state of the system, and $u_j(t) \in \mathbb{R}^{r_j}$ is a control input to the jth-channel of the system.

For this system, consider the set of all stabilizing state-feedback gains

$$\mathcal{K}_N \subseteq \left\{ (K_1, K_2, \ldots, K_N) \in \prod_{j=1}^{N} \mathcal{K}_j \subseteq \prod_{j=1}^{N} \mathbb{R}^{r_j\times n} \,\Big|\, \mathrm{Sp}\left(A + \sum_{j=1}^{N} B_j K_j \right) \in \mathbb{C}^- \right\}, \tag{2}$$

Let us define the following matrices that will be later used in Theorem 1.

$$E = [\underbrace{I_{n\times n}\ I_{n\times n} \cdots I_{n\times n}}_{(N+1)\ times}], \quad [A, B]_{U,\tilde{L}} = [\underbrace{AU\, B_1L_1\ B_2L_2 \cdots B_NL_N}_{(N+1)\ times}],$$

and

$$\langle U, \widetilde{W} \rangle = \mathrm{block\,diag}\{\underbrace{U, \overbrace{W_1, W_2, \ldots, W_N}^{=\widetilde{W}}}_{(N+1)\ times}\}.$$

Then, we can characterize the set $\mathcal{K}_N$ in terms of a set of dilated LMIs as follows:

Theorem 1 *For any stabilizable pair* $\left(A, [\,B_1\ B_2 \cdots B_N\,]\right)$*, there exist* $X \in \mathbb{S}^n_+$*,* $\epsilon > 0$*,* $U \in \mathrm{GL}_n(\mathbb{R})$*,* $W_j \in \mathrm{GL}_n(\mathbb{R})$ *and* $L_j \in \mathbb{R}^{r_j\times n}$ *for* $j = 1, 2, \ldots, N$ *such that*

$$\begin{bmatrix} 0_{n\times n} & XE \\ E^TX & 0_{(N+1)n\times(N+1)n} \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} [A, B]_{U,\tilde{L}} \\ -\langle U, \widetilde{W} \rangle \end{bmatrix} \left[\, E^T\ \ \epsilon I_{(N+1)n\times(N+1)n} \,\right] \right) \prec 0, \tag{3}$$

Moreover, for any family of N-tuples $(L_1, L_2, \ldots, L_N)$ and $(W_1, W_2, \ldots, W_N)$ as above, and setting $K_j = L_j W_j^{-1} \in \mathcal{K}_j$ for each $j = 1, 2, \ldots, N$, then $\left(A + \sum_{j=1}^{N} B_j K_j\right)$ is a Hurwitz matrix.[1]

Proof Note that

$$\begin{bmatrix} [A, B]_{U,\widetilde{L}} \\ -\langle U, \widetilde{W} \rangle \end{bmatrix}^{\perp} = \begin{bmatrix} I_{n\times n} & [A, B]_{U,\widetilde{L}} \langle U, \widetilde{W} \rangle^{-1} \end{bmatrix}, \quad (4)$$

$$\begin{bmatrix} E \\ \epsilon I_{(N+1)n\times(N+1)n} \end{bmatrix}^{\perp} = \begin{bmatrix} \epsilon I_{(N+1)n\times(N+1)n} & -E \end{bmatrix}. \quad (5)$$

Then, eliminating $\langle U, \widetilde{W} \rangle$ from (3) by using these matrices, we have two inequalities

$$\begin{bmatrix} I_{n\times n} & [A, B]_{U,\widetilde{L}} \langle U, \widetilde{W} \rangle^{-1} \end{bmatrix} \begin{bmatrix} 0_{n\times n} & XE \\ E^T X & 0_{(N+1)n\times(N+1)n} \end{bmatrix} \begin{bmatrix} I_{n\times n} \\ (\langle U, \widetilde{W} \rangle^{-1})^T [A, B]_{U,\widetilde{L}}^T \end{bmatrix}$$
$$= \mathrm{He}\left((A + \sum_{i=1}^{N} B_i K_i) X \right) \prec 0, \quad (6)$$

$$\begin{bmatrix} \epsilon I_{(N+1)n\times(N+1)n} & -E \end{bmatrix} \begin{bmatrix} 0_{n\times n} & XE \\ E^T X & 0_{(N+1)n\times(N+1)n} \end{bmatrix} \begin{bmatrix} \epsilon I_{(N+1)n\times(N+1)n} \\ -E^T \end{bmatrix}$$
$$= -2\epsilon(N+1)X \prec 0. \quad (7)$$

Hence, we see that (6) and (7) state exactly the Lyapunov stability condition with $X \in \mathbb{S}_+^n$ and state-feedback gains $K_j = L_j W_j^{-1} \in \mathcal{K}_j$ for $j = 1, 2, \ldots, N$.

Suppose the system in (1) is stable with state-feedback gains $K_j = L_j W_j^{-1} \in \mathcal{K}_j$ for $W_j \in \mathrm{GL}_n(\mathbb{R})$, $j = 1, 2, \ldots, N$. Then, there exists a sufficiently small $\epsilon > 0$ that satisfies

$$\mathrm{He}\left((A + \sum_{j=1}^{N} B_j K_j) X \right) + \frac{1}{2}\epsilon \left[A, B \right]_{X,\widetilde{L}} \langle X, \widetilde{X} \rangle \left[A, B \right]_{X,\widetilde{L}}^T \prec 0, \quad (8)$$

where

$$\langle X, \widetilde{X} \rangle = \text{block diag} \{ \underbrace{X, X, \ldots, X}_{(N+1)\ times} \} \text{ and } [A, B]_{X,\widetilde{L}} = [\underbrace{AX\ B_1 L_1\ B_2 L_2\ \cdots\ B_N L_N}_{(N+1)\ times}].$$

Note that $\langle X, \widetilde{X} \rangle \succ 0$ and $\langle X, \widetilde{X} \rangle E^T = E^T X$, employing the Schur complement for (8), then we have

$$\begin{bmatrix} \mathrm{He}\left((A + \sum_{j=1}^{N} B_j K_j) X \right) & \epsilon [A, B]_{X,\widetilde{L}} \langle X, \widetilde{X} \rangle \\ \epsilon \langle X, \widetilde{X} \rangle ([A, B]_{X,\widetilde{L}})^T & -2\epsilon \langle X, \widetilde{X} \rangle \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & XE \\ E^T X & 0_{(N+1)n\times(N+1)n} \end{bmatrix}$$
$$+ \mathrm{He}\left(\begin{bmatrix} [A, B]_{X,\widetilde{L}} \langle U, \widetilde{W} \rangle^{-1} \\ -I_{(N+1)n\times(N+1)n} \end{bmatrix} \langle X, \widetilde{X} \rangle \begin{bmatrix} E^T & \epsilon I_{(N+1)n\times(N+1)n} \end{bmatrix} \right) \prec 0.$$

This means that (3) holds with $\langle U, \widetilde{W} \rangle = \langle X, \widetilde{X} \rangle$ for $U \in \mathrm{GL}_n(\mathbb{R})$. □

[1] Recently, a similar dilated LMIs condition has been investigated by Fujisaki and Befekadu [13] in the context of reliable decentralized stabilization for multi-channel systems (see [29], [10] and references therein for a review of dilated LMI technique).

Note that Theorem 1 is a generalization of the square-dilated LMI technique that has been considered in [13] in the context of reliable stabilization for multi-channel systems. In fact, if we multiply equation (3) from the left side by

$$\Gamma = \begin{bmatrix} (N+1)I_{n\times n} & 0_{n\times(N+1)n} \\ 0_{n\times n} & E \end{bmatrix}, \tag{9}$$

and from the right side by its transpose matrix Γ^T. Finally, making use of the following relation $EE^T = (N+1)I$ and set $W_j \to W$ for $j = 1, 2, \cdots, N$ and $U \to W$ (which also gives us the condition $\langle W, \widetilde{W}\rangle E^T = E^T W$), then (3) reduces to

$$\begin{bmatrix} 0 & (N+1)X \\ (N+1)X & 0 \end{bmatrix} + \text{He}\left(\begin{bmatrix} \left(AW + \sum_{j=1}^N B_j L_j\right) \\ -W \end{bmatrix} \right. \\ \left. \times \begin{bmatrix} (N+1)I_{n\times n} & \epsilon_j I_{n\times n} \end{bmatrix} \right) \prec 0, \tag{10}$$

which is basically the square dilated LMI condition presented in [13], i.e., if we further let $\epsilon \to (N+1)\epsilon'$, then we have

$$\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} + \text{He}\left(\begin{bmatrix} \left(AW + \sum_{j=1}^N B_j L_j\right) \\ -W \end{bmatrix} \begin{bmatrix} I_{n\times n} & \epsilon' I_{n\times n} \end{bmatrix} \right) \prec 0. \tag{11}$$

Note that if we set $K_j = L_j W^{-1}$, $j = 1, 2, \ldots, N$, for any $W \in \text{GL}_n(\mathbb{R})$ and a family of N-tuple $(L_1, L_2, \ldots, L_N)$ as above, then $\left(A + \sum_{j=1}^N B_j K_j\right)$ is Hurwitz matrix. Moreover, here it should be noted that a common $W \in \text{GL}_n(\mathbb{R})$ is used for all $\{L_j\}_{j=1}^N$.

Remark 1 We remark that this new dilated LMI framework which is stated above in Theorem 1 provides a sufficient condition that guaranteeing *decentralized* control/strategy spaces $\mathcal{K}_N \triangleq \prod_{j=1}^N \mathcal{K}_j$ (with independent access for each player's strategy space $\mathbb{R}^{r_j \times n} \supseteq \mathcal{K}_j$, $\forall j \in \{1, 2, \cdots, N\}$). Such a solution framework will, therefore, be used (or exploited) to characterize the problem of robust stability in the context of *non-fragile* Nash equilibria via a game-theoretic interpretation.

Consider next a multi-channel system with a perturbation term, i.e.,

$$\dot{x}(t) = \left(A + u_\rho A^\delta\right) x(t) + \sum_{j=1}^N B_j u_j(t), \tag{12}$$

where $u_\rho \in [-\rho, \rho]$, $\rho \in \mathbb{R}_+$ is the uncertainty level and $A^\delta \in \mathbb{R}^{n\times n}$ is a fixed perturbation term in the system. Here we assume that the perturbed matrix $(A + u_\rho A^\delta)$ lies in a compact uncertainty set $\mathcal{U}_\rho \subset \mathbb{R}^{n\times n}$.[2]

In what follows, we assume there exists a set of stabilizing state-feedback gains $\mathcal{K}_N$ that guarantees the stability of the system in (1) and this set is completely characterized via a solution of (3). Then, we will estimate an upper bound $\hat{\rho} \in \mathbb{R}_+$ on the uncertainty level for which the state-feedback gains preserve robust (or *non-fragile*) stability property of the perturbed multi-channel system.

[2] Note that the existence of a solution for state trajectories is well-defined and it is always upper semicontinuous in x_0 (e.g., see references [9] and [21]).

Lemma 1 *Let $X \in \mathbb{S}_+^n$, $\epsilon > 0$, $U \in \mathrm{GL}_n(\mathbb{R})$, $W_j \in \mathrm{GL}_n(\mathbb{R})$ and $L_j \in \mathbb{R}^{r_j \times n}$, $j = 1, 2, \dots, N$ satisfy Theorem 1. Suppose $\alpha > 0$, $\beta \geq 1$ and $Z \in \mathbb{S}_+^n$, then there exist an upper bound $\hat{\rho} \in \mathbb{R}_+$ and $Y \in \mathbb{S}_+^n$ such that*

$$\beta^{-1} Z \preceq Y \preceq Z, \tag{13}$$

$$\begin{bmatrix} I \\ \langle U, \widetilde{W} \rangle^{-1} E^T \end{bmatrix}^T \begin{bmatrix} u_{\hat{\rho}} \,\mathrm{He}((A^{\delta})^T Y) & Y \, [A, B]_{U,\widetilde{L}} \\ ([A, B]_{U,\widetilde{L}})^T Y & 0 \end{bmatrix} \begin{bmatrix} I \\ \langle U, \widetilde{W} \rangle^{-1} E^T \end{bmatrix} \preceq -\alpha Z. \tag{14}$$

Moreover, the perturbed multi-channel system in (12) *is stable for all instances of perturbation $u_{\hat{\rho}} \in [-\hat{\rho}, \hat{\rho}]$ in the system.*

Proof To prove the above theorem, we require the following system

$$\begin{aligned} \dot{x}(t) &= \left(A + u_{\rho} A^{\delta} + \sum_{j=1}^{N} B_j K_j \right) x(t) + 0_{n \times 1} \tilde{u}(t), \\ \tilde{y}(t) &= x(t) + 0_{n \times 1} \tilde{u}(t), \end{aligned} \tag{15}$$

to satisfy certain dissipativity property for all instances of perturbation in the system.

Define the following supply rate

$$w_{[\alpha, Z]}(\tilde{y}(t), \tilde{u}(t)) = \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}^T \begin{bmatrix} -\alpha Z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}, \tag{16}$$

with $Z \in \mathbb{S}_+^n$ and $\alpha > 0$. We clearly see that if the system in (15) is stable for all instances of perturbation, then the following *dissipation inequality* will hold

$$V(x(0)) + \int_0^t w_{[\alpha, Z]}(\tilde{y}(t), \tilde{u}(t)) dt \geq V(x(t)), \tag{17}$$

for all $t \geq 0$ with non-negative quadratic storage function $V(x(t)) = x(t)^T Y x(t)$, $Y \in \mathbb{S}_+^n$ that satisfies $V(0) = 0$.

Condition (17) with (16) further implies the following

$$\mathrm{He}\left(\, (A + u_{\rho} A^{\delta} + \sum_{j=1}^{N} B_j K_j)^T Y \, \right) \preceq -\alpha Z. \tag{18}$$

Therefore, there exists an upper bound $\hat{\rho} \in \mathbb{R}_+$ for which the dissipativity condition in (18) will hold true for all instances of perturbation in the system.

Then, we have the following result

$$\begin{aligned} &\mathrm{He}\left(\left([A, B]_{U,\widetilde{L}} \langle U, \widetilde{W} \rangle^{-1} E^T + u_{\hat{\rho}} A^{\delta} \right)^T Y \right) = \\ &\begin{bmatrix} I \\ \langle U, \widetilde{W} \rangle^{-1} E^T \end{bmatrix}^T \begin{bmatrix} u_{\hat{\rho}} \,\mathrm{He}\left((A^{\delta})^T Y\right) & Y [A, B]_{U,\widetilde{L}} \\ \left([A, B]_{U,\widetilde{L}}\right)^T Y & 0 \end{bmatrix} \begin{bmatrix} I \\ \langle U, \widetilde{W} \rangle^{-1} E^T \end{bmatrix} \preceq -\alpha Z, \end{aligned} \tag{19}$$

with $u_{\hat{\rho}} \in [-\hat{\rho},\ \hat{\rho}]$.[3]

On the other hand, let us define the following matrix interval set in $\mathbb{S}_+^n$

$$\mathcal{I}_{[\beta,Z]} = \left\{ Y \,\middle|\, \beta^{-1} Z \preceq Y \preceq Z \right\}, \tag{20}$$

where $Z \in \mathbb{S}_+^n$ and $\beta \geq 1$ are assumed to be known $a\ priori$. Suppose that Y satisfies the conditions in (13) and (14), then the trajectories of the perturbed closed-loop system

$$\dot{x}(t) = \left(A + u_\rho A^\delta + \sum_{j=1}^{N} B_j K_j \right) x(t),$$

satisfy

$$\begin{aligned} \frac{d}{dt}(x^T(t) Y x(t)) &= x^T(t)\, \mathrm{He}\left(\left(A + u_\rho A^\delta + \sum_{j=1}^{N} B_j K_j \right)^T Y \right) x(t), \\ &\leq -\alpha x^T(t) Z x(t), \\ &\leq -\alpha x^T(t) Y x(t). \end{aligned} \tag{21}$$

Hence, condition (21) stating, equivalently, that $Y \in \mathcal{I}_{[\beta,Z]}$ is a dissipativity certificate with supply rate (16) for all instances of perturbation in (15) (e.g., see references [2], [6]). □

Remark 2 We remark that the parameter $\alpha > 0$ determines the long-term behavior of the system, whereas the parameter $\beta \geq 1$ bounds its short-term or transient behavior. In general, these parameters can be chosen so as to guarantee the robust stability of the system with acceptable decay and transient behavior [16].

Note that if there exists a solution set X for Lemma 1 that gives a minimum distance between X and the set $\mathcal{I}_{[\beta,Z]}$, i.e., $\varrho(X, Y) \triangleq \inf_{Y \in \mathcal{I}_{[\beta,Z]}} \|X - Y\|$, then we essentially have a weak-optimal solution. This solution is unique since $\mathcal{I}_{[\beta,Z]}$ is a non-empty compact and convex set [23]. Moreover, finding a lower upper bound $\hat{\rho} \in \mathbb{R}_+$ and Y from a non-empty compact and convex set $\mathcal{I}_{[\beta,Z]}$ is equivalent to solving the verification problem, i.e., the constrained dissipativity control problem (e.g., see reference [15]).

In the next section, we will see that such additional information structure, i.e., the dissipativity property, about the system is indeed useful in the context of game-theoretic framework.

3 Main results

In this section, we establish an equivalence result between the set of non-fragile state-feedback gains corresponding to constrained dissipativity problem and the feedback Nash equilibrium. Specifically, we consider two fundamental problems in this framework: (i) we

[3] Note that the upper bound $\hat{\rho}$ continuously depends (*in the weak sense*) on x_0 and K_j, $j = 1, 2, \ldots, N$.

first isolate a condition guaranteeing that the control/strategy space is sufficiently *decentralized* to make the game-theoretic scenario/interpretation sensible, and (ii) we then provide a sufficient condition for the existence of *non-fragile* feedback Nash equilibrium, where the individual players have different objective functions that are associated with certain information structures, i.e., the dissipativity inequalities, of the following system.

$$\begin{aligned}\dot{x}(t) &= \left(A + u_{\rho_j} A_j^{\delta} + \sum_{i=1}^{N} B_i K_i\right) x(t) + 0_{n\times 1}\tilde{u}(t),\\ \tilde{y}(t) &= x(t) + 0_{n\times 1}\tilde{u}(t),\end{aligned} \tag{22}$$

where $u_{\rho_j} \in [-\rho_j, \rho_j]$, $\rho_j \in \mathbb{R}_+$ and $A_j^{\delta} \in \mathbb{R}^{n\times n}$ are the uncertainty levels and the perturbation terms associated with the jth-player, respectively. We further assume that each perturbed system matrix $(A + u_{\rho_j} A_j^{\delta})$ lies in a compact uncertainty set $\mathcal{U}_{\rho_j} \subset \mathbb{R}^{n\times n}$ for $j = 1, 2, \ldots, N$ and $(K_1, K_2, \ldots, K_N) \in \mathcal{K}_N$.

Next it will be convenient to identify each objective function $J_j \colon \mathbb{R}^n \times [-\rho_j, \rho_j] \times \mathcal{K}_N \to \mathbb{R}_+$ with related function

$$\mathbb{R}^n \times [-\rho_j, \rho_j] \times \mathcal{K}_N \to \mathbb{R}_+ \colon (x_0, u_{\rho_j}, K_{\neg j}) \mapsto J_j(x_0, u_{\rho_j}, K_{\neg j}), \tag{23}$$

for all $j = 1, 2, \ldots, N$.

Then, we can specify a game Γ in strategic form, i.e., the feedback Nash game, by the following data:

$$\Gamma\left(\mathcal{N}, \mathcal{K}_N, (J_j)_{j\in\mathcal{N}}, \Big((A + u_{\rho_j} A_j^{\delta}), [\,B_1,\, B_2,\, \ldots\, B_N\,]\Big)_{j\in\mathcal{N}}\right),$$

where $\mathcal{N} \triangleq \{1, 2, \ldots, N\}$ is the players set.

Therefore, for such a game in strategic form, an N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$, (i.e., $K^* \triangleq (K_1^*, K_2^*, \ldots, K_N^*)$) is called a feedback Nash equilibrium, if for all $j \in \{1, 2, \ldots, N\}$, for all $K_j \in \mathbb{R}^{r_j\times n}$, all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ and each $x_0 \in \mathbb{R}^n$, we have

$$J_j(x_0, u_{\hat{\rho}_j}, K_{\neg j}^*) \le J_j(x_0, u_{\hat{\rho}_j}, K^*), \tag{24}$$

where $K_{\neg j}^* \triangleq (K_1^*, \ldots, K_{j-1}^*, K_j, K_{j+1}^*, \ldots, K_N^*) \in \mathcal{K}_N$.[4]

In the following, we assume that the strategy space for each player is restricted to linear time-invariant state-feedback gains, and the resulting multi-channel closed-loop system is also assumed to be stable for all (or some) initial conditions $x_0 \in \mathbb{R}^n$.

Introduce the following set of supply rate functions

$$\mathcal{W} = \left\{ w_{[\alpha_j, Z_j]}(\tilde{y}(t), \tilde{u}(t)) \,\middle|\, w_{[\alpha_j, Z_j]}(\tilde{y}(t), \tilde{u}(t)) = \begin{bmatrix}\tilde{y}(t)\\ \tilde{u}(t)\end{bmatrix}^T \begin{bmatrix}-\alpha_j Z_j & 0\\ 0 & I\end{bmatrix} \begin{bmatrix}\tilde{y}(t)\\ \tilde{u}(t)\end{bmatrix} \right\}, \tag{25}$$

[4] In this paper, the game is essentially defined in the framework of an incomplete information, since the jth-player's objective function involves different uncertainty information, i.e., u_{ρ_j}, about the system. However, we remark that the jth-player decides his own strategy by solving the optimization problem with the opponents' strategies $(K_1^*, \ldots, K_{j-1}^*, K_{j+1}^*, \ldots, K_N^*)$ fixed.

for $j = 1, 2, \ldots, N$, and a matrix interval set $\mathcal{I}_{[\beta_j, Z_j]}$ in $\mathbb{S}_+^n$

$$\mathcal{I}_{[\beta_j, Z_j]} = \left\{ Y_j \;\middle|\; \beta_j^{-1} Z_j \preceq Y_j \preceq Z_j \right\}, \tag{26}$$

where $\alpha_j > 0$, $\beta_j \geq 1$ and $Z_j \in \mathbb{S}_+^n$ for $j = 1, 2, \ldots, N$.

In light of Lemma 1 and above discussion, we have the following theorem which provides a sufficient condition for the existence of feedback Nash equilibria.

Theorem 2 *Let $W_j \in \mathrm{GL}_n(\mathbb{R})$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$. Assume that $\alpha_j > 0$, $\beta_j \geq 1$ and $Z_j \in \mathbb{S}_+^n$ for $j = 1, 2, \ldots, N$. Then, there exit $X_j \in \mathbb{S}_+^n$, $U_j \in \mathrm{GL}_n(\mathbb{R})$, $j = 1, 2, \ldots, N$ and an N-tuple $(L_1^*, L_2^*, \ldots, L_N^*) \in \prod_{j=1}^N \mathbb{R}^{r_j \times n}$ such that*

$$\begin{bmatrix} 0 & X_j E \\ E^T X_j & 0 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} [A, B]_{U_j, \widetilde{L}^*_{\neg j}} \\ -\langle U_j, \widetilde{W} \rangle \end{bmatrix} \begin{bmatrix} E^T & \epsilon_j I \end{bmatrix} \right) \prec 0, \tag{27}$$

where, for some $L_j \in \mathbb{R}^{r_j \times n}$, $j = 1, 2, \ldots, N$,

$$[A, B]_{U_j, \widetilde{L}^*_{\neg j}} = \begin{bmatrix} AU_j & B_1 L_1^* & \cdots & B_{j-1} L_{j-1}^* & B_j L_j & B_{j+1} L_{j+1}^* & \cdots & B_N L_N^* \end{bmatrix},$$

and

$$\langle U_j, \widetilde{W} \rangle = \mathrm{block\,diag}\{U_j, W_1, \ldots, W_{j-1}, W_j, W_{j+1}, \ldots, W_N\}.$$

Further, there exist $Y_j \in \mathcal{I}_{[\beta_j, Z_j]}$ and $\hat{\rho}_j(x_0, K^*)$ for $j = 1, 2, \ldots, N$ such that

$$\sup_{(x_0, u_{\rho_j}, K_j) \in \mathbb{R}^n \times [-\rho_j, \rho_j] \times \mathbb{R}^{r_j \times n}} J_j(x_0, u_{\rho_j}, K^*_{\neg j}) \ni \hat{\rho}_j(x_0, K^*), \tag{28}$$

for which the perturbed systems in (22) are robustly stable for all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ with $K_j^* \in \underset{K_j \in \mathbb{R}^{r_j \times n}}{\arg\sup}\, J_j(x_0, u_{\hat{\rho}_j}, K^*_{\neg j}) \{\triangleq \hat{\rho}_j(x_0, K^*)\}$ for all $j = 1, 2, \ldots, N$.[5]

Proof Suppose all the perturbed systems in (22) satisfy the following dissipativity inequalities

$$V_j(x(0)) + \int_0^t w_{[\alpha_j, Z_j]}(\tilde{y}(t), \tilde{u}(t)) dt \geq V_j(x(t)), \tag{29}$$

for all $t \geq 0$ with non-negative quadratic storage functions $V_j(x(t)) = x(t)^T Y_j x(t)$ and $Y_j \in \mathcal{I}_{[\beta_j, Z_j]}$ that satisfy $V_j(0) = 0$ for $j = 1, 2, \ldots, N$.

Thus, the trajectories of each perturbed closed-loop system (i.e., for $j = 1, 2, \ldots, N$)

$$\dot{x}(t) = \left(A + u_{\rho_j} A_j^\delta + \sum_{i=1}^N B_i K_i^* \right) x(t),$$

[5] In general, simultaneously solving a set of optimization problems, i.e., solving (28) together with (27), is not easy since it is a non-convex optimization problem which involves finding a solution satisfying at the intersection of a set of constrained quadratic functionals [42] (c.f. Remark 3, Section 2 above).

satisfy

$$\frac{d}{dt}\left(x^T(t)Y_j x(t)\right) = x^T(t)\,\mathrm{He}\left(\left(A + u_{\rho_j}A_j^{\delta} + \sum_{i=1}^{N} B_i K_i^*\right)^T Y_j\right)x(t),$$
$$\leq -\alpha_j x^T(t) Z_j x(t),$$
$$\leq -\alpha_j x^T(t) Y_j x(t). \tag{30}$$

for all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ in the system.

Then, the rest of the proof follows the same lines as that of Theorem 1. In fact, replacing the following

$$[A, B]_{U,\widetilde{L}} \to [A, B]_{U_j, \widetilde{L}^*_{\neg j}}, \quad \langle U, \widetilde{W} \rangle \to \langle U_j, \widetilde{W} \rangle \quad \text{and} \quad X \to X_j,$$

in Theorem 1 immediately gives the condition in (27) of Theorem 2. Note that K_j^* and K_j are given by

$$K_j^* = L_j^* W_j^{-1} \qquad \text{and} \qquad K_j = L_j W_j^{-1},$$

for $j = 1, 2, \ldots, N$.

Moreover, the N-tuple $(Y_1, Y_2, \cdots, Y_N) \in \prod_{j=1}^N \mathcal{I}_{[\beta_j, Z_j]}$ is a collection of dissipativity certificates corresponding to a set of supply rates (25) for all instances of perturbation in (22). □

We next present a more realistic game-theoretic interpretation in terms of the upper uncertainty bounds $\hat{\rho}_j \in \mathbb{R}_+$ for all $j = 1, 2, \ldots, N$ that describe the N-tuple uncertainty set $(u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \cdots, u_{\hat{\rho}_N}) \in \prod_{j=1}^N [-\hat{\rho}_j, \hat{\rho}_j]$ together with the existence of stabilizing state-feedback gains that provide a sufficient condition for obtaining a set of feedback Nash equilibria.

Now, we will prove the equivalence of the following statements:

(i). $\exists K^* \in \mathcal{K}_N,\ \forall x_0,\ \forall u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j],\ \forall K^*_{\neg j} \in \mathcal{K}_N,\ \forall j \in \{1, 2, \ldots, N\}$ such that

$$J_j(x_0, u_{\hat{\rho}_j}, K^*_{\neg j}) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*). \tag{31}$$

(ii). The dilated LMIs condition in (27) and the dissipativity inequalities of (29) with a set of supply rates $\mathcal{W}$ in (25) completely describes the set of non-fragile stabilizing state-feedback gains.

The equivalence between (i) and (ii) leads to characterization of feedback Nash equilibria over an infinite-time horizon in terms of stabilizing solutions of a set of dilated LMIs.

Furthermore, the exact characterization of the feedback Nash equilibria is given by the following two theorems.

Theorem 3 *Let $W_j \in \mathrm{GL}_n(\mathbb{R})$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$. Suppose $X_j \in \mathbb{S}^n_+$, $U_j \in \mathrm{GL}_n(\mathbb{R})$, $L_j^* \in \mathbb{R}^{r_j \times n}$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$ satisfy the dilated LMIs condition in* (27)*. Then, there exists an N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ feedback Nash equilibrium with respect to the upper uncertainty bounds $\hat{\rho}_j \in \mathbb{R}_+$ for $j = 1, 2 \ldots, N$ of* (28).

Proof The first part of this theorem is already provided in Theorem 2, i.e., from the standard argument of the stabilizability of the pair $(A, [\, B_1 \; B_2 \; \cdots \; B_N \,])$, we can always find an N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ and for all $K_j = L_j W_j^{-1} \in \mathbb{R}^{r_j \times n}$ and $j = 1, 2, \ldots, N$ such that (27) holds. Applying (28) of Theorem 2 together with the dissipativity certificates $Y_j \in \mathcal{I}_{[\beta_j, Z_j]}$ of (26) and a set of supply rate functions $\mathcal{W}$ of (25). Then, for a fixed $(x_0, K^*) \in \mathbb{R}^n \times \mathcal{K}_N$, we will obtain an upper bound $\hat{\rho}_j \in \mathbb{R}_+$ for all instances of perturbation in (22) and so that

$$J_j(x_0, u_{\hat{\rho}_j}, K_{\neg j}^*) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*),$$

for all $j = 1, 2, \ldots, N$.

Hence, we immediately see that the N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ satisfies the feedback Nash equilibrium.[6] □

Remark 3 The class of admissible strategies for all players are generated through a set of individual objective functions that are induced from dissipativity inequalities of (29) with a set of supply rates (25).

Theorem 4 *Suppose the N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ is a feedback Nash equilibrium with respect to the values of the objective functions of* (28). *Assume that $W_j \in \mathrm{GL}_n(\mathbb{R})$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$. Then, there exists a solution set $X_j \in \mathbb{S}_+^n$, $U_j \in \mathrm{GL}_n(\mathbb{R})$ and $L_j^* \in \mathbb{R}^{r_j \times n}$ for $j = 1, 2, \ldots, N$ that satisfies the dilated LMIs condition of* (27).

Proof Suppose the N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ is a feedback Nash equilibrium such that

$$J_j(x_0, u_{\hat{\rho}_j}, K_{\neg j}^*) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*),$$

where the value for the continuous objective function $J_j \colon \mathbb{R}^n \times [-\rho_j, \rho_j] \times \mathcal{K}_N \to \mathbb{R}_+$ is claimed as

$$\sup_{(x_0, u_{\rho_j}, K_j) \in \mathbb{R}^n \times [-\rho_j, \rho_j] \times \mathbb{R}^{r_j \times n}} J_j(x_0, u_{\rho_j}, K_{\neg j}) \ni \hat{\rho}_j(x_0, K),$$

[6] Here we remark that a strong version of fixed-point theorem is required to establish the existence of feedback Nash equilibria for the game Γ, which is defined on compact topological spaces with continuous objective functions (e.g., see [14]). To this end, if we introduce the following continuous map $\Phi_{[x_0, u_{\hat{\rho}}]} \colon \mathcal{K}_N \times \mathcal{K}_N \to \mathbb{R}$ defined by

$$\Phi_{[x_0, u_{\hat{\rho}}]}(K, \bar{K}) = \sum_{j=1}^{N} \Big(J_j(x_0, u_{\hat{\rho}_j}, K) - J_j(x_0, u_{\hat{\rho}_j}, K_{\neg j}) \Big),$$

where $K = \big(K_1, K_2, \ldots, K_N \big) \in \mathcal{K}_N$, $\bar{K} = \big(\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_N \big) \in \mathcal{K}_N$, $K_{\neg j} = \big(K_1, \ldots, \bar{K}_j \ldots, K_N \big) \in \mathcal{K}_N$, $j = 1, 2, \ldots, N$ and $u_{\hat{\rho}} \triangleq (u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \ldots, u_{\hat{\rho}_N}) \in \prod_{j=1}^{N} [-\hat{\rho}_j, \hat{\rho}_j]$. Note that for such a map whose fixed-point is an equilibrium is called a Nash map for the game Γ, i.e., if the N-tuple $\big(K_1^*, K_2^*, \ldots, K_N^* \big)$ is a feedback Nash equilibrium, then $J_j(x_0, u_{\hat{\rho}_j}, K_{\neg j}^*) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*)$ with $K_j \in \mathcal{K}_j$ for all $j \in \{1, 2, \ldots, N\}$. This further shows that the map Φ satisfies $\Phi_{[x_0, u_{\hat{\rho}}]}(K^*, K) \geq 0$ for any arbitrary $K = \big(K_1, K_2, \ldots, K_N \big) \in \mathcal{K}_N$. Therefore, the feedback Nash equilibrium K^* is an equilibrium point, i.e., a fixed point, for the map $\Phi_{[x_0, u_{\hat{\rho}}]}(., .)$.

with $K_j^* \in \underset{K_j \in \mathbb{R}^{r_j \times n}}{\arg\sup}\, J_j(x_0, u_{\hat{\rho}_j}, K_{\neg j}^*) \{\triangleq \hat{\rho}_j(x_0, K^*)\}$ for all $j \in \{1, 2, \ldots, N\}$.

Then, we can always find a solution set that satisfies the dilated LMIs condition in (27) for which the closed-loop systems in (22) are robustly stable for all instances of perturbations $(u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \cdots, u_{\hat{\rho}_N}) \in \prod_{j=1}^{N} [-\hat{\rho}_j, \hat{\rho}_j]$. □

Remark 4 Note that all closed-loop systems in (22) satisfy the dissipative inequality properties of (29) with a set of supply rates (25) for all $j \in \{1, 2, \ldots, N\}$ and instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$.

Finally, the feedback Nash equilibrium has a strong time consistency property. This fact corresponds to the information structure that is associated with the dissipative inequalities of the system where the equilibrium trajectory $x_{eq}(t)$ (or the equilibrium point $x(0) = x_0$) of the system if it is truncated part in the time interval $[T, \infty)$, where $T > 0$, asymptotically represents an equilibrium (c.f. references [40], [43]).[7] This further implies any (sub-)game starting at $t = T$, does not depend on the initial condition $x_{eq}(T)$ (e.g., see references [27], [19]). Moreover, the game, where the class of admissible strategies for all players is induced from a solution set of the individual objective functions (23), is an infinite-time horizon game. Thus, this game has non-unique feedback Nash equilibrium solutions that are associated with a set of non-fragile stabilizing state-feedback gains of Theorem 2.

Remark 5 Note that the equivalence between (i) and (ii) (i.e., Theorem 3: (ii) ⇒ (i) and Theorem 4: (i) ⇒ (ii)) leads exactly to characterization of the feedback Nash equilibrium via a set of non-fragile stabilizing state-feedback solutions of the dilated LMIs.

4 Illustrative example

As an illustration, let us consider the following simple example with nominal system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Next, we are interested in finding the largest uncertainty levels for which there exist instance-independent feedback Nash equilibria with the following structural information:

(i) for each agent the perturbation term is assumed to vary independently and at most $\hat{\rho}_j$ times the nominal value of the system matrix A, i.e., $u_{\rho_j} A$ with $u_{\rho_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ for $j = 1, 2, 3$.

(ii) the matrix intervals $\mathcal{I}_{[\beta_j, Z_j]} = \{Y \mid \beta_j^{-1} Z_j \preceq Y \preceq Z_j\} \subset \mathbb{S}_+^3$ for $j = 1, 2, 3$

$$Z_j \triangleq Y_{[req]} = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix},$$

[7] Note that the stability behavior is considered here over an infinite-time horizon.

where $\beta \triangleq (\beta_1, \beta_2, \beta_3) = (10, 10, 10)$. Note that the matrix $Y_{[req]}$ can be associated with a minimum possible cost, i.e., $\tilde{x}^T Y_{[req]} \tilde{x}$, which is required to bring the nominal system from an equilibrium to (any) given state $\tilde{x}$, when an LQR minimizing cost functional $J_{LQR}(x, u) = \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} u_i^2$ is used for the nominal system.

(iii) the supply rate functions for $j = 1, 2, 3$

$$w_{[\alpha_j, Z_j]}(\tilde{y}(t), \tilde{u}(t)) = \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}^T \begin{bmatrix} -\alpha_j Z_j & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix},$$

where $\alpha \triangleq (\alpha_1, \alpha_2, \alpha_3) = (0.35, 0.25, 0.45)$.

Here, the corresponding upper bounds $\hat{\rho} \triangleq \{\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3\}$ on the perturbation levels, that are computed together with the dissipativity certifications (Y_1, Y_2, Y_3) from the compact set $\prod_{j=1}^{3} \mathcal{I}_{[\beta_j, Z_j]}$ are given by

$$\hat{\rho}_1 = 5.4251, \qquad \hat{\rho}_2 = 3.8923, \qquad \text{and} \qquad \hat{\rho}_3 = 4.3576.$$

Moreover, the corresponding instance-independent feedback Nash strategies are given by $K_1^* = \begin{bmatrix} -0.9398 & 0.0120 & -0.0808 \end{bmatrix}$, $K_2^* = \begin{bmatrix} -0.0096 & -0.8912 & 0.0096 \end{bmatrix}$ and $K_3^* = \begin{bmatrix} -0.0808 & -0.0120 & -0.9336 \end{bmatrix}$ with $\epsilon_1 = 1$, $\epsilon_2 = 1$ and $\epsilon_3 = 1$. Note that the corresponding eigenvalues of the nominal closed-loop system when the instance-independent feedback Nash strategies (K_1^*, K_2^*, K_3^*) placed in the system are given by $\lambda_1 = -1.0205$ and $\lambda_2, \lambda_2 = -0.8751 \pm j1.4294$.

We finally remark that for very large values of β_j, $j = 1, 2, 3$, the instance-independent feedback Nash strategies are slightly different from the one computed above. However, the upper bounds $\hat{\rho} \triangleq (\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)$ on the perturbation levels approximately remained the same. This fact basically corresponds with relaxation of the strict conditions on the matrices Y_j for $j = 1, 2, 3$, i.e., Y_j can be any positive definite matrix from $\mathcal{I}_{[0, Z_j]} = \{Y_j \mid 0 \prec Y_j \preceq Z_j\} \subset \mathbb{S}_+^3$.

5 Concluding remarks

In this paper, we have looked the problem of state-feedback stabilization for a multi-channel system in a game-theoretic framework, where the class of admissible strategies for the players is induced from a solution set of the objective functionals that are realized through certain dissipativity inequality property of the system. In such a scenario, we characterized the feedback Nash equilibria via a set of non-fragile stabilizing state-feedback gains corresponding to a family of constrained dissipativity problems. Moreover, we showed that the existence of a weak-optimal solution to the family of constrained dissipativity problems is a sufficient condition for the existence of a feedback Nash equilibrium, with the latter having a nice property of strong time consistency.

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