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# Characterization of graphs with rainbow connection number $m-2$ and $m-3$*

Xueliang Li    Yuefang Sun    Yan Zhao

*Center for Combinatorics and LPMC-TJKLC*
*Nankai University, Tianjin 300071*
*P.R. China*
lxl@nankai.edu.cn    zhaoyan2010@mail.nankai.edu.cn
syf@cfc.nankai.edu.cn

**Abstract**

A path in an edge-colored graph, where adjacent edges may be colored the same, is a rainbow path if no two edges of it are colored the same. A nontrivial connected graph $G$ is rainbow connected if there is a rainbow path connecting any two vertices, and the rainbow connection number of $G$, denoted by $rc(G)$, is the minimum number of colors that are needed in order to make $G$ rainbow connected. Chartrand et al. showed that $G$ is a tree if and only if $rc(G) = m$, and it is easy to see that $G$ is not a tree if and only if $rc(G) \leq m-2$, where $m$ is the number of edges of $G$. So an interesting problem arises: Characterize the graphs $G$ with $rc(G) = m-2$. In this paper, we resolve this problem. Furthermore, we also characterize the graphs $G$ with $rc(G) = m-3$.

## 1 Introduction

All graphs in this paper are finite, undirected and simple. We follow the terminology and notation of Bondy and Murty [1]. Let $G$ be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \ldots, \ell\}$, $\ell \in \mathbb{N}$, of the edges of $G$, where adjacent edges may be colored the same. A path is a *rainbow path* if no two edges of it are colored the same. An edge-colored graph $G$ is *rainbow connected* if any two vertices are connected by a rainbow path. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, we define the *rainbow connection number* of a connected graph $G$, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make $G$ rainbow connected.

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If $G_1$ is a connected spanning subgraph of $G$, then $rc(G) \leq rc(G_1)$. Chartrand et al. [3] obtained that $rc(G) = 1$ if and only if $G$ is complete, and that $rc(G) = m$ if and only if $G$ is a tree, as well as that a cycle with $k > 3$ vertices has rainbow connection number $\lceil \frac{k}{2} \rceil$, and a triangle has rainbow connection number 1. Also notice that, clearly, $rc(G) \geq \text{diam}(G)$, where $\text{diam}(G)$ denotes the diameter of $G$. For more information on rainbow connections, we refer to [4, 6]. In an edge-colored graph $G$, we use $c(e)$ to denote the color of edge $e$ and for a subgraph $G_2$ of $G$, $c(G_2)$ denotes the set of colors of edges in $G_2$.

Since $rc(G) = m$ if and only if $G$ is a tree, $rc(G) \neq m-1$ and $G$ is not a tree if and only if $rc(G) \leq m - 2$ (Observation 3 below), then there is an interesting problem: Characterize the graphs with $rc(G) = m-2$. In this paper, we resolve this problem. Furthermore, we also characterize the graphs $G$ with $rc(G) = m - 3$.

We use $V(G)$, $E(G)$ for the set of vertices and edges of $G$, respectively. A *pendant edge* of $G$ is an edge incident to a vertex of degree 1. The girth of $G$, denoted by $g(G)$, is the length of a smallest cycle in $G$. A block of $G$ is a maximal connected subgraph of $G$ that does not have any cut vertex. So every block of a nontrivial connected graph is either a $K_2$ or a 2-connected subgraph. All the blocks of a graph $G$ form a block decomposition of $G$. A *rooted tree* $T(x)$ is a tree $T$ with a specified vertex $x$, called the *root* of $T$. Let $L(x)$ denote the set of leaves of $T(x)$ and $|L(x)| = l(x)$. If $T(x)$ is a trivial tree, then $l(x) = 0$. We let $P_n$ and $C_n$ be the path and cycle with $n$ vertices, respectively. And $xPy$ denotes a path from $x$ to $y$. Let $[t] = \{1, \ldots, t\}$ denote the set of the first $t$ natural numbers. For a set $S$, $|S|$ denotes the cardinality of $S$.

## 2 Some basic results

We first give an observation which will be useful in the sequel.

**Observation 1.** [5] If $G$ is a connected graph and $\{E_i\}_{i\in[t]}$ is a partition of the edge set of $G$ into connected subgraphs $G_i = G[E_i]$, then

$$rc(G) \leq \sum_{i=1}^{t} rc(G_i).$$

□

We now give a necessary condition for an edge-colored graph to be rainbow connected. If $G$ is rainbow connected under some edge-coloring, then for any two cut edges (if they exist) $e_1 = u_1u_2$ and $e_2 = v_1v_2$, there must exist some $1 \leq i, j \leq 2$, such that any $u_i - v_j$ path must contain edge $e_1, e_2$. So we have:

**Observation 2.** If $G$ is rainbow connected under some edge-coloring $c$ where $e_1$ and $e_2$ are any two cut edges, then $c(e_1) \neq c(e_2)$.

For a connected graph $G$, if it is a tree, then $rc(G) = m$; if it contains a unique cycle of length $k$, then we give the cycle a rainbow coloring using $\lceil \frac{k}{2} \rceil$ colors (if the cycle is a triangle, we just need one color) and color each other edge with a fresh color. Then by Observation 1, we have $rc(G) \leq (m - k) + \lceil \frac{k}{2} \rceil \leq m - 2$. So we have the following observation.

**Observation 3.** Let $G$ be a connected graph with $m$ edges. Then $rc(G) \neq m - 1$ and $G$ is not a tree if and only if $rc(G) \leq m - 2$. Moreover, if $G$ contains a cycle of length $k (k \geq 4)$, then $rc(G) \leq m - \lfloor \frac{k}{2} \rfloor$.

For a connected graph $G$, if it contains two edge-disjoint 2-connected subgraphs $B_1$ and $B_2$, then by Observation 3, we give $B_1$ and $B_2$ a rainbow coloring using $|E(B_1)| - 2$ and $|E(B_2)| - 2$ colors, respectively, and color each other edge with a fresh color. Then by Observation 1, we have $rc(G) \leq m - 4$. So the following lemma holds.

**Lemma 1.** *Let $G$ be a connected graph with $m$ edges. If it contains two edge-disjoint 2-connected subgraphs, then $rc(G) \leq m - 4$.*

To *subdivide* an edge $e$ is to delete $e$, add a new vertex $x$, and join $x$ to the ends of $e$. Any graph derived from a graph $G$ by a sequence of edge subdivisions is called a *subdivision* of $G$. Given a rainbow coloring of $G$, if we subdivide an edge $e = uv$ of $G$ by $xu$ and $xv$, then we assign $xu$ the same color as $e$ and assign $xv$ a new color, which also make the subdivision of $G$ rainbow connected. Hence, the following lemma holds.

**Lemma 2.** *Let $G$ be a connected graph, and $H$ be a subdivision of $G$. Then $rc(H) \leq rc(G) + |E(H)| - |E(G)|$.*

The $\Theta$-*graph* is a graph consisting of three internally disjoint paths with common end vertices and of lengths $a$, $b$, and $c$, respectively, such that $a \leq b \leq c$. Then $a + b + c = m$.

**Lemma 3.** *Let $G$ be a $\Theta$-graph with $m$ edges. If $m = 5$, then $rc(G) = m - 3$; otherwise, $rc(G) \leq m - 4$.*

*Proof.* Let the three internally disjoint paths be $P_1, P_2, P_3$ with the common end vertices $u$ and $v$, and the lengths of $P_1, P_2, P_3$ be $a, b, c$, respectively, where $a \leq b \leq c$. If $m = 5$, we color $uP_1v$ with color 1, $uP_2v$ with colors $1, 2$, and $uP_3v$ with colors $2, 1$. The resulting coloring makes $G$ rainbow connected. Thus, $rc(G) \leq m - 3$. Since $\text{diam}(G) = 2$, it follows that $rc(G) = m - 3$. For $m \geq 6$, we first consider the graph $\Theta_1$ with $a = 1$, $b = 2$ and $c = 3$. We color $uP_1v$ with color 1, $uP_2v$ with colors $1, 1$, and $uP_3v$ with colors $2, 1, 2$. Next we consider the graph $\Theta_2$ with $a = 2$, $b = 2$ and $c = 2$. We color $uP_1v$ with colors 1,2, $uP_2v$ with colors $2, 1$, and $uP_3v$ with colors $2, 2$. The resulting colorings make $\Theta_1$ and $\Theta_2$ rainbow connected. For a general $\Theta$-graph $G$ with $m \geq 6$, it is a subdivision of $\Theta_1$ or $\Theta_2$, hence by Lemma 2, $rc(G) \leq m - 4$. □

## 3 Characterizing unicyclic graphs with $rc(G) = m - 2$ and $m - 3$

In this section we first give an observation about unicyclic graphs which will be used frequently. Let $G$ be a connected unicyclic graph with the unique cycle $C = v_1v_2 \dots v_sv_1$. For brevity, orient $C$ clockwise. Then $G$ has the structure as follows: a tree, denoted by $T(v_i)$, is attached at each vertex $v_i$ of $C$. Note that, $T(v_i)$ may be trivial. Let $i \neq j$. If $e_i = x_iy_i(e_j = x_jy_j)$ is a pendant edge which belongs to a tree $T(v_i)(T(v_j))$. Then there is a unique path $x_iP_iv_i(x_jP_jv_j)$ from $x_i(x_j)$ to $v_i(v_j)$. Since $v_i$ and $v_j$ divide $C$ into two segments $v_iCv_j$ and $v_jCv_i$, there are exactly two paths between $x_i$ and $x_j$ in $G$. Let $c = \{1, 2, \dots, \ell\}$ be an edge coloring of $G$. Since each edge in $G \setminus E(C)$ is a cut edge, by Observation 2, they must obtain distinct colors. It is easy to see that $|c(x_iP_iv_i) \cap c(C)| \leq 1$. In the process of coloring, we always first color $G \setminus E(C)$ with $[t]$ colors, then color $C$, where $t = |E(G) \setminus E(C)|$. Thus, after coloring $E(G) \setminus E(C)$, the unique path $x_iP_iv_i$ can be viewed as a pendant edge and every $T(v_i)$ will be a star with the center vertex $v_i$. Suppose $|c(x_iP_iv_i) \cap c(C)| = 1$ and $|c(x_jP_jv_j) \cap c(C)| = 1$, then we can adjust the colors of cut edges such that $c(e_i) = 1$ and $c(e_j) = 2$. Thus, $1, 2 \in v_iCv_j$ or $1, 2 \in v_jCv_i$, namely, 1,2 can only be assigned in the same path from $v_i$ to $v_j$. Moreover, another path from $v_i$ to $v_j$ should be rainbow. We summarize the above argument into an observation.

**Observation 4.** Let $G$ be a connected unicyclic graph with the unique cycle $C = v_1v_2 \dots v_sv_1$, and let $c = \{1, 2, \dots, \ell\}$ be an edge coloring of $G$. Let $p \in T(v_i)$ and $q, r \in T(v_j)$.

(i) If $p, q \in C$, then they are in the same path from $v_i$ to $v_j$ and the other path from $v_i$ to $v_j$ should be rainbow.

(ii) If $q, r$ are in the unique path from a vertex $x$ of $V(G) \setminus V(C)$ to $v_j$, then $q$ and $r$ can not both belong to $C$.

In this section we only deal with unicyclic graphs. According to the girth of $G$, we introduce some graph classes and discuss them by some lemmas. Note that, $l(v_i)$ is the number of leaves of the tree attached at the vertex $v_i$ from the unique cycle of $G$.

Let $i$ be an integer with $1 \leq i \leq 3$ and the addition is performed modulo 3. Let $\mathcal{G} = \{G : m = n, g(G) = 3\}$, $\mathcal{G}_1 = \{G : G \in \mathcal{G}, l(v_i) \geq 1, l(v_{i+1}) \geq 1, l(v_{i+2}) \geq 1, \ or \ l(v_i) \geq 3\}$, $\mathcal{G}_2 = \{G : G \in \mathcal{G}, l(v_i) = 0, l(v_{i+1}) \leq 2, l(v_{i+2}) \leq 2\}$. Obviously, $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$.

**Lemma 4.** *Let $G$ be a graph belonging to $\mathcal{G}$. If $G \in \mathcal{G}_1$, then $rc(G) = m - 3$; otherwise $rc(G) = m - 2$.*

*Proof.* Let the unique cycle of $G$ be $C = v_1v_2v_3v_1$. Suppose $G \in \mathcal{G}_1$, by Observation 2, each edge of $G \setminus E(C)$ must obtain a distinct color, color them with a set $[m-3]$ of colors. We consider two cases. Without loss of generality, first suppose that $e_i = x_iy_i$

is a pendant edge in $T(v_i)$ that is assigned color $i$, where $1 \leq i \leq 3$. Set $c(v_1v_2) = 3$, $c(v_2v_3) = 1$, $c(v_3v_1) = 2$. Next suppose that $e_j = x_jy_j$ is a pendant edge of $T(v_1)$ that is assigned color $j$, where $1 \leq j \leq 3$. Color $E(C)$ with 1,2,3, respectively. It is easy to show that these two colorings are rainbow, and in these two cases, $rc(G) = m-3$.

If $G \in \mathcal{G}_2$, by Observation 3, $rc(G) \leq m-2$. By Observation 4, we know that at most two colors for $G \setminus E(C)$ can be assigned to $C$. Thus, we need a fresh color for $C$, and it follows that $rc(G) \geq m-2$. Therefore, $rc(G) = m-2$. □

Let $i$ be an integer with $1 \leq i \leq 4$ and the addition is performed modulo 4. Set $\mathcal{H} = \{G : m = n, g(G) = 4\}$. Then $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$, where $\mathcal{H}_1 = \{G : G \in \mathcal{H}, l(v_i) = l(v_{i+2}) = 0, l(v_{i+1}) \leq 1, l(v_{i+3}) \leq 1\}$, $\mathcal{H}_2 = \{G : G \in \mathcal{H}, l(v_i) \geq 4, \textit{or } l(v_i) \geq 1, l(v_{i+1}) \geq 2, l(v_{i+2}) \geq 1\}$, and $\mathcal{H}_3$ is the set of the rest unicyclic graphs with girth 4.

**Lemma 5.** *Let $G$ be a graph belonging to $\mathcal{H}$. If $G \in \mathcal{H}_1$, then $rc(G) = m-2$; if $G \in \mathcal{H}_2$, then $rc(G) = m-4$; if $G \in \mathcal{H}_3$, then $rc(G) = m-3$.*

*Proof.* Let the unique cycle of $G$ be $C = v_1v_2v_3v_4v_1$. By Observation 2, each edge of $G \setminus E(C)$ must obtain a distinct color, this costs $m-4$ colors, thus $rc(G) \geq m-4$. Color $G \setminus E(C)$ with a set $[m-4]$ of colors. Suppose $G \in \mathcal{H}_1$. By Observation 3, $rc(G) \leq m-2$. By Observation 4, we know that at least two colors different from $c(G \setminus E(C))$ should be assigned to $C$, so it follows that $rc(G) \geq m-2$. Hence, $rc(G) = m-2$.

Suppose $G \in \mathcal{H}_2$. First let $e_i = x_iy_i$ be a pendant edge in $T(v_1)$ that is assigned color $i$, where $1 \leq i \leq 4$. Color $E(C)$ with $1, 2, 3, 4$, respectively. Next suppose that $e_j = x_jy_j$ is a pendant edge that is assigned color $j$ such that $1 \in T(v_1)$, $2, 3 \in T(v_2)$ and $4 \in T(v_3)$, where $1 \leq j \leq 4$. Set $c(v_1v_2) = 4$, $c(v_2v_3) = 1$, $c(v_3v_4) = 3$, $c(v_1v_4) = 2$. It is easy to show that these two colorings are rainbow, and in these two cases, $rc(G) = m-4$.

If $G \in \mathcal{H}_3$, by Observation 4, we check one by one that at least one color different from $c(G \setminus E(C))$ should be assigned to $C$, thus $rc(G) \geq m-3$. If $e_1$ and $e_2$ are two pendant edges in a tree (say $T(v_1)$) that are assigned colors 1 and 2, respectively, then set $c(v_1v_2) = m-3$, $c(v_2v_3) = 1$, $c(v_3v_4) = 2$, $c(v_1v_4) = m-3$. By symmetry, it remains to consider the case that $l(v_1) = l(v_2) = l(v_3) = 1$. Suppose that $e_i = x_iy_i$ is a pendant edge in $T(v_i)$ that is assigned color $i$, where $1 \leq i \leq 3$. Set $c(v_1v_2) = 3$, $c(v_2v_3) = 1$, $c(v_3v_4) = m-3$, $c(v_1v_4) = 2$. It is easy to show that these two colorings are rainbow, and in these two cases, $rc(G) = m-3$. □

Let $i$ be an integer with $1 \leq i \leq 5$ and the addition is performed modulo 5. Set $\mathcal{J} = \{G : m = n, g(G) = 5\}$ and $\mathcal{J} = \mathcal{J}_1 \cup \{C_5\} \cup \mathcal{J}_2$, where $\mathcal{J}_1 = \{G : G \in \mathcal{J}, l(v_i) \leq 2, l(v_{i+2}) \leq 1, l(v_{i+1}) = l(v_{i+3}) = l(v_{i+4}) = 0 \textit{ or } l(v_i) \leq 1, l(v_{i+1}) \leq 1, l(v_{i+2}) \leq 1, l(v_{i+3}) = l(v_{i+4}) = 0\}$, and $\mathcal{J}_2$ is the set of the rest unicyclic graphs with girth 5.

**Lemma 6.** *Let $G$ be a graph belonging to $\mathcal{J}$. If $G$ is isomorphic to a cycle $C_5$, then $rc(G) = m-2$. If $G \in \mathcal{J}_1$, then $rc(G) = m-3$. If $G \in \mathcal{J}_2$, then $rc(G) \leq m-4$.*

*Proof.* Let the unique cycle of $G$ be $C = v_1v_2v_3v_4v_5v_1$. If $G$ is isomorphic to a cycle $C_5$, it is easy to see that $rc(G) = m - 2$. Suppose $G \in \mathcal{J}_1$. Suppose $e_1$ is a pendant edge of $T(v_1)$ that is assigned color 1. Set $c(v_1v_2) = m-4$, $c(v_2v_3) = m-3$, $c(v_3v_4) = 1$, $c(v_4v_5) = m - 4$, $c(v_1v_5) = m - 3$. Thus $rc(G) \leq m - 3$. On the other hand, since it costs $m - 5$ colors for $G \setminus E(C)$, and by Observation 4, we know that at least two colors different from $c(G \setminus E(C))$ should be assigned to $C$, it follows that $rc(G) \geq m - 3$. Therefore, $rc(G) = m - 3$.

Suppose $G \in \mathcal{J}_2$. Without loss of generality, we consider the following three cases. If $l(v_i) \geq 3$ for some $i$ with $1 \leq i \leq 5$, then we may suppose that $e_1$, $e_2$ and $e_3$ are the three pendant edges of $T(v_1)$ that are assigned colors 1,2,3, respectively. Set $c(v_1v_2) = m - 4$, $c(v_2v_3) = 3$, $c(v_3v_4) = 2$, $c(v_4v_5) = 1$, $c(v_1v_5) = m - 4$. If $l(v_i) = 2$, then we may suppose that $e_1$, $e_2$ are the two pendant edges of $T(v_1)$ that are assigned colors 1,2, respectively, and $e_3$ is a pendant edge of $T(v_2)$ that is assigned color 3. Set $c(v_1v_2) = m - 4$, $c(v_2v_3) = 1$, $c(v_3v_4) = 2$, $c(v_4v_5) = m - 4$, $c(v_1v_5) = 3$. It remains to consider the case that $l(v_i) \leq 1$ for each $i$. Without loss of generality, let $l(v_1) = l(v_2) = l(v_4) = 1$. Suppose that $e_i$ is a pendant edge that is assigned color $i$ such that $e_1 \in T(v_1)$, $e_2 \in T(v_2)$ and $e_3 \in T(v_4)$, where $1 \leq i \leq 3$. Set $c(v_1v_2) = 3$, $c(v_2v_3) = m - 4$, $c(v_3v_4) = 1$, $c(v_4v_5) = 2$, $c(v_1v_5) = m - 4$. It is easy to show that these three colorings are rainbow, and in these three cases, $rc(G) \leq m - 4$. $\square$

Let $i$ be an integer with $1 \leq i \leq 6$ and the addition is performed modulo 6. Set $\mathcal{L} = \{G : m = n, g(G) = 6\}$ and $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_1 = \{G : G \in \mathcal{L}, l(v_i) \leq 1, l(v_{i+3}) \leq 1, l(v_{i+1}) = l(v_{i+2}) = l(v_{i+4}) = l(v_{i+5}) = 0\}$, $\mathcal{L}_2$ is the set of the rest unicyclic graphs with girth 6.

**Lemma 7.** *Let $G$ be a graph belonging to $\mathcal{L}$. If $G \in \mathcal{L}_1$, then $rc(G) = m - 3$; otherwise $rc(G) \leq m - 4$.*

*Proof.* Let the unique cycle of $G$ be $C = v_1v_2v_3v_4v_5v_6v_1$. By Observation 2, each edge of $G \setminus E(C)$ must obtain a distinct color, this costs $m-6$ colors, thus $rc(G) \geq m-6$. Color $G \setminus E(C)$ with a set $[m - 6]$ of colors. Suppose $G \in \mathcal{L}_1$. Set $c(v_1v_2) = m - 5$, $c(v_2v_3) = m-4$, $c(v_3v_4) = m-3$, $c(v_4v_5) = m-5$, $c(v_5v_6) = m-4$, $c(v_1v_6) = m-3$. By Observation 2, $rc(G) \leq m - 3$. On the other hand, by Observation 4, we know that at least three colors different from $c(G \setminus E(C))$ should be assigned to $C$, it follows that $rc(G) \geq m - 3$. Therefore, $rc(G) = m - 3$.

Suppose $G \in \mathcal{L}_2$. If $l(v_i) \geq 2$, then we may suppose that $e_1$ and $e_2$ are the two pendant edges of $T(v_1)$ that are assigned colors 1,2, respectively. Set $c(v_1v_2) = m-5$, $c(v_2v_3) = m - 4$, $c(v_3v_4) = 1$, $c(v_4v_5) = 2$, $c(v_5v_6) = m - 5$, $c(v_1v_6) = m - 4$. It remains to consider the case that $l(v_i) \leq 1$ for each $i$. Suppose $l(v_1) = l(v_2) = 1$. Let $e_1$ and $e_2$ be the two pendant edges that are assigned colors 1,2, respectively, such that $e_1 \in T(v_1)$ and $e_2 \in T(v_2)$. Set $c(v_1v_2) = m - 5$, $c(v_2v_3) = m - 4$, $c(v_3v_4) = 1$, $c(v_4v_5) = 2$, $c(v_5v_6) = m - 5$, $c(v_1v_6) = m - 4$. Without loss of generality, let $l(v_1) = l(v_3) = 1$. Suppose that $e_1$ and $e_2$ are the two pendant edges that are assigned colors 1,2, respectively, such that $e_1 \in T(v_1)$ and $e_2 \in T(v_3)$. Set $c(v_1v_2) = m - 5$, $c(v_2v_3) = m - 4$, $c(v_3v_4) = 1$, $c(v_4v_5) = m - 5$, $c(v_5v_6) = m - 4$,

$c(v_1v_6) = 2$. It is easy to show that these three colorings are rainbow, and in these three cases, $rc(G) \leq m - 4$. □

## 4 Characterizing graphs with $rc(G) = m - 2$ and $m - 3$

Now we are ready to characterize the graphs with $rc(G) = m-2$ and $rc(G) = m-3$.

**Theorem 1.** *$rc(G) = m - 2$ if and only if $G$ is isomorphic to a cycle $C_5$ or belongs to $\mathcal{G}_2 \cup \mathcal{H}_2$.*

*Proof.* Suppose that $G$ is a graph with $rc(G) = m - 2$. By Lemma 1, $G$ contains a unique 2-connected subgraph. By Lemma 3, $G$ contains no $\Theta$-graph as a subgraph. It follows that $G$ is a unicyclic graph. By Observation 3, the girth of $G$ is at most 5. The cases that the girth of $G$ is 3,4 and 5 have been discussed in Lemmas 4, 5 and 6, respectively. We conclude that $G$ must be isomorphic to a graph shown in our theorem.

Conversely, By Lemmas 4, 5 and 6, the result holds. □

Let $\mathcal{M}$ be a class of graphs where in each graph a path is attached at each vertex of degree 2 of $K_4 - e$, respectively. Note that, the path may be trivial.

**Theorem 2.** *$rc(G) = m - 3$ if and only if $G$ is isomorphic to a cycle $C_7$ or belongs to $\mathcal{G}_1 \cup \mathcal{H}_3 \cup \mathcal{J}_1 \cup \mathcal{L}_1 \cup \mathcal{M}$.*

*Proof.* Suppose that $G$ is a graph with $rc(G) = m - 3$. By Lemma 1, $G$ contains a unique 2-connected subgraph $B$. Set $V(B) = \{v_1, \ldots, v_s\}$, then $G$ has the structure as follows: a tree, denoted by $T(v_i)$, is attached at each vertex $v_i$ of $B$. If $B$ is exactly a cycle, then by Observation 3, the girth of $G$ is at most 7. The cases that the girth of $G$ is 3,4,5 and 6 have been discussed in Lemmas 4, 5, 6 and 7, respectively. It remains to deal with the case that the girth of $G$ is 7. If $G$ is not isomorphic to a cycle $C_7$, then suppose that $e_1$ is a pendant edge of $T(v_1)$ that is assigned color 1. Color $G \setminus E(B)$ with a set $[m-7]$ of colors and set $c(v_1v_2) = m-6$, $c(v_2v_3) = m-5$, $c(v_3v_4) = m-4$, $c(v_4v_5) = 1$, $c(v_5v_6) = m-6$, $c(v_6v_7) = m-5$, $c(v_1v_7) = m-4$. By Observation 1, we have $rc(G) \leq m - 4$.

So $B$ is not a cycle. By Lemma 3, $G$ contains no $\Theta$-graph except a $K_4 - e$ as a subgraph. We first claim that $B$ is isomorphic to a $K_4 - e$. If $B$ is isomorphic to a $K_4$, we first color the edges of $G \setminus E(B)$ with $m - 6$ colors, then give each edge of $B$ the same new color, this costs $m - 5$ colors totally, it is easy to check that this coloring is rainbow, and in this case, $rc(G) \leq m - 5$, a contradiction. Set $V(K_4 - e) = \{v_1, v_2, v_3, v_4\}$, and $E(K_4 - e) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$. If $G \notin \mathcal{M}$, then $l(v_i) \geq 1$ or $l(v_j) \geq 2$ where $i = 1$ *or* 3, $j = 2$ *or* 4. If $l(v_1) \geq 1$, suppose that $e_1$ is a pendant edge of $T(v_1)$ that is assigned color 1. Assign color 1 to $v_2v_3$ and $m - 4$ to each other edge of $K_4 - e$. If $l(v_2) \geq 2$, suppose that $e_1$ and $e_2$ are two pendant edges of $T(v_2)$ that are assigned colors 1 and 2, respectively. Set $c(v_1v_2) = c(v_2v_3) = c(v_1v_3) = m - 4$, $c(v_3v_4) = 1$, $c(v_1v_4) = 2$. In both cases,

$rc(G) \leq m-4$. We conclude that $G$ must be isomorphic to a graph shown in our theorem.

Conversely, if $G$ is isomorphic to a cycle $C_7$, then $rc(G) = m-3$. If $G \in \mathcal{M}$, it is easy to see that at least two new colors different from $c(G \setminus E(B))$ should be assigned to $B$. Since each edge of $G \setminus E(B)$ must obtain a distinct color, this costs $m-5$ colors, it follows that $rc(G) \geq m-3$. Set $c(v_1v_2) = c(v_3v_4) = c(v_1v_3) = m-4$, $c(v_2v_3) = c(v_1v_4) = m-3$, thus $rc(G) \leq m-3$. Therefore, $rc(G) = m-3$. By Lemmas 4, 5, 6 and 7, the result holds. □

**Remark:** We have also characterized the graphs $G$ with $rc(G) = m-4$. But, the proof is similar to the above ones, and very long and tedious, and therefore not written down here.

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