

## Characterization of linear complementarity problems as linear programs

— [Source link](#) 

Olvi L. Mangasarian

**Institutions:** University of Oxford

**Published on:** 01 Jan 1978

**Topics:** Linear complementarity problem, Mixed complementarity problem, Diagonally dominant matrix, Linear programming and Nonnegative matrix

Related papers:

- [Complementary pivot theory of mathematical programming](#)
- [Linear complementarity problems solvable by A single linear program](#)
- [On solving linear complementarity problems as linear programs](#)
- [Bimatrix Equilibrium Points and Mathematical Programming](#)
- [Simplified Characterizations of Linear Complementarity Problems Solvable as Linear Programs](#)

Share this paper:    

View more about this paper here: <https://typeset.io/papers/characterization-of-linear-complementarity-problems-as-bydjawf80p>

CHARACTERIZATION OF LINEAR COMPLEMENTARITY  
PROBLEMS AS LINEAR PROGRAMS

by

O. L. Mangasarian

Computer Sciences Technical Report #271

May 1976



The University of Wisconsin  
Computer Sciences Department  
1210 West Dayton Street  
Madison, Wisconsin 53706

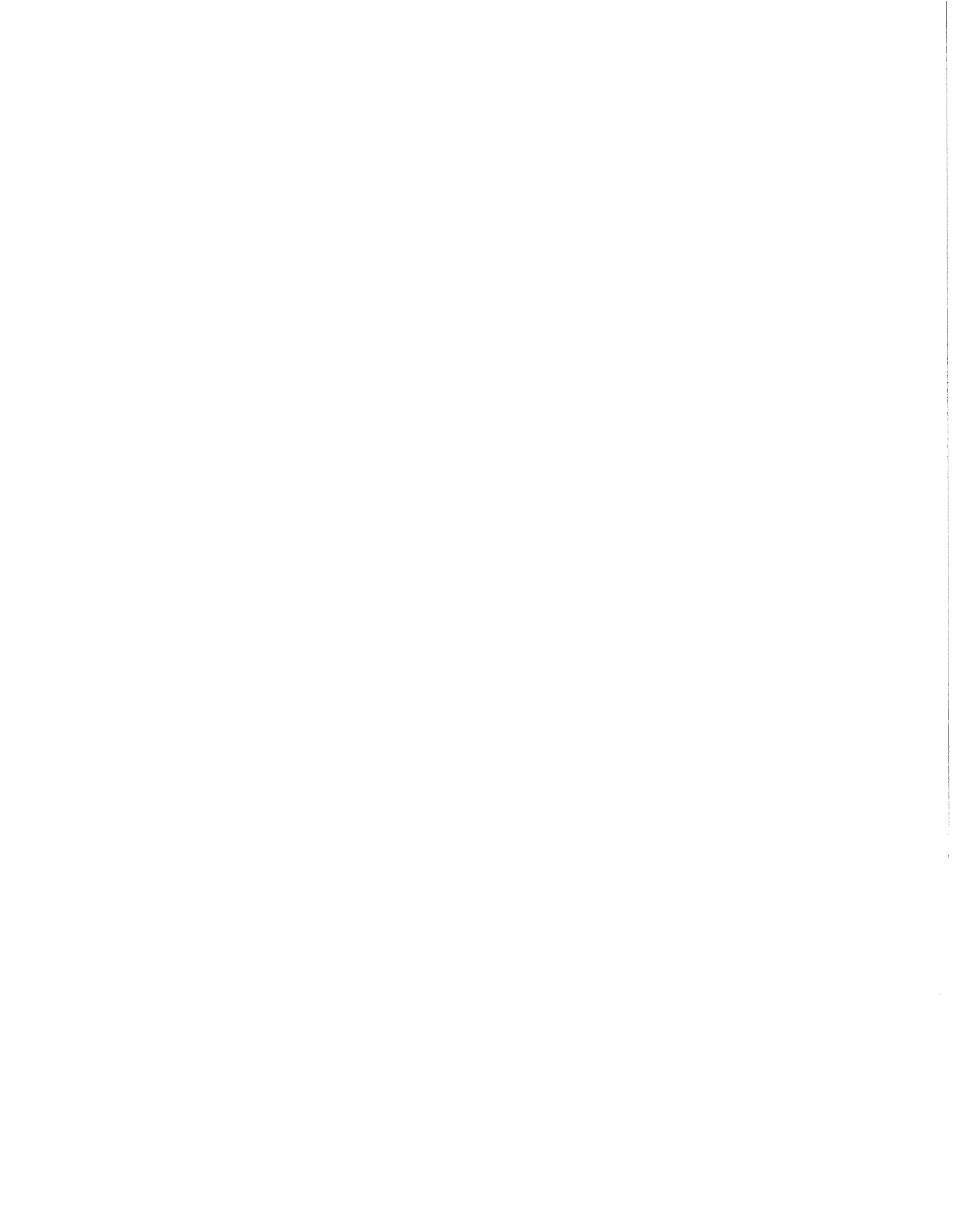
Received: May 12, 1976

CHARACTERIZATION OF LINEAR COMPLEMENTARITY

PROBLEMS AS LINEAR PROGRAMS

by

O. L. Mangasarian



CHARACTERIZATION OF LINEAR COMPLEMENTARITY  
PROBLEMS AS LINEAR PROGRAMS<sup>1</sup>

O. L. Mangasarian

Oxford University Computation Laboratory  
Oxford, England<sup>2</sup>

ABSTRACT

It is shown that the linear complementarity problem of finding an  $n$ -by-1 vector  $x$  such that  $Mx + q \geq 0$ ,  $x \geq 0$ , and  $x^T(Mx+q) = 0$ , where  $M$  is a given  $n$ -by- $n$  real matrix and  $q$  is a given  $n$ -by-1 vector, is solvable if and only if the linear program: minimize  $p^T x$  subject to  $Mx + q \geq 0$ ,  $x \geq 0$ , is solvable, where  $p$  is an  $n$ -by-1 vector which satisfies certain conditions. Furthermore each solution of the linear program, solves the linear complementarity problem. For a number of special cases including those when  $M$  has nonpositive off-diagonal elements, or when  $M$  is strictly or irreducibly diagonally dominant, or when  $M$  is a positive matrix with a dominant diagonal columnwise,  $p$  is very easily determined and the linear complementarity problem can be solved as an ordinary linear program. Examples of linear complementarity problems are given which can be solved as linear programs, but not by Lemke's method or the principal pivoting method.

---

<sup>1</sup>Research supported by Science Research Council Grant B/RG/4079.7, National Science Foundation Grant DCR-74-20584 and the Wisconsin Alumni Research Foundation.

<sup>2</sup>Permanent address: Computer Sciences Department, University of Wisconsin, 1210 West Dayton Street, Madison, Wisconsin 53706, U.S.A.



1. INTRODUCTION

We consider the linear complementarity problem of finding an  $x$  in  $R^n$  such that

(1)  $Mx + q \geq 0, x \geq 0, x^T(Mx+q) = 0$  (LCP)

where  $M$  is a given  $n$ -by- $n$  real matrix and  $q$  is a given vector in  $R^n$ . In earlier works [10,11,3] it was shown that for special cases of  $M$  a solution to the linear complementarity (1) can be obtained by solving the linear program

(2) minimize  $p^T x$  subject to  $w = Mx + q \geq 0, x \geq 0$  (LP)

for an easily determined  $p$  in  $R^n$ . The principal aim of this work is to show that for any  $n$ -by- $n$  real matrix  $M$ , the linear complementarity problem (1) is solvable if and only there exists a  $p$  in  $R^n$  satisfying certain conditions (Theorem 1). Furthermore for such  $p$ , each solution of the linear program (1) solves the linear complementarity problem. Because of the presence of bilinear conditions (6c,d) among the otherwise linear conditions (6a,b,e,f) which  $p$  must satisfy, it is not easy in general to determine  $p$  for an arbitrary  $M$ . However for a number of special cases including those when  $M$  is a  $Z$ -matrix, that is a real square matrix with nonpositive off-diagonal elements, or when  $M$  is strictly or irreducibly diagonally dominant,  $p$  can be easily determined (Table 1) and the linear complementarity problem (1) can be solved as an ordinary linear program (2).

It will be convenient to state here for later reference the dual, program to the linear program (2), which is

(3) maximize  $-q^T y$  subject to  $v = -Ny + p \geq 0, y \geq 0$ . (DP)

In Section 2 of the paper we establish some preliminary results including the fact the linear complementarity problem is equivalent to minimizing a piecewise linear concave function on a polyhedral set contained in the nonnegative orthant (Lemma 1), and consequently if it has a solution, it has one on a vertex of the polyhedral set (Lemma 2). In Section 3 we establish our principal results which include (Theorem 1) a characterization of the linear complementarity problem (1) as a linear program (2), and a number of special cases. In Section 4, the results of Section 3 are extended to other cases by employing an equivalent slack linear complementarity problem (8). Many of the specific cases derived in this paper are tabulated for convenience in Table 1. Finally in Section 5 we give two numerical examples of linear complementarity problems which are solvable as linear programs but not by Lemke's method [8,2] or the principal pivoting method [2].

We define here the notation employed in the paper and mention some of the well known results that will be used. All matrices and vectors considered are real ones. If  $A$  is an  $n$ -by- $m$  matrix then this is denoted as  $A \in R^{n \times m}$ , and  $A_i$  and  $A_{i,j}$  denote respectively the  $i$ -th row and  $j$ -th column of  $A$ . Sometimes we shall use the notation  $Z_1, Z_2$  to denote specific matrices. This will be made clear from the context. The letter  $I$  will denote a diagonal matrix of ones. If  $x \in R^n$ , then  $x_i$  denotes its  $i$ -th element, and  $(x)_+ = \max\{0, x_i\}$ . If  $M \in R^{n \times n}$ , then  $\text{diag } M$  is the matrix obtained from  $M$  by setting all its off-diagonal elements to zero. Similarly off-diag  $M$  is a matrix obtained from  $M$  by setting its diagonal equal to zero. If  $c \in R^n$ , then  $C = \text{diag } c$  is a matrix in  $R^{n \times n}$  with zero off-diagonal elements and  $C_{ij} = c_i, i=1, \dots, n$ . If  $A \in R^{n \times m}$ , then  $|A|$  denotes the matrix obtained from  $A$  by replacing each  $A_{ij}$  by  $|A_{ij}|$ . The



spectral radius of a square matrix  $M$  is the maximum of the absolute values of its eigenvalues and is denoted by  $\rho(M)$ . The letter  $e$  will denote a vector of ones in  $R^n$  or  $R^m$ . A square matrix is said to be irreducible if there exists no simultaneous permutation of rows and columns that will create a block of zeros in a southwest corner [12, pp. 102-105]. A square matrix with nonpositive off-diagonal elements is said to be a  $Z$ -matrix or it is said to belong to  $Z$ . Two subclasses,  $K$  and  $K_0$ , of  $Z$  will be used. These classes have been extensively investigated in [5,6,7]. The class  $K$  can be characterized in many ways of which we mention two. A matrix  $M$  is in  $K$  [5, Theorem 4.3] if it is in  $Z$  and either (a)  $M^{-1}$  exists and  $M^{-1} \geq 0$ , or (b) There exists a vector  $r \geq 0$  such that  $Mr > 0$  or  $r^T M > 0$ . A matrix  $M$  is in  $K_0$  [5, Theorem 5.1] if it is in  $Z$  and if  $M + \delta I \in K$  for each  $\delta > 0$ . It follows that  $K \subset K_0 \subset Z$ .

### 2. PRELIMINARIES

We begin by establishing, not in the most direct way, that every solvable linear complementary problem has a solution at a vertex of its feasible set

$$(4) \quad S = \{x | Mx + q \geq 0, x \geq 0\}$$

We first establish the equivalence of the linear complementarity problem (1) to the minimization of a piecewise linear concave function on  $S$ .

Lemma 1 Each solution  $\bar{x}$  of the linear complementarity problem (1) satisfies

$$(5) \quad 0 = \sum_{i=1}^n \bar{x}_i - (\bar{x}_i - (M\bar{x} + q)_+)_+ = \min_{x \in S} \sum_{i=1}^n x_i - (x_i - (Mx + q)_+)_+$$

and conversely.

Proof First observe that for  $x \in S$  the objective function of (5) is nonnegative, because

$$x_i - (x_i - Mx + q)_+ = (Mx + q)_+ \geq 0 \quad \text{if } x_i - (Mx + q)_+ \geq 0$$

$$x_i - (x_i - Mx + q)_+ = x_i \geq 0 \quad \text{if } x_i - (Mx + q)_+ < 0$$

Suppose now that  $\bar{x}$  solves (1). Then  $\bar{x} \in S$  and  $\bar{x}_i (M\bar{x} + q)_+ = 0$  for  $i = 1, \dots, n$ . Hence

$$\bar{x}_i - (\bar{x}_i - (M\bar{x} + q)_+)_+ = -(M\bar{x} + q)_+ = 0 \quad \text{for } \bar{x}_i = 0$$

$$\bar{x}_i - (\bar{x}_i - (M\bar{x} + q)_+)_+ = \bar{x}_i - \bar{x}_i = 0 \quad \text{for } (M\bar{x} + q)_+ = 0.$$

Thus  $\bar{x}$  solves (5). Conversely if  $\bar{x}$  solves (5) then  $\bar{x}_i - (\bar{x}_i - (M\bar{x} + q)_+)_+ = 0$  for  $i = 1, \dots, n$ , and

$$(M\bar{x} + q)_+ = 0 \quad \text{if } \bar{x}_i - (M\bar{x} + q)_+ \geq 0$$

$$\bar{x}_i = 0 \quad \text{if } \bar{x}_i - (M\bar{x} + q)_+ < 0$$

Hence  $\bar{x}$  solves (1).  $\square$

Remark 1 Because  $(x_i - (Mx+q)_i)_+$  is the maximum of two linear functions:  $x_i - (Mx+q)_i$  and 0, it is a piecewise linear convex function. Hence the objective function of (5) is a piecewise linear concave function, which is indeed a difficult function to minimize because it may have many local minima which are not global. Thus Lemma 1 shows that the linear complementarity problem is inherently a difficult nonconvex problem, but it also shows that

Lemma 2 If the linear complementarity problem (1) has a solution, it has a solution at a vertex of S.

Proof Because S is contained in the nonnegative orthant, it does not contain any straight lines (that go to infinity at both ends) and hence by Rockafellar's theorem [14, Corollary 32.3.4] the concave minimization problem (5) must have a solution at a vertex of S, which by Lemma 1 must also solve (1).  $\square$

Proposition 1 If the linear complementarity problem (1) has a solution, then there exists a p in  $R^n$  such that the linear program (2) has a (unique) solution  $\bar{x}$  that also solves the linear complementarity problem (1).

Proof By Lemma 2 there exists a vertex  $\bar{x}$  of S which solves (1). Because  $\bar{x}$  is a vertex of S, there exist subsets (not necessarily disjoint) J and L of  $\{1, \dots, n\}$  such that  $\bar{x}_{i \in J} = 0$ ,  $(M\bar{x}+q)_{i \in L} = 0$ , and  $\{I_{i \in J}, M_{i \in L}\}$  are linearly independent. The desired p for the linear program (2) is determined by

$$p^T x = \sum_{i \in J} x_i + \sum_{i \in L} (Mx)_i$$

If  $\bar{x}$  is not a unique solution of the linear program (2) with the above p, then there exists an  $\hat{x} \neq \bar{x}$  such that

$$\hat{x}_i = 0 \text{ for } i \in J, \text{ and } (M\hat{x}+q)_i = 0 \text{ for } i \in L$$

Hence

$$\hat{x} - \bar{x} \neq 0, \quad I_{i \in J}(\hat{x} - \bar{x}) = 0, \quad M_{i \in L}(\hat{x} - \bar{x}) = 0$$

which contradicts the linear independence of  $\{I_{i \in J}, M_{i \in L}\}$ . Hence  $\bar{x}$  is the unique solution of the linear program (2) with the p given above.  $\square$

3. PRINCIPAL RESULTS

We establish in this section the conditions that  $p$  of the linear program (2) must satisfy in order that each solution of (2) is also a solution of the linear complementarity problem (1).

Theorem 1 The linear complementarity problem (1) has a solution if and only if the linear program (2) is solvable for a  $p$  which together with vectors  $r, s, c, d$ , all in  $R^n$ , and matrices  $Z_1, Z_2, Y_1, Y_2$  all

in  $R^{n \times n}$  satisfy the conditions

(6a)  $p = r + M^T s, (r, s) \geq 0$

(6b)  $MZ_1 = Z_2 + qc^T$

(6c)  $M^T(Y_1 - sc^T) + (Y_2 - rc^T) = 0$

(6d)  $r^T Z_1 + s^T Z_2 - q^T (Y_1 - sc^T) = d^T$

(6e)  $\text{diag } d + \text{diag } (Y_1 + Y_2) > 0$

(6f)  $Z_1, Z_2 \in Z, Y_1, Y_2, c, d \geq 0$

Furthermore, each solution of the linear program (2), with a vector  $p$  satisfying (6) above, is also a solution of the linear complementarity problem (1).

Proof

$$\langle \text{LCP}(1) \text{ has solution} \rangle \Leftrightarrow \langle \exists p: x \text{ solves LP}(2) \text{ and } x \text{ solves LCP}(1). \rangle$$

(By Proposition (1))

$$\Leftrightarrow \langle \exists p: (x, w) \text{ solve LP}(2), (y, v) \text{ solve DP}(3), \text{ and for } i = 1, \dots, n, v_i + y_i = 0 \Rightarrow x_i = 0 \text{ or } w_i = 0 \rangle$$

(When the optimal dual variable components  $v_i, y_i$  satisfy  $v_i + y_i > 0$ , it follows automatically from the complementarity condition  $y_i w_i + x_i v_i = 0$ , that  $x_i = 0$  or  $w_i = 0$ .)

$$\Leftrightarrow \langle \exists p: (x, w) \text{ solve LP}(2), (y, v) \text{ solve DP}(3), \text{ and for } i = 1, \dots, n, v_i + y_i \leq 0 \Rightarrow x_i \leq 0 \text{ or } w_i \leq 0 \rangle$$

(Because  $v_i, y_i, x_i, w_i \geq 0$ )

$$\Leftrightarrow \langle \exists p: w = Mx + q \geq 0, x \geq 0, v = -M^T y + p \geq 0, y \geq 0, p^T x + q^T y \leq 0, \text{ and for } i = 1, \dots, n, v_i + y_i \leq 0 \Rightarrow x_i \leq 0 \text{ or } w_i \leq 0 \rangle$$

(The first set of conditions are the standard linear programming optimality conditions)

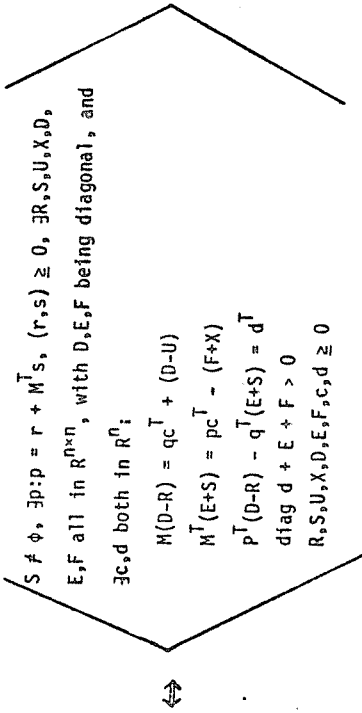
$$\Leftrightarrow \langle \exists p: p = r + M^T s, (r, s) \geq 0, S \neq \emptyset, \text{ and for each } i \in \{1, \dots, n\} w = Mx + q \geq 0, x \geq 0, v = -M^T y + p \geq 0, y \geq 0, p^T x + q^T y \leq 0, v_i + y_i \leq 0, v_i > 0, x_i > 0, w_i > 0 \rangle$$

Has no solution  $x, w, y, v$

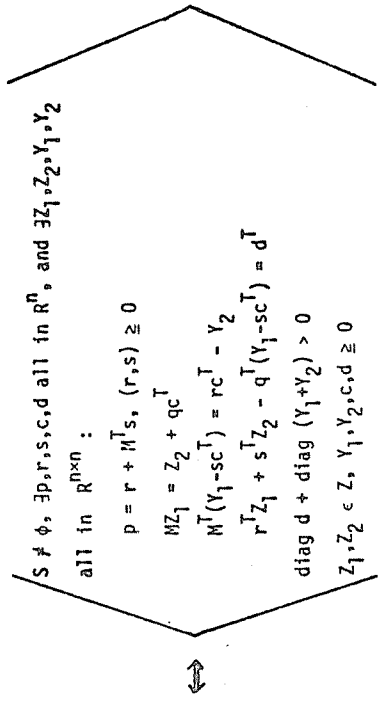
(The first condition is equivalent to dual feasibility which together with  $S \neq \emptyset$  is equivalent to the solvability of the linear program (2).)

$$\Leftrightarrow \langle \exists p: p = r + M^T s, (r, s) \geq 0, S \neq \emptyset \text{ and for each } i \in \{1, \dots, n\} -M^T y + p \zeta \geq 0, Mx + q \zeta \geq 0, y \geq 0, x \geq 0, (-I_1 + M_1^T) y - p_1 \zeta \geq 0, M_1 x + q_1 \zeta > 0, I_1 x > 0, -q^T y - p^T x \geq 0, \zeta > 0, \zeta \in R \rangle$$

Has no solution  $x, y, \zeta$

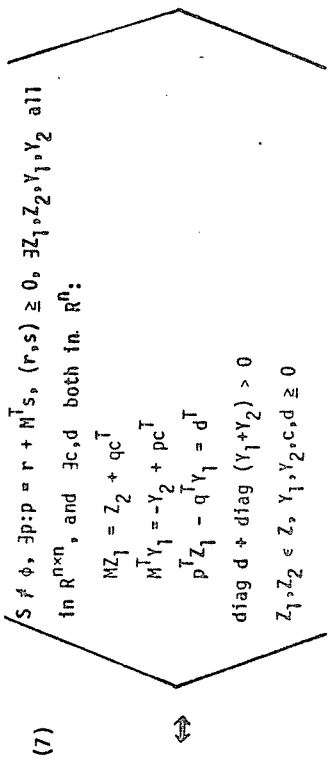


(By Motzkin's theorem of the alternative [9])



(Follows by substitution of  $r + M^T s$  for  $p$ .)

□



(Follows by setting  $Z_1 = D - R,$

$Z_2 = D - U, Y_1 = E + S, Y_2 = F + X.$ )

Remark 2 Conditions (7) are an alternative equivalent statement of  $S \neq \phi$  and conditions (6). Note that the presence of the bilinear conditions (6c,d) among the otherwise linear conditions (6a,b,e,f) precludes the possibility of easily solving (6) for the unknowns  $p, r, s, c, d, Z_1, Z_2, Y_1$  and  $Y_2$  for a general  $M$ . However for important special cases, this is possible, as is indicated in the subsequent results.

All the results of [10,11] can be derived from the above theorem. For example Theorem 1 of [11] can be obtained from the above theorem by replacing in (7) the condition  $\text{diag } d + \text{diag } (Y_1 + Y_2) > 0$  by the more stringent condition  $d > 0$ . Theorem 2 of [11] follows by setting  $Y_1 = Y_2 = 0$ , and  $c = 0$  in (7), or the equivalent Theorem 1 of [10] follows by setting  $Y_1 = Y_2 = 0$  and  $c = 0$  in (6).

Another interesting special case may be obtained from Theorem 1 above by setting  $Y_1 = sc^T$ ,  $Y_2 = rc^T$  and  $C = \text{diag } c$ . This gives the following theorem which again is a generalization of Theorem 1 of [10].

Theorem 2 If  $S \neq \emptyset$  and there exist  $r, s$  in  $R^n$ ,  $Z_1, Z_2$  in  $R^{n \times n}$  such that

$$\begin{aligned}
MZ_1 &= Z_2 + qc^T \\
r^T Z_1 + s^T Z_2 &\geq 0 \\
r^T (Z_1 + C) + s^T (Z_2 + C) &> 0, \quad C = \text{diag } c \\
Z_1, Z_2 &\in Z, \quad c, r, s \geq 0
\end{aligned}$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with  $p = r + M^T s$ .

Setting  $c = \delta e$ ,  $C = \delta I$  in the above where  $\delta$  is some positive number and imposing the more stringent condition  $(r+s)^T c > 0$  we obtain

Corollary 1 If  $S \neq \emptyset$  and there exist  $\delta$  in  $R$ ,  $r, s$  in  $R^n$ ,  $Z_1, Z_2$  in  $R^{n \times n}$  such that

$$\begin{aligned}
MZ_1 &= Z_2 + \delta qe^T \\
r^T Z_1 + s^T Z_2 &\geq 0 \\
r + s &> 0, \quad \delta > 0 \\
Z_1, Z_2 &\in Z, \quad r, s \geq 0
\end{aligned}$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with  $p = r + M^T s$ .

By noting that if  $Z_1 \in K_0$  and  $Z_1$  is irreducible, then there exists an  $r > 0$  such that  $r^T Z_1 \geq 0$  [5, Theorem 5.8], the following corollary is an immediate consequence of Corollary 1.

Corollary 2 If  $S \neq \emptyset$  and there exist  $\delta \in R$ ,  $Z_1, Z_2$  in  $R^{n \times n}$  such that

$$\begin{aligned}
MZ_1 &= Z_2 + \delta qe^T \\
(Z_1, Z_2) &\in Z, \quad \delta > 0 \\
Z_1 \text{ or } Z_2 &\text{ irreducible and belongs to } K_0
\end{aligned}$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with

$$\begin{aligned}
p &= r, \quad r^T Z_1 \geq 0, \quad r > 0 \text{ (if } Z_1 \in K_0 \text{ and irreducible)} \\
p &= M^T s, \quad s^T Z_2 \geq 0, \quad s > 0 \text{ (if } Z_2 \in K_0 \text{ and irreducible)}
\end{aligned}$$

Another special case follows from Corollary 1 by noting that if  $Z_1 \in Z$ ,  $MZ_1 = \bar{Z}_2 \in Z$  and the off-diagonal elements of the rows  $(\bar{Z}_2)_i$  are negative when  $q_i < 0$ , then by taking  $\delta$  small enough we have that  $\bar{Z}_2 - \delta qe^T \in Z$ . By taking  $s = 0$  and renaming  $\bar{Z}_2$  (for simplicity)  $Z_2$  we obtain the following

Corollary 3 If  $S \neq \emptyset$  and there exist  $r$  in  $R^n$  and  $Z_1, Z_2$  in  $R^{n \times n}$  such that

$$\begin{aligned}
MZ_1 &= Z_2 \\
r^T Z_1 &\geq 0 \\
r &> 0, \quad Z_1, Z_2 \in Z \\
(Z_2)_ij &< 0 \text{ if } j \neq i \text{ and } q_i < 0
\end{aligned}$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with  $p = r$ .

Note that the conditions of Corollary 3 above [5, Theorem 5,4] imply that  $Z_1 \in K_0$ . Furthermore the conditions  $Z_1 \in Z$ ,  $r^T Z_1 \geq 0$ ,  $r > 0$  are implied [5, Theorem 5,8] if we make the assumption that  $Z_1 \in K_0$  and  $Z_1$  is irreducible.

Corollary 4 Let  $n \geq 3$  and let  $M$  be a positive matrix which is diagonally dominant column-by-column, that is  $M_{jj} \geq \sum_{i \neq j} M_{ij}$ ,  $j = 1, \dots, n$ . Then the linear complementarity problem

(1) has a solution which can be obtained by solving the linear program (2) with  $p = e$ .

Proof Without loss of generality assume that  $M$  has been normalized so that it has ones along its diagonal. Let  $M = I + F$  where

$F \in R^{n \times n}$ ,  $F_{ij} = 0$ ,  $F_{ij} = M_{ij} > 0$  for  $i \neq j$ . Then  $e^T(i-F) \geq 0$ . Set  $Z_1 = I - F$  in Corollary 3. Then  $Z_2 = MZ_1 = I - F^2$ . For  $i \neq j$ ,

the  $ij$ th element of  $F^2$  is the scalar product of two nonnegative vectors in  $R^n$ ,  $n \geq 3$ ,  $F_i$  and  $F_j$ , each of which containing exactly one zero element, and so  $(F^2)_{ij} > 0$ . Consequently  $(Z_2)_{ij} = (-F^2)_{ij} < 0$  for  $i \neq j$  and the assumptions of Corollary 3 are satisfied with  $r = e$ ,  $Z_1 = I - F$  and  $Z_2 = I - F^2$ .  $\square$

Corollary 5 Let  $S \neq \emptyset$  and let  $f: R^{n \times n} \rightarrow R^{n \times n}$ . If  $f(M) \geq 0$  and  $I + Mf(M) - M \geq 0$  and either

- (a)  $\rho(f(M)) < 1$  ( $p=r \geq 0$ ,  $r^T(I-f(M)) > 0$ )
- (b)  $\rho(I+Mf(M)-M) < 1$  ( $p=M^T s$ ,  $s \geq 0$ ,  $s^T M(I-f(M)) > 0$ )

or

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with the  $p$  indicated above.

Proof: Set in Theorem 2,  $c = 0$ ,  $Z_1 = I - f(M)$ , and  $Z_2 = M(I-f(M))$ . For case (a) take  $s = 0$  and by [6, Theorem 2,1],  $Z_1 \in K$  and hence there exists  $r \geq 0$  such that  $r^T Z_1 = r^T(I-f(M)) > 0$ . Similarly for case (b) take  $r = 0$  and again  $Z_2 \in K$  and hence there exists  $s \geq 0$  such that  $s^T Z_2 = s^T M(I-f(M)) > 0$ .  $\square$

Corollary 6 Let  $S \neq \emptyset$ , let  $f: R^{n \times n} \rightarrow R^{n \times n}$ , let  $x \in R^{n \times n}$ ,  $x \geq 0$ , and let  $M = I + f(x)$ . If  $f(x) \leq (I+f(x))x$  and either

- (a)  $\rho(x) < 1$  ( $p=r \geq 0$ ,  $r^T(I-x) > 0$ )
- (b)  $\rho(x+f(x)x-f(x)) < 1$  ( $p=M^T s$ ,  $s \geq 0$ ,  $s^T(I-x+f(x)-f(x)x) > 0$ )

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with the  $p$  indicated above.

Proof: Set in Theorem 2,  $c = 0$ ,  $Z_1 = I - x$ , and  $Z_2 = I - (x-f(x)+f(x)x)$ . For case (a) take  $s = 0$  and by [6, Theorem 2,1],  $Z_1 \in K$  and hence there exists  $r \geq 0$  such that  $r^T Z_1 = r^T(I-x) > 0$ . Similarly for case (b) take  $r = 0$  and again  $Z_2 \in K$  and hence there exists  $s \geq 0$  such that  $s^T Z_2 = s^T(I-x+f(x)-f(x)x) > 0$ .  $\square$

Let such that

then the ...  
be obtained by solving the linear program (2) with the  $p$  indicated above.

4. EXTENSIONS AND SPECIAL CASES

In order to enlarge the class of matrices for which the linear complementarity problem can be easily set up as a linear program we consider the following linear complementarity problem with a slack variable  $x_0$  in  $R^m$  which is slightly more general than considered earlier [11]

$$(8) \begin{bmatrix} w \\ w_0 \end{bmatrix} = \begin{bmatrix} M & A \\ E & B \end{bmatrix} \begin{bmatrix} x \\ x_0 \end{bmatrix} + \begin{bmatrix} q \\ 0 \end{bmatrix} \geq 0, \begin{bmatrix} x \\ x_0 \end{bmatrix} \geq 0, x^T w + x_0^T w_0 = 0$$

where  $A \in R^{n \times m}$ ,  $E \in R^{m \times n}$  and  $B \in R^{m \times m}$ .

Lemma 3 Let  $E \geq 0$  and let  $B$  be strictly copositive, that is  $x^T B x > 0$  whenever  $0 \leq x \neq 0$ . If  $x$  solves the linear complementarity problem (1) then  $(x, x_0=0)$  solves the slack linear complementarity problem (8). If  $(x, x_0)$  solves (8) then  $x_0 = 0$ , and moreover  $x$  solves (1).

Proof If  $x$  solves (1) then obviously  $(x, x_0=0)$  solves (8). If  $(x, x_0)$  solves (8) then

$$0 = x_0^T w_0 = x_0^T E x + x_0^T B x_0 \geq x_0^T B x_0$$

Since  $x_0 \geq 0$  and  $B$  is strictly copositive, it follows that  $x_0 = 0$  and  $x$  solves (1).  $\square$

By combining this lemma with the previous results we obtain a more general result than [11, Theorem 3].

Theorem 3 If  $S \neq \emptyset$  and there exist matrices  $Z_1, Z_2$  in  $R^{n \times n}$ ,  $B, Z_3, Z_4$  in  $R^{m \times m}$ ,  $A, H, N$  in  $R^{n \times m}$  and  $E, G, L$  in  $R^{m \times n}$  such that  $E, G, H, L, N \geq 0$ ,  $B$  is strictly copositive,  $Z_1, Z_2, Z_3, Z_4 \in Z$  and such that

$$(9) \begin{cases} MZ_1 - AG = Z_2, & MH - AZ_3 = N \\ -EZ_1 + BG = L & -EH + BZ_3 = Z_4 \end{cases}$$

and

$$(10) \begin{pmatrix} Z_1 & -H \\ -G & Z_3 \\ Z_2 & -N \\ -L & Z_4 \end{pmatrix} \begin{pmatrix} r \\ r_0 \\ s \\ s_0 \end{pmatrix} > 0, \text{ for some } r, r_0, s, s_0 \geq 0 \\ (r, r_0, s, s_0) \in R^{n+m+n+m}$$

then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with

$$(11) \quad p = r + M^T s + E^T s_0$$

Proof By applying Theorem 2 with  $c = 0$  to the slack linear complementarity problem (8) we have that (8) has a solution which can be obtained by solving the linear program

$$\begin{aligned} & \text{minimize } p^T x + p_0^T x_0 \\ & \text{subject to } \begin{bmatrix} M & A \\ E & B \end{bmatrix} \begin{bmatrix} x \\ x_0 \end{bmatrix} + \begin{bmatrix} q \\ 0 \end{bmatrix} \geq 0, \begin{bmatrix} x \\ x_0 \end{bmatrix} \geq 0 \end{aligned}$$

where

$$\begin{bmatrix} p \\ p_0 \end{bmatrix} = \begin{bmatrix} r \\ r_0 \end{bmatrix} + \begin{bmatrix} M^T & E^T \\ A^T & B^T \end{bmatrix} \begin{bmatrix} s \\ s_0 \end{bmatrix}$$

Since each solution  $(x, x_0)$  of this linear program solves (8), we have by Lemma 3 that  $x_0 = 0$  and  $x$  solves (1). Since  $x_0 = 0$  at each solution of the above linear program, we can set  $x_0 = 0$  at the outset which reduces the above linear program to (2).  $\square$

Corollary 7 Let  $m = n$ . Theorem 3 holds with condition (10) replaced by any one of the following four more special conditions

(12a)  $s = s_0 = 0, Z_1 - G = Z_5, Z_3 - H = Z_6, Z_5, Z_6 \in K.$

$r = r_0 \geq 0, r^T Z_5 > 0, r^T Z_6 > 0$

(12b)  $r_0 = s = 0, (E+I)Z_1 - BG = Z_5, BZ_3 - (E+I)H = Z_6$

$Z_5, Z_6 \in K, r = s_0 \geq 0, r^T Z_5 > 0, r^T Z_6 > 0$

(12c)  $r = s_0 = 0, MZ_1 - (A+I)G = Z_5, (A+I)Z_3 - MH = Z_6$

$Z_5, Z_6 \in K, r_0 = s \geq 0, s^T Z_5 > 0, s^T Z_6 > 0$

(12d)  $r = r_0 = 0, (H+E)Z_1 - (A+B)G = Z_5, (A+B)Z_3 - (H+E)H = Z_6$

$Z_5, Z_6 \in K, s = s_0 \geq 0, s^T Z_5 > 0, s^T Z_6 > 0$

We give now special cases, most of which are generalizations of cases given earlier [10, 11].

Theorem 4 Let  $S \neq \phi$ , and let  $M$  satisfy any of the conditions below. Then the linear complementarity problem (1) has a solution which can be obtained by solving the linear program (2) with the  $p$  indicated

(a)  $M = Z_2 + ab^T, Z_2 \in K, a, b \in R^n, 0 \neq a \geq 0, 0 \neq b \geq 0,$

$p_i = (1+\delta)b_i$  for  $b_i > 0, p_i = \delta$  for  $b_i = 0$ , for sufficiently small  $\delta > 0$ . (See (13) below.)

(b)  $M = 2Z_2 - Z_5, Z_2 \in Z, Z_5 \in K, Z_2 \geq Z_5, p = M^T s,$

where  $s \geq 0$  and  $s^T Z_5 > 0$ .

(c)  $D = |F| \in K$  where  $D = \text{diag } M, F = \text{off-diag } M, p = M^T s, s \geq 0$  and  $s^T(D-|F|) > 0$ .

(d)  $M_{jj} > \sum_{i \neq j} |M_{ij}|, j = 1, \dots, n, p = M^T e$

(e)  $M_{ii} > \sum_{j \neq i} |M_{ij}|, i = 1, \dots, n$

or

$M_{ii} \geq \sum_{j \neq i} |M_{ij}|, i = 1, \dots, n, M$  irreducible, strict inequality holding for at least one  $i$

or

$M_{jj} \geq \sum_{i \neq j} |M_{ij}|, j = 1, \dots, n, M$  irreducible, strict inequality holding for at least one  $j$

Here  $p = M^T s$ , where  $s \geq 0, s^T(D-|F|) > 0, D = \text{diag } M$  and  $F = \text{off-diag } M$ .

(f)  $M = I + \sum_{i=1}^k \alpha_i X^i, I \geq \alpha_i \geq \alpha_{i+1} \geq 0, i = 1, \dots, k-1,$

$k \geq 1, X \in R^{n \times n}, X \geq 0, \rho(X) < 1, p = r \geq 0, r^T(I-X) > 0.$

If  $k = \infty$  then it is required in addition that  $\rho(X) < \bar{\rho}$  where  $\bar{\rho}$  is the radius of convergence of the scalar power series

$\sum_{i=1}^{\infty} \alpha_i X^i.$

(g)  $M = e^X$ , or  $M = I + \sinh X$ , or  $M = \cosh X, X \in R^{n \times n}, X \geq 0,$

$\rho(X) < 1, p = r \geq 0, r^T(I-X) > 0.$

(h)  $M \geq 0, \rho(M) < \frac{1}{2}, 2M \leq (I-M)^{-1}, p = r \geq 0, r^T(2I-(I-M)^{-1}) > 0.$

Proof (a) Set in Theorem 3,  $s = 0, s_0 = 0, Z_1 = I, E = 0, B = I, A = a, G = b^T$ , and  $H = h \geq 0$  where  $h$  satisfies  $Z_2 h > 0$  which is possible because  $Z_2 \in K$ . Define

$Z_5 = Z_3 - b^T h$



and note that  $Z_5$  and  $Z_3$  are real numbers here. Conditions (9) are satisfied provided that

$$N = Z_2 h - Z_5 a \geq 0$$

This is achieved by taking  $Z_5 = \min_i \frac{(Z_2 h)_i}{a_i} > 0$ . To satisfy (10)

we need

$$r^T - r_0 b^T > 0 \text{ and } -r^T h + r_0 (Z_5 + b^T h) > 0$$

These inequalities are satisfied if we take  $r_0 = 1$ ,

$r_i = (1+\delta)b_i$  for  $b_i > 0$  and  $r_i = \delta$  for  $b_i = 0$  where

$$(13) \quad 0 < \delta < Z_5 / (b^T h + \sum h_j) = \min_i \frac{(Z_2 h)_i}{(b^T h + \sum h_j) a_i} > 0$$

(b) Set in Corollary 7 and condition (12c) thereof

$A = B = Z_1 = H = I, Z_2 = Z_3 = Z_4 = Z_5 = Z_6, L = N = G = Z_2 = Z_5$  and  $E = 0$ .

(c) Set in part (b) of this theorem  $Z_2 = D - (-F)_+$ ,

$Z_5 = D - |F| = D - (F)_+ - (-F)_+$ , where  $((F)_+)_i = \max\{0, F_{ij}\}$ .

(d) Set  $s = e$  in part (c) of this theorem.

(e) From [12, Theorem 6.2.17] we have that  $D - |F| \in K$  and hence we can apply part (c) of this theorem.

(f) Set in Corollary 6 part (a),  $f(x) = \sum_{i=1}^k \alpha_i x_i^1$ . If  $k = \infty$ , the requirement that  $\rho(x) < \bar{\rho}$  is sufficient for the power series  $\sum_{i=1}^{\infty} \alpha_i x_i^1$

to converge [1, Theorem 4-1-1].

(g) These are special cases of part (f) of this theorem and for each case of which  $\bar{\rho} = \infty$ .

(h) Set in Corollary 5, part (a),  $f(M) = \sum_{i=1}^{\infty} M^i$ . Then

$$I + Mf(M) - M = (I-M)^{-1} - 2M \geq 0, \text{ and } \rho(f(M)) = \sum_{i=1}^{\infty} (\rho(M))^i = \frac{\rho(M)}{1-\rho(M)} < 1,$$

since  $\rho(M) < \frac{1}{2}$ . We also have that  $I - f(M) = 2I - (I-M)^{-1} \in K$ , because  $\rho(f(M)) < 1$ .  $\square$

Remark 3 Cases (a), (d) and the first case of (e), of Theorem 4 above, are generalizations of corresponding cases of Theorem 6 of [11]. These generalizations have also been obtained independently in [3] by a different approach. Note that in case (a), if  $b > 0$ , then  $p = (1+\delta)b$  and hence the constant factor  $(1+\delta)$  can be ignored in solving the linear program (2). This reduces to Theorem 6(a) of [11].

Remark 4 In solving the linear program (2) associated with case (a) of Theorem 4, instead of calculating the upper bound on  $\delta$  given by (13), which may involve a linear program for determining  $h$ , we can start with an arbitrary positive value of  $\delta$  and solve (2) for successive decreasing values of  $\delta$  using the techniques of parametric linear programming [4] until we arrive at a solution of the linear complementarity problem (1).

We give in Table 1 a convenient summary of some of the cases for which the linear complementarity problem (1) is solvable as a linear program together with the conditions that  $M$  and  $p$  must satisfy.

TABLE 1 (Continued)

Matrix M of (1)	Conditions on M	Vector p of (2)	Matrix M of (1)	Conditions on M	Vector p of (2)
$M = Z_2 Z_1^{-1}$	$Z_1, Z_2 \in Z$ $r^T Z_1 + s^T Z_2 > 0$ $r, s \geq 0$	$p = r + M^T s$	M	$M_{ij} > \sum_{j \neq i}  M_{ij} , i=1, \dots, n$	$p = M^T s, s \geq 0$
M	$M \in Z$	$p > 0$	M	$M_{ij} \geq \sum_{j \neq i}  M_{ij} , i=1, \dots, n$	$s^T (D -  F ) > 0$
M	$M^{-1} \in Z$	$p = M^T s, s > 0$	M	strict inequality holds for at least one i, M irreducible	$D = \text{diag } M$
$M = Z_2 + ab^T$	$Z_2 \in K$ $0 \neq a \geq 0$ $0 \neq b \geq 0$	$p_1 = (1+\delta)b_1$ for $b_1 > 0$ $p_i = \delta$ for $b_i = 0$ $\delta > 0$ , sufficiently small (See (13))	$MZ_1 = Z_2 + \delta qe^T$	$M_{jj} \geq \sum_{i \neq j}  M_{ij} , j=1, \dots, n$	$F = \text{off-diag } M$
$M = 2Z_2 - Z_5$	$Z_2 \in Z, Z_5 \in K$ $Z_2 \geq Z_5$	$p = M^T s, s \geq 0$ $s^T Z_5 > 0$	$MZ_1 = Z_2 + \delta qe^T$	strict inequality holds for at least one j, M irreducible	$p = r + M^T s$
M	$D =  F  \in K$ $D = \text{diag } M$ $F = \text{off-diag } M$	$p = M^T s, s \geq 0$ $s^T (D -  F ) > 0$	$MZ_1 = Z_2 + \delta qe^T$	$Z_1, Z_2 \in Z, \delta > 0$ $r^T Z_1 + s^T Z_2 \geq 0$ $r, s \geq 0, r + s > 0$	$p = r > 0$ $r^T Z_1 \geq 0$
M	$M > 0, n \geq 3$ $M_{jj} \geq \sum_{i \neq j} M_{ij}, j = 1, \dots, n$	$p = e$	$MZ_1 = Z_2$	$Z_1, Z_2 \in Z, \delta > 0$ $Z_2 \in K_0, Z_2$ irreducible	$p = M^T s, s > 0$ $s^T Z_2 > 0$ $p = r$
M	$M_{jj} > \sum_{i \neq j}  M_{ij} , j=1, \dots, n$	$p = M^T e$	$M = I + \sum_{i=1}^k \alpha_i x_i x_i^T$	$Z_1, Z_2 \in Z, r^T Z_1 \geq 0, r > 0$ $(Z_2)_{ij} < 0$ for $\alpha_i < 0$ and $j \neq i$	$p = r \geq 0$ $r^T (I - X) > 0$

(For  $k = \infty, \rho(X) < \bar{\rho}$  is also required, where  $\bar{\rho}$  is the radius of convergence of  $\sum_{i=1}^{\infty} \alpha_i x_i x_i^T$ )

TABLE 1 (Continued)

Matrix M of (1)

$$\left. \begin{aligned} M &= e^X \\ M &= I + \sinh X \\ M &= \cosh X \end{aligned} \right\}$$

M

$X \geq 0, \rho(X) < 1$

$M \geq 0, \rho(M) < \frac{1}{2}$

$2M \leq (I-M)^{-1}$

Vector p of (2)

$p = r \geq 0$

$r^T(I-X) > 0$

$p = r \geq 0$

$r^T(2I - (I-M)^{-1}) > 0$

5. NUMERICAL EXAMPLES

We give now some examples for which the linear complementarity problem can be solved by a linear program but not by Lemke's method [8,2] or the principal pivoting procedure [2].

Example 1

$$M = \begin{pmatrix} 0 & 3 & 4 \\ 1 & -1 & 0 \\ 0 & -1 & -3 \end{pmatrix} \quad q = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

This example satisfies the conditions of Theorem 2 with  $c = s = e, r = 0, Z_1 = -I$  and

$$Z_2 = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

By using  $p = M^T e$  and solving the linear program (2) we obtain

$$\begin{array}{r} w_1 = 0 \\ w_2 = -1 \\ w_3 = 0 \end{array} \quad \begin{array}{r} -x_1 \\ -x_2 \\ -x_3 \end{array} \quad \begin{array}{r} 1 \\ -4 \\ 0 \end{array} \quad \begin{array}{r} 1 \\ 0 \\ 1 \end{array}$$

$$\begin{array}{r} x_3 = 1/5 \\ x_1 = 2/5 \\ x_2 = 2/5 \end{array}$$

Three pivots of dual simplex algorithm

Hence the solution of the linear program which is  $x_1 = 2/5, x_2 = 2/5, x_3 = 1/5, w_1 = w_2 = w_3 = 0$ , solves the linear complementarity problem. Application of Lemke's algorithm by adding one artificial variable  $x_0$

$$\begin{array}{r} w_1 = 0 \\ w_2 = -1 \\ w_3 = 0 \\ w_0 = 0 \end{array} \quad \begin{array}{r} -x_1 \\ -x_2 \\ -x_3 \\ -x_0 \end{array} \quad \begin{array}{r} -4 \\ 0 \\ 3 \\ -1 \end{array} \quad \begin{array}{r} 1 \\ 0 \\ 1 \\ 1 \end{array} \quad \begin{array}{r} -2 \\ 0 \\ 1 \\ 0 \end{array}$$

to make the problem feasible, and pivoting on the circled -1 above gives

$$\begin{array}{cccc} -x_1 & -x_2 & -x_3 & -w_1 & 1 \\ x_0 = & 0 & 3 & 4 & -1 & 2 \\ w_2 = & -1 & 4 & 4 & -1 & 2 \\ w_3 = & 0 & 4 & 7 & -1 & 3 \\ w_0 = & 0 & 3 & 4 & -1 & 2 \end{array}$$

The complement of  $w_1$  which just became nonbasic is  $x_1$  and can be made arbitrarily large and hence we have encountered a ray.

If we apply the principal pivoting method, then in the tableau

$$\begin{array}{cccc} -x_1 & -x_2 & -x_3 & 1 \\ w_1 = & 0 & -3 & -4 & -2 \\ w_2 = & -1 & 1 & 0 & 0 \\ w_3 = & 0 & 1 & 3 & 1 \end{array}$$

$x_1$  is the driving variable and  $w_1$  is the distinguished variable. Again by increasing  $x_1$  an unbounded (infeasible) ray is generated.

Example 2

$$M = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \quad q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This example which is due to Ponstein [13] arose from different considerations regarding the linear complementarity problem. This example satisfies the conditions of Theorem 2 with  $r^T = (0 \ 1)$ ,

$$s^T = (1 \ 0), c^T = (0 \ 2),$$

$$z_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, z_2 = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix},$$

and  $p^T = (-1 \ 2)$ . Hence the linear program (2) with  $p^T = (-1 \ 2)$  has the (unique) solution  $x_1 = 1, x_2 = 0, w_1 = 0, w_2 = 1$  which also solves the linear complementarity problem. Using Lemke's procedure we have the tableaux

$$\begin{array}{cccc} -x_1 & -x_2 & -x_0 & 1 \\ w_1 = & 1 & -1 & -1 & 1 \\ w_2 = & -2 & 1 & 0 & -1 \\ w_3 = & 0 & 0 & -1 & 0 \end{array} \quad \begin{array}{cccc} -x_1 & -x_2 & -w_2 & 1 \\ w_1 = & 3 & -2 & -1 & 2 \\ x_1 = & 2 & -1 & -1 & 1 \\ w_0 = & 2 & -1 & -1 & 1 \end{array}$$

and an unbounded ray has been obtained by letting  $x_2 \rightarrow \infty$ . Using the principal pivoting algorithm we have

$$\begin{array}{cccc} -x_1 & -x_2 & 1 \\ w_1 = & 1 & -1 & 1 \\ w_2 = & -2 & 1 & -1 \end{array} \quad \begin{array}{cccc} -x_1 & -w_2 & 0 \\ w_1 = & -1 & 1 & 0 \\ x_2 = & -2 & 1 & -1 \end{array}$$

and the next tableau would be identical to the first one and hence the method cycles.

REFERENCES

1. S. Barnett & C. Storey: "Matrix methods in stability theory," Nelson, London, 1970.
2. R. W. Cottle & G. B. Dantzig: "Complementary pivot theory of mathematical programming," *Linear Algebra and Its Appl.* 1, 1968, 103-125.
3. R. W. Cottle & J.-S. Pang: "On solving linear complementarity problems as linear programs," Technical Report SOL-76-5, March 1976, Systems Optimization Laboratory, Stanford University, Stanford, California.
4. G. B. Dantzig: "Linear programming and extensions," Princeton University Press, Princeton, 1963.
5. M. Fiedler & V. Pták: "On matrices with non-positive off-diagonal elements and positive principal minors," *Czech. Math. J.* 12, 1962, 382-400.
6. M. Fiedler & V. Pták: "Some results on matrices of class K and their application to the convergence rate of iteration procedures," *Czech. Math. J.* 16, 1966, 260-273.
7. M. Fiedler & V. Pták: "Some generalizations of positive definiteness and monotonicity," *Numer. Math.* 9, 1966, 163-172.
8. C. E. Lemke: "Bimatrix equilibrium points and mathematical programming," *Management Science* 11, 1965, 681-689.
9. O. L. Mangasarian: "Nonlinear programming," McGraw-Hill, New York, 1969.
10. O. L. Mangasarian: "Linear complementarity problems solvable by a single linear program," *Tech. Report No. 237, Comp. Sci. Dept., Univ. of Wis., Madison, Jan. 1975, forthcoming in Math. Programming.*

11. O. L. Mangasarian: "Solution of linear complementarity problems by linear programming," in "Numerical analysis Dundee 1975," G. W. Watson (ed), *Lecture Notes in Mathematics* No. 506, Springer-Verlag, Berlin, 1976, 166-175.
12. J. M. Ortega: "Numerical analysis a second course," Academic Press, New York, 1972.
13. J. Ponstein: Private communication, March 5, 1976.
14. R. T. Rockafellar: "Convex analysis," Princeton Univ. Press, Princeton, New Jersey, 1970.