Characterization of Matrix-exponential Distributions

Mark William Fackrell

Thesis submitted for the degree of

Doctor of Philosophy

in

Applied Mathematics

at

The University of Adelaide

(Faculty of Engineering, Computer and Mathematical Sciences)

School of Applied Mathematics

November 18, 2003

Contents

Signed Statement **vi**

Acknowledgements **vii**

Dedication **viii**

Abstract **ix**

1 Introduction **1**

2 Phase-type Distributions **8**

2.1 Introduction 8

2.2 Continuous Phase-type Distributions 11

2.3 Discrete Phase-type Distributions 16

2.4 Characterization of Phase-type Distributions 18

2.5 Closure Properties of Phase-type Distributions 24

2.6 Concluding Remarks 28

3 Parameter Estimation and Distribution Approximation with Phase-type Distributions **29**

3.1 Introduction 29

3.2 Parameter Estimation and Distribution Approximation Methods for Phase-type Distributions 31

3.3 Problems with Phase-type Parameter Estimation and Distribution Approximation Methods . . . 37
3.4 Concluding Remarks . . . 43

4 Parameter Estimation and Distribution Approximation in the Laplace-Stieltjes Transform Domain 44
4.1 Introduction . . . 44
4.2 Preliminaries . . . 47
4.3 Harris and Marchal's Method 1 . . . 49
4.4 Harris and Marchal's Method 2 . . . 55
4.5 Problems With Parameter Estimation and Distribution Approximation in the Laplace-Stieltjes Transform Domain . . . 59

5 Matrix-exponential Distributions 61
5.1 Introduction . . . 61
5.2 Matrix-exponential Distributions . . . 63
5.3 The Physical Interpretation of Matrix-exponential Distributions . . . 65
5.4 Matrix-exponential Representations . . . 69
5.5 Distribution Functions . . . 74
5.6 Characterization of Matrix-exponential Distributions . . . 80

6 The Region Ω_p 89
6.1 The Region Ω_3 . . . 89
6.2 The Constraint $g(\mathbf{x}, u) = 0$ as $u \to \infty$ . . . 103
6.3 The Region Ω_p . . . 107
6.4 Comparing the Classes of Matrix-exponential and Phase-type Distributions . . . 110

7 An Algorithm for Identifying Matrix-exponential Distributions 113
7.1 Introduction . . . 113
7.2 The Work of Dehon and Latouche . . . 114

7.3 The Matrix-exponential Identification Algorithm 120
7.4 Examples 125
7.5 Another Parameterization of Ω_3 130
7.6 The Boundedness of Ω_p 144
7.7 Concluding Remarks 147

8 An Alternative Algorithm for Identifying Matrix-exponential Distributions 149

8.1 Introduction 149
8.2 The Matrix-exponential Identification Problem 150
8.3 Semi-infinite Programming 154
8.4 The Algorithm 158
8.5 Examples 161
8.6 Problems and Suggested Improvements 164

9 Fitting with Matrix-exponential Distributions 165

9.1 Introduction 165
9.2 Fitting Matrix-exponential Distributions to Data 166
9.3 Examples 170

10 Conclusion 182

Bibliography 185

List of Figures

4.3.1 Histogram of the *PH* data 52
4.3.2 Empirical cumulative distribution of the *PH* data 52
4.3.3 *ELST* of the *PH* data and fitted *RLT* 53
4.3.4 Adjusted transform fit 56
4.3.5 Adjusted density fit 56
4.3.6 Adjusted distribution fit 57
4.4.1 *ELST* of the *PH* data and fitted *RLT* 58

6.1.1 Plots of Ω_3 for various configurations of the zeros of $b(\lambda)$ 91
6.1.2 Plots of $\partial\Omega_3$ for various configurations of the zeros of $b(\lambda)$ 104
6.3.1 Diagram of the sets P_3, P_4, P_5, and P_∞ 110

7.2.1 Diagram of C_3 showing T_3 and the arrangement of the points that represent the distributions F_1, F_2, F_3, F_{12}, F_{13}, F_{23}, and F_{123} 117
7.3.1 Diagram of Ω_3 showing the points P, Q, and X 121
7.4.1 Diagram of Ω_3 for Example 1 126
7.4.2 Graph of $r(u)$ versus u for Example 1 127
7.4.3 Diagram of Ω_3 and Σ_3 for Example 2 128
7.4.4 Graph of $r(u)$ versus u for Example 2 129
7.4.5 Graph of $r(u)$ versus u for Example 3 131
7.5.1 Diagram of Ω_3 showing the points O, P, R, and S 133
7.5.2 Diagram of Ω_3 showing the points O, P, and R 144
7.6.1 Diagram of the curve Z and its convex hull $C(Z)$ 146

9.3.1 Histogram of the shifted inter-eruption times of the Old Faithful geyser data set . . . 171
9.3.2 Empirical cumulative distribution of the shifted inter-eruption times of the Old Faithful geyser data set . . . 172
9.3.3 Density functions for the three *ME* and one *PH* fits plotted with the histogram of the data . . . 176
9.3.4 Distribution functions for the three *ME* and one *PH* fits with the empirical cumulative distribution of the data . . . 177
9.3.5 Density functions for the three *ME* and one *PH* approximations plotted with the density function of the uniform distribution on $(1, 2)$ . . 180
9.3.6 Distribution functions for the three *ME* and one *PH* approximations with the distribution function for the uniform distribution on $(1, 2)$. 181

Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

SIGNED: .... DATE: ....

Acknowledgements

I would like to extend my sincere thanks to my two supervisors Prof Peter Taylor and Dr Nigel Bean for their tireless support and encouragement over the past four and a half years.

Thanks also go to Dr David Green, Dr Andre Costa, and Kate Kennedy for their patient assistance in many matters throughout the course of this PhD.

The staff and students of the Teletraffic Research Centre have provided a supportive and friendly environment in which to study and I wish to express my gratitude to them.

Thanks go to Associate Prof Andrew Eberhard of the Department of Mathematics and Statistics, RMIT University, Melbourne, for suggesting the semi-infinite programming approach that led to Chapters 8 and 9. I would also like to express my gratitude to Prof Lang White of the Department of Electrical and Electronic Engineering, University of Adelaide, for his advice and encouragement, particularly in the vital, early stages of candidature.

The funding for this PhD research was provided by a Federal Government Australian Postgraduate Award scholarship and a Teletraffic Research Centre top-up scholarship. I am grateful to both funding bodies for their financial assistance without which this research would not have been possible.

I would also like to thank the two examiners of this thesis who provided prompt feedback and valuable reports.

And last, but certainly not least, a big thankyou to my wife Jenny and son Matthew for enduring much throughout the course of this PhD.

Dedication

This thesis is dedicated to Associate Professor William (Bill) Henderson (1943–2001) who was a truly inspirational applied probabilist.

Abstract

A random variable that is defined as the absorption time of an evanescent finite-state continuous-time Markov chain is said to have a *phase-type* distribution. A phase-type distribution is said to have a *representation* $(\boldsymbol{\alpha}, \boldsymbol{T})$ where $\boldsymbol{\alpha}$ is the initial state probability distribution and $\boldsymbol{T}$ is the infinitesimal generator of the Markov chain. The distribution function of a phase-type distribution can be expressed in terms of this representation. The wider class of *matrix-exponential* distributions have distribution functions of the same form as phase-type distributions, but their representations do not need to have a simple probabilistic interpretation. This class can be equivalently defined as the class of all distributions that have rational Laplace-Stieltjes transform. There exists a one-to-one correspondence between the Laplace-Stieltjes transform of a matrix-exponential distribution and a representation $(\boldsymbol{\beta}, \boldsymbol{S})$ for it where $\boldsymbol{S}$ is a companion matrix.

In order to use matrix-exponential distributions to fit data or approximate probability distributions the following question needs to be answered:

> *"Given a rational Laplace-Stieltjes transform, or a pair $(\boldsymbol{\beta}, \boldsymbol{S})$ where $\boldsymbol{S}$ is a companion matrix, when do they correspond to a matrix-exponential distribution?"*

In this thesis we address this problem and demonstrate how its solution can be applied to the abovementioned fitting or approximation problem.