Characterization of metric spaces whose free space is isometric to ℓ_1*

Aude Dalet Pedro L. Kaufmann Antonín Procházka

Abstract

We characterize metric spaces whose Lipschitz free space is isometric to ℓ_1. In particular, we show that the Lipschitz free space over an ultrametric space is not isometric to $\ell_1(\Gamma)$ for any set Γ. We give a lower bound for the Banach-Mazur distance in the finite case.

1 Introduction

An $\mathbb{R}$-*tree* (T,d) is a metric space which is geodesic (i.e. for every pair of points $x, y \in T$ there is an isometry $\phi : [0, d(x,y)] \to T$ with $\phi(0) = x$ and $\phi(d(x,y)) = y$) and satisfies the *4-point condition*:

$$\forall\, a, b, c, d \in T \quad d(a,b) + d(c,d) \leq \max\left\{d(a,c) + d(b,d), d(b,c) + d(a,d)\right\}.$$

A space which satisfies just the 4-point condition is called 0-*hyperbolic*. Clearly, a subset of an $\mathbb{R}$-tree is 0-hyperbolic. The converse is also true [4, 7], so we will use terms "0-hyperbolic" and "subset of an $\mathbb{R}$-tree" interchangeably. Moreover, for every 0-hyperbolic M there exists a unique (up to isometry) minimal $\mathbb{R}$-tree which contains M, we will denote it $\operatorname{conv}(M)$. Thus one can define the Lebesgue measure $\lambda(M)$ of M which is independent of any particular tree containing M. We will say that M is negligible if $\lambda(M) = 0$. A. Godard [9] has proved that a

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metric space M is 0-hyperbolic if and only if $\mathcal{F}(M)$, the *Lipschitz free space over M* (see the definition in the next section), is isometric to a subspace of some $L_1(\mu)$. In this paper we are interested in metric spaces whose free space is isometric to (a subspace of) ℓ_1. By the above, such spaces must be 0-hyperbolic, and it is also easy to see that they must be negligible (if not the free space will contain L_1).

So let M be a separable negligible complete metric space which is a subset of an $\mathbb{R}$-tree. One can ask two questions:

- When is $\mathcal{F}(M)$ isometric to ℓ_1?
- When is $\mathcal{F}(M)$ isometric to a subspace of ℓ_1?

Concerning the first question, the results of A. Godard point to the relevance of branching points of $\operatorname{conv}(M)$. We recall that a point $b \in T$ is a branching point of a tree T if $T \setminus \{b\}$ has at least three connected components. A sufficient condition for $\mathcal{F}(M) \equiv \ell_1$ is that M contain all the branching points of $\operatorname{conv}(M)$ [9, Corollary 3.4]. The main result of this paper (Theorem 5) claims that this is also a necessary condition. We give two different proofs – one is based on properties of the extreme points of $B_{\mathcal{F}(M)}$ and the other on properties of the extreme points of $B_{\mathrm{Lip}_0(M)}$ (Theorem 4).

For certain finite 0-hyperbolic spaces M we have a third proof which also allows to compute a simple lower bound for the Banach-Mazur distance between $\mathcal{F}(M)$ and $\ell_1^{|M|-1}$ (Proposition 9).

As far as the second question is concerned, it is obviously enough that M be a subset of a metric space N such that $\mathcal{F}(N) \equiv \ell_1$. We will show that this is the case when M is compact, 0-hyperbolic and negligible (Proposition 8). We do not know whether one can drop the assumption of compactness in general.

This paper is an outgrowth of a shorter preprint in which we have shown that for any ultrametric space M, the free space $\mathcal{F}(M)$ is never isometric to ℓ_1 (Corollary 6) answering a question posed by M. Cúth and M. Doucha in a draft of [5]. In the meantime, this question has been independently answered in [5].

2 Preliminaries

As usual, for a metric space M with a distinguished point $0 \in M$, the *Lipschitz free space* $\mathcal{F}(M)$ is the norm-closed linear span of $\{\delta_x : x \in M\}$ in the space $\mathrm{Lip}_0(M)^*$, where the Banach space $\mathrm{Lip}_0(M) = \{f \in \mathbb{R}^M : f \text{ Lipschitz}, f(0) = 0\}$ is equipped with the norm $\|f\|_L := \sup\left\{\dfrac{f(x)-f(y)}{d(x,y)} : x \neq y\right\}$. It is well known that
$\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$ isometrically. More about the very interesting class of Lipschitz-free spaces can be found in [10].

To prove that a Lispchitz-free space is not isometric to ℓ_1, we will exhibit two extreme points of its unit ball at distance less than one. For this purpose we will use the notion of *peaking function at* (x,y), $x \neq y$, which is a function $f \in \mathrm{Lip}_0(M)$ such that $\frac{f(x)-f(y)}{d(x,y)} = 1$ and for every open set U of $\{(x,y) \in M \times M, x \neq y\}$

containing (x,y) and (y,x), there exists $\delta > 0$ with

$$(z,t) \notin U \Rightarrow \frac{|f(z)-f(t)|}{d(z,t)} \leq 1-\delta.$$

This definition is equivalent to: $\frac{f(x)-f(y)}{d(x,y)} = 1$ and if $(u_n)_{n\in\mathbb{N}}, (u_n)_{n\in\mathbb{N}} \subset M$, then

$$\lim_{n\to+\infty} \frac{f(u_n)-f(v_n)}{d(u_n,v_n)} = 1 \Rightarrow \lim_{n\to+\infty} u_n = x \text{ and } \lim_{n\to+\infty} v_n = y.$$

Moreover in [11, Proposition 2.4.2], the following is proved:

Proposition 1. *Let (M,d) be a complete metric space and $x \neq y$ in M. If there is a function $f \in \mathrm{Lip}_0(M)$ peaking at (x,y), then $\frac{\delta_x-\delta_y}{d(x,y)}$ is an extreme point of the unit ball of $\mathrm{Lip}_0(M)^*$. In particular, it is an extreme point of the unit ball of $\mathcal{F}(M)$.*

Given an $\mathbb{R}$-tree (T,d) and $x,y \in T$, the *segment* $[x,y]$ is defined as the range of the unique isometry $\phi_{x,y}$ from $[0,d(x,y)] \subset \mathbb{R}$ into T which maps 0 to x and $d(x,y)$ to y.

We recall that for every 0-hyperbolic space M, there exists an $\mathbb{R}$-tree T such that $M \subset T$. The set $\bigcup\{[x,y] : x,y \in M\} \subset T$ is then also an $\mathbb{R}$-tree. It is clearly a minimal $\mathbb{R}$-tree containing M; it is unique up to an isometry and will be denoted $\mathrm{conv}(M)$. Simple examples show that $\mathrm{conv}(M)$ does not have to be complete when M is. This does not present any difficulty in what follows.

A point $b \in T$ is said to be a *branching point* if there are three distinct points $x,y,z \in T\setminus\{b\}$ with $[x,b]\cap[y,b] = [x,b]\cap[z,b] = [y,b]\cap[z,b] = \{b\}$. We say that the branching point b is witnessed by x,y,z. The set of all branching points of T is denoted $Br(T)$. If M is 0-hyperbolic, the set of all branching points of $\mathrm{conv}(M)$ is denoted $Br(M)$.

A subset A of T is *measurable* if $\phi_{x,y}^{-1}(A)$ is Lebesgue-measurable, for every x and y in T. For a segment $S = [x,y]$ in T and A measurable, we denote $\lambda_S(A) := \lambda(\phi_{x,y}^{-1}(A))$, with λ the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{R}$ be the set of subsets of T that can be written as a finite union of disjoint segments. For $R = \bigcup_{k=1}^{r} S_k \in \mathcal{R}$, define $\lambda_R(A) := \sum_{k=1}^{r} \lambda_{S_k}(A)$ and finally, set $\lambda_T(A) := \sup_{R\in\mathcal{R}} \lambda_R(A)$. If M is 0-hyperbolic, we put simply $\lambda(M) := \lambda_{\mathrm{conv}(M)}(M)$. We say that M is *negligible* if $\lambda(M) = 0$.

Given two points x and y in T, we will denote $\pi_{xy} : T \to [x,y]$ the metric projection onto the segment $[x,y]$. It is well known and easily seen that π_{xy} is non-expansive (see [1, 3]).

Finally, we recall that a metric space (M,d) is *ultrametric* if $d(x,y) \leq \max\{d(x,z), d(y,z)\}$ for any $x,y,z \in M$.

3 Isometries with ℓ_1

Let us start by characterizing precisely when there exists a function peaking at (x,y) for points $x,y \in M \subset T$.

Proposition 2. *Let (M,d) be a complete subset of an $\mathbb{R}$-tree and $x,y \in M$, $x \neq y$. The following assertions are equivalent*

(i) There is $f \in \mathrm{Lip}_0(M)$ peaking at (x,y).

(ii) $M \cap [x,y] = \{x,y\}$ and for every $p \in \{x,y\}$,

$$\liminf_{u,v\to p} \frac{d(\pi_{xy}(u),u)+d(\pi_{xy}(v),v)}{d(\pi_{xy}(u),\pi_{xy}(v))} > 0,$$
$$(\textit{with the convention that } \frac{\alpha}{0} = +\infty). \quad (1)$$

(iii) $M \cap [x,y] = \{x,y\}$ and for every $p \in \{x,y\}$,

$$\liminf_{u\to p} \frac{d(\pi_{xy}(u),u)}{d(\pi_{xy}(u),p)} > 0, \ (\textit{with the convention that } \frac{\alpha}{0} = +\infty). \quad (2)$$

Proof. (ii) $\Rightarrow$ (i) Let us first suppose that x,y satisfy (1) and $[x,y]\cap M = \{x,y\}$. For any $u \in M$ we define $f(u) = d(y,\pi_{xy}(u))$. Then $\frac{f(x)-f(y)}{d(x,y)} = 1$ and $\|f\|_L = 1$. Consider $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset M$ such that $\lim_{n\to+\infty} \frac{f(x_n)-f(y_n)}{d(x_n,y_n)} = 1$. We thus have for n large enough

$$d(y,\pi_{xy}(x_n)) = f(x_n) > f(y_n) = d(y,\pi_{xy}(y_n)). \quad (3)$$

It follows

$$1 = \lim_{n\to+\infty} \frac{f(x_n)-f(y_n)}{d(x_n,y_n)} = \lim_{n\to+\infty} \frac{d(\pi_{xy}(x_n),\pi_{xy}(y_n))}{d(x_n,\pi_{xy}(x_n)) + d(\pi_{xy}(x_n),\pi_{xy}(y_n)) + d(\pi_{xy}(y_n),y_n)}$$

and in particular

$$\lim_{n\to\infty} \frac{d(x_n,\pi_{xy}(x_n)) + d(\pi_{xy}(y_n),y_n)}{d(\pi_{xy}(x_n),\pi_{xy}(y_n))} = 0. \quad (4)$$

Since $\lim_{n\to+\infty} d(x_n,\pi_{xy}(x_n)) = \lim_{n\to+\infty} d(y_n,\pi_{xy}(y_n)) = 0$, the sets of cluster points of the sequences $((\pi_{xy}(x_n),\pi_{xy}(y_n)))_{n\in\mathbb{N}} \subset [x,y]^2$ and $((x_n,y_n))_{n\in\mathbb{N}} \subset M^2$ coincide. By compactness of $[x,y]^2$ there exists such a cluster point $(u,v) \in [x,y]^2$. Since the space M is complete, $(u,v) \in M^2$, and therefore $(u,v) \in \{(y,x),(x,x),(y,y),(x,y)\}$. Clearly, (3) implies $(u,v) \neq (y,x)$, and (1) together with (4) imply that $(u,v) \neq (x,x)$ and $(u,v) \neq (y,y)$. We thus get that (x_n) converges to x and (y_n) converges to y which proves that f is peaking at (x,y).

(i) $\Rightarrow$ (iii) If there is $z \in M \cap (x,y)$, then $\frac{\delta_x-\delta_y}{d(x,y)}$ is a convex combination of $\frac{\delta_x-\delta_z}{d(x,z)}$ and $\frac{\delta_z-\delta_y}{d(z,y)}$ so by Proposition 1, there cannot be a peaking function at (x,y).

Next assume that $[x,y] \cap M = \{x,y\}$ but there is a sequence $(u_n)_{n\in\mathbb{N}} \subset M$ converging to x and

$$\lim_{n\to+\infty} \frac{d(\pi_{x,y}(u_n),u_n)}{d(\pi_{x,y}(u_n),x)} = 0.$$

Let $f \in S_{\mathrm{Lip}_0(M)}$ be such that $\frac{f(x)-f(y)}{d(x,y)} = 1$. Let $\widetilde{f}$ be a 1-Lipschitz extension of f to $[x,y]$. Then

$$\begin{aligned}
|f(x)-f(u_n)| &\geq |f(x)-\widetilde{f}(\pi_{xy}(u_n))| - |\widetilde{f}(\pi_{xy}(u_n)) - f(u_n)| \\
&= d(x,\pi_{xy}(u_n)) - |\widetilde{f}(\pi_{xy}(u_n)) - f(u_n)| \\
&\geq d(x,\pi_{xy}(u_n)) - d(\pi_{xy}(u_n),u_n) \\
&\geq d(x,u_n) - 2d(\pi_{xy}(u_n),u_n)
\end{aligned}$$

It follows that

$$\lim_{n\to+\infty} \frac{|f(x)-f(u_n)|}{d(x,u_n)} = 1$$

and f is not peaking at (x,y).

(iii) $\Rightarrow$ (ii) Finally, since

$$\frac{d(u,\pi_{xy}(u)) + d(v,\pi_{xy}(v))}{d(\pi_{xy}(u),\pi_{xy}(v))} \geq \min\left\{\frac{d(\pi_{xy}(u),u)}{d(\pi_{xy}(u),p)}, \frac{d(\pi_{xy}(v),v)}{d(\pi_{xy}(v),p)}\right\}$$

we get

$$\liminf_{u\to p} \frac{d(\pi_{xy}(u),u)}{d(\pi_{xy}(u),p)} = 0$$

if the liminf in (1) is 0 for some $p \in \{x,y\}$. ∎

For the dual version of the proof we will need the following simple lemma which is valid in any metric space (see also [8] for a different proof).

Lemma 3. *Let (M,d) be any metric space and suppose that $0 \in A \subset M$. If $f \in \operatorname{ext}\left(B_{\mathrm{Lip}_0(A)}\right)$, then $f_S, f_I \in \operatorname{ext}\left(B_{\mathrm{Lip}_0(M)}\right)$ where*

$$f_S(x) := \sup_{z\in A} f(z) - d(z,x) \quad \textit{and} \quad f_I(x) := \inf_{z\in A} f(z) + d(z,x)$$

for $x \in M$.

Note that f_S resp. f_I above are the smallest resp. the largest 1-Lipschitz extensions of f (which basically gives the proof).

Proof. Let us give a proof for f_S. The proof for f_I is similar. Clearly $f_S(x) = f(x)$ for $x \in A$ and f_S is 1-Lipschitz as a supremum of 1-Lipschitz functions. Let $f_S = \frac{p+q}{2}$, $p, q \in B_{\mathrm{Lip}_0 M}$. If $x \in A$, then $p(x) = q(x) = f(x)$ as $f \in \mathrm{ext}\left(B_{\mathrm{Lip}_0 A}\right)$. If $x \in M \setminus A$, then $\forall\, z \in A$:

$$f(z) - p(x) = p(z) - p(x) \leq d(z, x).$$

Thus

$$f_S(x) = \sup_{z \in A} f(z) - d(z, x) \leq p(x).$$

By the same argument $f_S(x) \leq q(x)$. So $f_S(x) = p(x) = q(x)$ for all $x \in M$. ∎

We are now ready to state and prove a statement about extreme points of the ball in $\mathcal{F}(M)$ and $\mathrm{Lip}_0(M)$ when M is 0-hyperbolic.

Theorem 4. *Let M be a complete subset of an $\mathbb{R}$-tree. If there is $b \in Br(M) \setminus M$ then*

a) there exist $\mu \neq \nu \in \mathrm{ext}\left(B_{\mathcal{F}(M)}\right)$ such that $\|\mu - \nu\| < 2$.

b) there exist $f \neq g \in \mathrm{ext}\left(B_{\mathrm{Lip}_0(M)}\right)$ such that $\|f - g\|_L < 2$.

Since the Lipschitz free space over the completion of M is isometric to the Lipschitz free space of M, the above completeness hypothesis is not restrictive.

Proof. **a)** Let the points $x', y', z' \in M$ witness that $b \in Br(M)$. For $p' \in \{x', y', z'\}$ we denote $M_{p'} = \left\{w \in M : \pi_{bp'}(w) \in]b, p']\right\}$. Then $M_{p'}$ is closed in M as $\pi_{bp'}$ is continuous and b is isolated from M. Notice that $p \in M_{p'}$ satisfies (2) if there is $\alpha > 0$ such that $d(w, \pi_{bp}(w)) \geq \alpha d(p, \pi_{bp}(w))$ for all $w \in M_{p'}$. We will show that for every $0 < \alpha < 1$ such a point p exists. Indeed let $\frac{1-\alpha}{1+\alpha} =: \beta > 0$ and set $f(w) := d(b, w)$. Then Ekeland's variational principle [6] ensures the existence of a point $p \in M_{p'}$ such that $f(p) \leq f(w) + \beta d(p, w)$ for all $w \in M_{p'}$. It follows that

$$\begin{aligned} & d(b, \pi_{bp}(w)) + d(\pi_{bp}(w), p) && \leq && d(b, \pi_{bp}(w)) + d(\pi_{bp}(w), w) + \beta d(p, w) \\ \Longrightarrow\quad & d(\pi_{bp}(w), p) && \leq && d(\pi_{bp}(w), w) + \beta(d(\pi_{bp}(w), w) \\ & && && \qquad + d(\pi_{bp}(w), p)) \\ \Longrightarrow\quad & \tfrac{1-\beta}{1+\beta} d(p, \pi_{bp}(w)) && \leq && d(w, \pi_{bp}(w)). \end{aligned}$$

Thus, we see that we can find $x, y, z \in M$ such that (iii) in Proposition 2 is satisfied for the segments $[p, q]$ where $p \neq q \in \{x, y, z\}$. Proposition 1 then yields that $\frac{\delta_p - \delta_q}{d(p,q)}$ is an extreme point of the unit ball of $\mathcal{F}(M)$. Assuming, as we may, that $d(x,z) \leq d(z,y) \leq d(x,y)$, we obtain

$$\begin{aligned}\left\|\frac{\delta_x-\delta_y}{d(x,y)}-\frac{\delta_z-\delta_y}{d(y,z)}\right\|_{\mathcal{F}(M)} &= \left\|\frac{1}{d(x,y)}\left[(\delta_x-\delta_z)+(\delta_z-\delta_y)\right]-\frac{\delta_z-\delta_y}{d(y,z)}\right\|_{\mathcal{F}(M)} \\ &= \left\|\left[\frac{1}{d(x,y)}-\frac{1}{d(y,z)}\right](\delta_z-\delta_y)+\frac{\delta_x-\delta_z}{d(x,y)}\right\|_{\mathcal{F}(M)} \\ &\leq d(z,y)\left[\frac{1}{d(y,z)}-\frac{1}{d(x,y)}\right]+\frac{d(x,z)}{d(x,y)} \\ &= 1+\frac{d(x,z)-d(z,y)}{d(x,y)} \leq 1.\end{aligned}$$

In conclusion, $\mu := \frac{\delta_x-\delta_y}{d(x,y)}$ and $\nu := \frac{\delta_z-\delta_y}{d(y,z)}$ are two extreme points of the unit ball of $\mathcal{F}(M)$ at distance less than or equal to 1.

b) We denote $\delta := \inf\{d(w,b) : w \in M\}$. Let x, y, z be 3 points witnessing the fact that b is a branching point. Two pointed metric spaces which differ only by the choice of the base point have isometric free spaces. This trivial observation allows us to assume that $x = 0$ and that, for a fixed $0 < \varepsilon < 1$, we have $d(b,z) < (1+\varepsilon)\delta$. Let $M_z = \{w \in M : \pi_{zb}(w) \in (b,z]\}$. Let us consider the closed nonempty set $F = \{w \in M_z : d(b,z) \leq (1+\varepsilon)\delta\}$. Given $0 < \alpha < 1$ and using Ekeland's variational principle as above, we may assume that z satisfies $d(w, \pi_{zb}(w)) \geq \alpha d(z, \pi_{zb}(w))$ for all $w \in F$. Clearly $d(w, \pi_{zb}(w)) \geq \alpha d(z, \pi_{zb}(w))$ for all $w \in M_z \setminus F$.

We define $f(\cdot) := d(0,\cdot)$ on M and then $g_2(\cdot) := d(0,\cdot)$ on $M \setminus M_z$, $g_1 := (g_2)_S$ on $(M \setminus M_z) \cup \{z\}$ and finally $g := (g_1)_I$ on M. Both $f, g \in \operatorname{ext}\left(B_{\mathrm{Lip}_0(M)}\right)$ by Lemma 3. The fact that M is a subset of an $\mathbb{R}$-tree helps to write g explicitly:

$$g(w) = \begin{cases} d(0,w), & w \in M \setminus M_z, \\ d(0,b) - d(b,z) + d(z,w), & w \in M_z. \end{cases}$$

It follows that $f(w) - g(w) = 0$ for $w \in M \setminus M_z$ and $f(w) - g(w) = 2d(b, \pi_{zb}(w))$ otherwise. We have

$$\begin{aligned}\|f-g\|_L &= \max\left\{\sup_{w_1 \in M_z, w_2 \notin M_z} \frac{2d(b,\pi_{zb}(w_1))}{d(w_1,w_2)},\right. \\ &\qquad\qquad \left.\sup_{w_1,w_2 \in M_z} \frac{2\left|d(w_1,\pi_{zb}(w_1)) - d(w_2,\pi_{zb}(w_2))\right|}{d(w_1,w_2)}\right\} \\ &\leq \max\left\{\frac{2(1+\varepsilon)\delta}{2\delta}, \frac{2}{1+\alpha}\right\} < 2\end{aligned}$$

■

Theorem 5. *Let (M,d) be a complete metric space. The Lipschitz free space over M is isometric to $\ell_1(\Gamma)$ if and only if M is of density $|\Gamma| - 1$ and is negligible subset of an $\mathbb{R}$-tree T which contains all the branching points of T.*

Proof. The sufficiency follows from [9, Theorem 3.2]. Conversely, let us assume that $\mathcal{F}(M) \equiv \ell_1(\Gamma)$. Then M is of density $|\Gamma| - 1$ and it must be 0-hyperbolic by [9, Theorem 4.2]. In this case $T = \mathrm{conv}(M)$. If $\lambda_T(M) > 0$, there is a set $A \subset [0,1]$ of positive measure such that A embeds isometrically into M. Then $L_1 \simeq \mathcal{F}(A) \subset \mathcal{F}(M) \equiv \ell_1(\Gamma)$ which is absurd. Since the extreme points of the ball (resp. dual ball) and their distances are preserved by bijective isometries we get by Theorem 4 a) (resp. b)) that $Br(M) \subset M$. ■

Corollary 6. *Let M be an ultrametric space of cardinality at least 3. Then $\mathcal{F}(M)$ is not isometric to $\ell_1(\Gamma)$ for any Γ.*

Proof. The completion of M stays clearly ultrametric. Thus it can be isometrically embedded into an $\mathbb{R}$-tree [4]. However ultrametric spaces do not contain the interior of any segment, much less branching points. ■

4 Isometries with subspaces of ℓ_1

We shall now deal with the second question, i.e. when is $\mathcal{F}(M)$ isometric to a subspace of ℓ_1.

Lemma 7. *Let M be a compact subset an $\mathbb{R}$-tree such that $\lambda(M) = 0$. Then $\lambda_{\mathrm{conv}(M)}(\overline{Br(M)}) = 0$ where the closure is taken in $\mathrm{conv}(M)$.*

Proof. Clearly $\lambda_{\mathrm{conv}(M)}(\overline{Br(M)} \cap M) = 0$. Assume that $\lambda_{\mathrm{conv}(M)}(\overline{Br(M)} \setminus M) > 0$. Then $\overline{Br(M)} \setminus M$ is uncountable. Hence there is some $\delta > 0$ such that $\overline{Br(M)} \cap \{x \in T : \mathrm{dist}(x, M) \geq \delta\}$ is uncountable and thus the set $Br(M) \cap \left\{x \in T : \mathrm{dist}(x, M) \geq \frac{\delta}{2}\right\}$ is infinite. We conclude that there is an infinite δ-separated family in M. This is absurd as M was supposed to be compact. ■

Proposition 8. *Let M be a compact subset of an $\mathbb{R}$-tree such that $\lambda(M) = 0$. Then $\mathcal{F}(M)$ is isometric to a subspace of ℓ_1.*

Proof. Since M is compact, $\mathrm{conv}(M)$ is compact and thus separable. Indeed, the mapping $\Phi : M \times M \times [0,1] \to \mathrm{conv}(M)$ defined by $\Phi(x, y, t) := \phi_{xy}(td(x, y))$ is continuous by [3, Theorem II.4.1]. Now

$$\mathcal{F}(M) \subseteq \mathcal{F}(Br(M) \cup M) \equiv \ell_1$$

by [9, Corollary 3.4] as $\lambda_{\mathrm{conv}(M)}(\overline{Br(M) \cup M}) = 0$ by the previous lemma. ■

We do not know if the above proposition is valid when M is supposed to be proper.

5 Banach-Mazur distance to ℓ_1^n

In the case of finite subsets of $\mathbb{R}$-trees we get the following quantitative result.

Proposition 9. *Let $M = \{x_0, x_1, \ldots, x_n\}$, $n \geq 2$, be a subset of a $\mathbb{R}$-tree. Let $x_0 = 0$ be the distinguished point. Let us suppose that*

$$0 < \operatorname{sep}(M) := \frac{1}{2} \inf \{d(x,y) + d(x,z) - d(y,z) : x,y,z \in M \text{ distinct}\}.$$

Then

$$d_{BM}(\mathcal{F}(M), \ell_1^n) > \left(1 - \frac{\operatorname{sep}(M)}{4 \operatorname{diam}(M)}\right)^{-1}.$$

The condition $\operatorname{sep}(M) > 0$ implies immediately that for each $x \neq y \in M$ we have $[x,y] \cap M = \{x,y\}$. For the proof we will need the following lemmas. The first one is inspired by [2, Lemma 2.3].

Lemma 10. *Let X be a Banach space. Let $C = \bigcap_{i=1}^n x_i^{*-1}(-\infty, 1)$ where $x_i^* \in X^*$. Let $A \subset X \setminus C$ have the following property: for every $x \neq y \in A$, we have $\frac{x+y}{2} \in C$. Then the cardinality $|A|$ of A is at most n.*

Proof. For $x \in A$ let $\varphi(x) := i$ for some $i \in \{1, \ldots, n\}$ such that $x_i^*(x) \geq 1$. Since $1 > x_{\varphi(x)}^*\left(\frac{x+y}{2}\right)$ it follows that $x_{\varphi(x)}^*(y) < 1$ for every $y \in A$, $y \neq x$. Thus φ is injective and the claim follows. ∎

Lemma 11. *Let $f_1, \ldots, f_{2n+1} \in S_Y$ such that $\left\|\frac{f_i+f_j}{2}\right\| \leq 1 - \varepsilon$ for some $\varepsilon > 0$ and all $1 \leq i \neq j \leq 2n+1$. Then $d_{BM}(Y, \ell_\infty^n) > (1-\varepsilon)^{-1}$.*

Proof. Let $T : Y \to \ell_\infty^n$ such that $\|f\| \leq \|Tf\|_\infty \leq (1+\varepsilon)\|f\|$. Then $\|Tf_i\| \geq 1$ and $\left\|\frac{Tf_i+Tf_j}{2}\right\| < 1, i \neq j$, which is in contradiction with the previous lemma as $B_{\ell_\infty^n}^O$ is the intersection of $2n$ halfspaces. ∎

Proof of Proposition 9. Given $0 \leq i \neq j \leq n$, we will denote $\pi_{ij} := \pi_{x_i x_j}$ the metric projection onto $[x_i, y_j]$. Further we define the function $f_{ij} : M \to \mathbb{R}$ as $f_{ij}(z) := d(x_j, \pi_{ij}(z))$ for $z \in M$. Observe that since $\operatorname{sep}(M) > 0$, this is the function peaking at (x_i, x_j) from the proof of Proposition 2. It is clear that $\left|\frac{f_{ij}(x) - f_{ij}(y)}{d(x,y)}\right| = 1$ if and only if $\{x,y\} = \{x_i, x_j\}$. We further have that

$$\left|\frac{f_{ij}(x) - f_{ij}(y)}{d(x,y)}\right| \leq \frac{d(x,y) - \operatorname{sep}(M)}{d(x,y)} \leq 1 - \frac{\operatorname{sep}(M)}{\operatorname{diam} M}$$

for any other couple $x \neq y \in M$. Hence $\left\|\frac{f_{ij}+f_{kl}}{2}\right\|_L \leq 1 - \frac{\operatorname{sep}(M)}{2 \operatorname{diam} M}$ for each $(i,j) \neq (k,l)$. Since $n \geq 2$, we have that $(n+1)n \geq 2n+1$ and the result follows by Lemma 11. ∎

Remark 12. Note that the lower bound given in Proposition 9 is not optimal. This can be seen when $M = \{0, x_1, x_2\}$ is equilateral. We also don't know if this result extends to infinite subsets of $\mathbb{R}$-trees.

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Université Franche-Comté
Laboratoire de Mathématiques UMR 6623
16 route de Gray
25030 Besançon Cedex
France
emails :aude.dalet@univ-fcomte.fr, antonin.prochazka@univ-fcomte.fr

Universidade Federal de São Paulo, Instituto de Ciência e Tecnologia,
Campus São José dos Campos - Parque Tecnológico,
Avenida Doutor Altino Bondensan, 500,
12247-016 São José dos Campos/SP, Brazil
email :plkaufmann@unifesp.br