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CHARACTERIZATION OF MINIMUM ABERRATION 2^{n-k} DESIGNS IN TERMS OF THEIR COMPLEMENTARY DESIGNS

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A general result is obtained that relates the word-length pattern of a 2^{n-k} design to that of its complementary design. By applying this result and using group isomorphism, we are able to characterize minimum aberration 2^{n-k} designs in terms of properties of their complementary designs. The approach is quite powerful for small values of $2^{n-k} - n - 1$. In particular, we obtain minimum aberration 2^{n-k} designs with $2^{n-k} - n - 1 = 1$ to 11 for *any* n and k.

1. Introduction. Two-level fractional factorial designs are the most commonly used designs for factorial experiments. The practical and theoretical importance of this class of designs has long been established [Box, Hunter and Hunter (1978)]. A 2^{n-k} design denotes a design with n factors, each at two levels, and 2^{n-k} runs and is a 2^{-k} fraction of the full factorial 2^n design. Since the fraction can be chosen in many different ways, an important concern is the choice of fractional factorial designs with good properties. The most commonly used criterion for design selection is the minimum aberration (MA) criterion first defined by Fries and Hunter (1980). (Its definition will be given in Section 2.) It includes the resolution criterion of Box and Hunter (1961) as a special case, and many design tables such as those in Box, Hunter and Hunter (1978) satisfy the MA criterion. A detailed discussion on the MA criterion can be found in Chen, Sun and Wu (1993).

Although computing enables us to find many MA designs, especially the small ones, it is important to have good theoretical results on the MA designs because they will give us more insight into the structure of the MA designs and also are not limited by the capacity of computing. Characterization of MA designs is a challenging theoretical problem. There is a handful of papers in the literature, for example, Fries and Hunter (1980), Franklin (1984), Chen and Wu (1991) and Chen (1992), which address this issue. Chen and Wu (1991) gave a theoretical characterization of MA 2^{n-k} designs with $k = 3$ and 4 and Chen (1992) solved the problem for $k = 5$ by using a combination of theoretical and computational tools. Because of the technical difficulties,

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H. Chen and Hedayat (1996) proposed a modified (and technically easier) version of the MA criterion and obtained some interesting results.

The main purpose of this paper is to propose a new approach for characterizing MA designs in terms of their complementary designs. As will be shown in Section 2, a 2^{n-k} design can be viewed as a subset s of n elements in a set of $2^{n-k}-1$ elements and its complementary design corresponds to the complementary set $\bar{s}$ of s. In Section 3 we present some general results (Theorems 1 and 2) to relate the word-length pattern of s to that of $\bar{s}$. (The definition of word-length pattern is given in Section 2.) If s is much larger than $\bar{s}$, we can study the properties of s by working with the much smaller and simpler problem for $\bar{s}$. Using some rules derived from Theorem 2 and group isomorphism, we are able to characterize in Section 4 MA 2^{n-k} designs with $2^{n-k}-n-1=3$ to 11 for *any* n and k.

2. Definitions, problem formulation and proposed approach. A 2^{n-k} fractional factorial design d is uniquely determined by k independent defining relations. A defining relation is given by a word of letters which are labels of factors denoted by $1, 2, \ldots, n$. The number of letters in a word is its *word-length* and the group generated by the k independent defining words is the defining contrast subgroup. The vector $W(d) = (A_1(d), A_2(d), \ldots, A_n(d))$ is called the *word-length* pattern of the design d, where $A_i(d)$ is the number of words of length i in the defining contrast subgroup. The resolution of a design is the smallest r satisfying $A_r \geq 1$. Two designs having the same resolution may have different word-length patterns. To further discriminate 2^{n-k} designs, Fries and Hunter (1980) proposed the minimum aberration criterion. For two designs d_1 and d_2, suppose r is the smallest value such that $A_r(d_1) \neq A_r(d_2)$. We say that d_1 has less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. If no design has less aberration than d_1, then d_1 is said to have minimum aberration.

To illustrate the approach taken in this paper, consider a 2^{4-1} design which involves eight runs and is defined by any four out of seven orthogonal columns 1, 2, 3, 12, 13, 23 and 123, where columns 1, 2 and 3 generate all seven columns and column 12 is the product between column 1 and column 2, and so on. See Table 1 for an illustration. In particular, the 2^{4-1} design with the defining contrast $I = 1234$, where I denotes the column of +'s, is defined by the columns 1, 2, 3 and 123. If we denote these four columns by c_1, c_2, c_3 and c_4, then obviously they satisfy the relation $c_1 c_2 c_3 c_4 = I$.

Generally, the choice of the defining contrast subgroup in a 2^{n-k} design amounts to choosing n columns out of all possible N $(= 2^m - 1)$ columns generated by the m independent columns $1, 2, \ldots, m$, where $m = n - k$. In coding theory, the collection of all the $2^m - 1$ columns corresponds to a Hamming code, denoted by $H_m = \{c_1, c_2, \ldots, c_{2^m-1}\}$. Let s be the set of n columns (or elements) from H_m that represents the 2^{n-k} design. (For the rest of the paper, we will use the terms "column" and "element" interchangeably.) A word of length i consists of i elements $c_{j_1}, \ldots, c_{j_i}$ from s such that $c_{j_1} \cdots c_{j_i} = I$.

TABLE 1

Run	1	2	3	12	13	23	123
1	−	−	−	+	+	+	−
2	−	−	+	+	−	−	+
3	−	+	−	−	+	−	+
4	−	+	+	−	−	+	−
5	+	−	−	−	−	+	+
6	+	−	+	−	+	−	−
7	+	+	−	+	−	−	−
8	+	+	+	+	+	+	+

Our approach is based on two techniques. The first is the use of isomorphism to reduce the number of searches for optimal solutions. Consider another 2^{4-1} design which takes up the columns 2, 12, 23 and 123. These four columns, denoted by d_1, d_2, d_3 and d_4, satisfy the relation $d_1 d_2 d_3 d_4 = I$ and the design is reduced to the previous one by the mapping: $2 \to 1$, $12 \to 2$, $23 \to 3$, $123 \to 123$. An isomorphism ϕ is a one–one mapping from H_m to H_m such that $\phi(c_{i_1} c_{i_2}) = \phi(c_{i_1})\phi(c_{i_2})$ for any c_{i_1} and c_{i_2}. Two 2^{n-k} designs, one consisting of the columns $c_1, \dots, c_n$ from H_m and the other of the columns $d_1, \dots, d_n$ from H_m, are said to be isomorphic if there is an isomorphic mapping ϕ that maps c_i to d_i, that is, $d_i = \phi(c_i)$ for $i = 1, \dots, n$. The two 2^{4-1} designs given before are isomorphic. Isomorphic designs are treated as the same design. For example, they are equivalent according to the aberration criterion.

Let us now turn our attention to the second technique. Consider a 2^{4-1} design of resolution III defined by $I = 124$, which is inferior to the 2^{4-1} design of resolution IV with $I = 1234$. The former has the columns $s_1 = \{1, 2, 3, 12\}$ out of H_3 and the latter has the columns $s_2 = \{1, 2, 3, 123\}$ out of H_3. If we look at the complement of s_1, $\bar{s}_1 = H_3 \setminus s_1 = \{13, 23, 123\}$ and, similarly, $\bar{s}_2 = H_3 \setminus s_2 = \{12, 13, 23\}$, the three elements in $\bar{s}_1$ are independent while those in $\bar{s}_2$ satisfy the relation $(12)(13)(23) = I$. Intuitively, one would argue that when the elements in the complementary set $\bar{s}$ are more “dependent,” those in the design given by s should be less “dependent” and thus may have less aberration. A rigorous version of this intuition turns out to be true and is supported by a general theory to be developed later. As will be seen later, this technique is particularly useful when $2^m - 1 - n = |\bar{s}|$ is much smaller than $n = |s|$, that is, when the number of factors n is much larger than 2^{n-k-1}, *half of the run size*.

The following lemma is crucial to the application of the two techniques.

LEMMA 1. *Suppose* $H_m = s_1 \cup \bar{s}_1 = s_2 \cup \bar{s}_2$. *If* $\bar{s}_1$ *and* $\bar{s}_2$ *are isomorphic, then* s_1 *and* s_2 *are also isomorphic.*

Its proof follows a simple group-theoretic argument. A simple result like Lemma 1 can be used to yield some very useful results as follows.

COROLLARY 1. *Any two 2^{n-k} designs with $n = 2^{n-k} - 2$ are isomorphic. The same holds for $n = 2^{n-k} - 3$.*

PROOF. Any 2^{n-k} design with $n = 2^{n-k} - 2$ is defined by a set s of $2^{n-k} - 2$ columns out of H_m, with $m = n - k$. Its complement $\bar{s}$ consists of only one column in H_m. Since any two sets of one column are isomorphic, the sets like s must be isomorphic to each other according to Lemma 1. Similarly, for $n = 2^{n-k} - 3$, the complementary set $\bar{s}$ consists of two distinct columns. Since any sets of two columns are isomorphic, any two designs with $n = 2^{n-k} - 3$ are also isomorphic. $\square$

The mathematical results in Lemma 1 and Corollary 1 can also be found in Chen (1990) in a different context. To further exploit the power of Lemma 1, we need to relate the word-length pattern of the design s to that of the complementary design $\bar{s}$. A general result along these lines will be given next.

3. Main results. As in Section 2, let H_m be the Hamming code generated by m independent columns, and let s be a 2^{n-k} design consisting of n columns from H_m, where $m = n - k$. Then its complementary design $\bar{s}$ consists of the remaining $2^m - 1 - n$ columns. In this section we establish the relationship between the word-length pattern $W(s) = (A_1(s), A_2(s), \dots)$ of the design s and the word-length pattern $W(\bar{s}) = (A_1(\bar{s}), A_2(\bar{s}), \dots)$ of the complementary design $\bar{s}$. The result is mathematically elegant as well as practically useful since it enables us to identify a design in terms of properties of its complementary design. Its value is particularly significant when applied to give an explicit characterization of minimum aberration designs as will be demonstrated in Section 4.

Our proof starts with the following combinatorial calculations. For a fixed s, let $N_q(p)$ denote the number of words of length q in H_m that consist of p elements from s and $q - p$ elements from $\bar{s}$. Thus we have $A_q(s) = N_q(q)$ and $A_q(\bar{s}) = N_q(0)$. Now, take p distinct elements $c_1, \dots, c_p$ from s and $q - p$ distinct elements $d_1, \dots, d_{q-p}$ from $\bar{s}$. Altogether there are $\binom{n}{p}\binom{f}{q-p}$ ways of drawing such elements, where $f = 2^m - 1 - n$, which can be classified according to the value of the product $c = c_1 \cdots c_p d_1 \cdots d_{q-p}$. It is obvious that one and only one of the following five situations can occur:

(i) $c = I$;
(ii) $c \in \{c_1, \dots, c_p\}$, say $c = c_i$;
(iii) $c \in s \setminus \{c_1, \dots, c_p\}$;
(iv) $c \in \{d_1, \dots, d_{q-p}\}$;
(v) $c \in \bar{s} \setminus \{d_1, \dots, d_{q-p}\}$.

Situation (i) gives a word of length q and there are $N_q(p)$ such words. Situation (ii) gives a word of length $q - 1$ but any other set of p elements $\{c_1, \dots, c_{i-1}, c', c_{i+1}, \dots, c_p\}$ with $c' \in s \setminus \{c_1, \dots, c_p\}$, in combination with $\{d_1, \dots, d_{q-p}\}$, also gives the same word $c_1 \cdots c_{i-1} c_{i+1} \cdots c_p d_1 \cdots d_{q-p}$. Since there are $n - p + 1$ subsets of p elements that can give the same word and the number of such words is $N_{q-1}(p-1)$, the total contribution to $\binom{n}{p}\binom{f}{q-p}$ by (ii) is $(n-p+1)N_{q-1}(p-1)$. Situation (iii) gives a word of length $q+1$, with $p-1$ columns from s and $q+p$ columns from $\bar{s}$. Any other set of p elements $\{c_1, \dots, c_{i-1}, c, c_{i+1}, \dots, c_p\}$ with $i = 1, \dots, p$, in combination with $\{d_1, \dots, d_{q-p}\}$, also gives the same word $c_1 \cdots c_p d_{q-p} c$. Noting that there are $N_{q+1}(p+1)$ such words, the total contribution to $\binom{n}{p}\binom{f}{q-p}$ by (iii) is $(p+1)N_{q+1}(p+1)$. Similar calculations for situations (iv) and (v) lead to the following identity:

$$\begin{aligned}\binom{n}{p}\binom{f}{q-p} &= N_q(p) + (n-p+1)N_{q-1}(p-1) \\ &\quad + (p+1)N_{q+1}(p+1) + (f-q+p+1)N_{q-1}(p) \\ &\quad + (q-p+1)N_{q+1}(p).\end{aligned} \tag{1}$$

We first note that $N_q(p) = 0$ for $q = 1, 2$ and $0 \le p \le q$. Next, let

$$\begin{aligned}A(p,q) = \binom{n}{p}\binom{f}{q-p} &- N_q(p) - (n-p+1)N_{q-1}(p-1) \\ &- (f-q+p+1)N_{q-1}(p).\end{aligned} \tag{2}$$

Then (1) can be rewritten as

$$(q-p+1)N_{q+1}(p) + (p+1)N_{q+1}(p+1) = A(p,q), \tag{3}$$

with $A(p,q)$ given in (2). Noting that $A(p,q)$ is determined by $N_{q'}(p)$, with $0 \le p \le q'$ and $q' \le q$, we now have the following theorem.

THEOREM 1. *Given the word-length pattern* $(N_3(0), N_4(0), \dots)$ *of the design* $\bar{s}$, *the vector* $(N_{q+1}(1), \dots, N_{q+1}(q+1))$ *for* $q = 2, 3, \dots$ *can be determined recursively by*

$$N_{q+1}(p+1) = (-1)^{p+1}\binom{q+1}{p+1}N_{q+1}(0) + B(p,q) \quad \text{for } p = 0, 1, \dots, q, \tag{4}$$

where

$$B(p,q) = (p+1)^{-1}\binom{q}{p}\sum_{j=0}^{p}(-1)^{p+j}A(j,q)\Big/\binom{q}{j}. \tag{5}$$

In particular, taking $p = q$ *in* (4) *gives*

$$N_{q+1}(q+1) = (-1)^{q+1}N_{q+1}(0) + B(q,q). \tag{6}$$

PROOF. The proof can be easily carried out by mathematical induction. Taking $p = 0$ in (4) gives

$$N_{q+1}(1) = -(q+1)N_{q+1}(0) + B(0,q),$$

which reduces to (3) with $p = 0$. Next assume that (4) holds for $p - 1$, that is,

$$N_{q+1}(p) = (-1)^p \binom{q+1}{p} N_{q+1}(0) + B(p-1, q). \tag{7}$$

Combining (7) with (3) and (5), we obtain (4). $\square$

By successively applying Theorem 1, we see that $N_{q+1}(q+1)$ can be expressed as a sum of a constant and a linear combination of $(N_3(0), N_4(0), \dots, N_{q+1}(0))$. In addition, it can be proved that the two leading coefficients have the same value $(-1)^{q+1}$. These are summarized as follows.

THEOREM 2. *The number $N_{q+1}(q+1)$ of words of length $q+1$ in the design s can be expressed as*

$$N_{q+1}(q+1) = C_0 + \sum_{j=3}^{q+1} C_j N_j(0),$$

or, equivalently,

$$A_{q+1}(s) = C_0 + \sum_{j=3}^{q+1} C_j A_j(\bar{s}), \tag{8}$$

where the coefficients $C_0, C_3, \dots, C_{q+1}$ are functions of n, f and q. Moreover, we have

$$C_{q+1} = C_q = (-1)^{q+1}. \tag{9}$$

PROOF. We only need to prove $C_q = (-1)^{q+1}$. From (6), the terms with $N_q(0)$ are contained in

$$B(q,q) = (q+1)^{-1} \sum_{j=0}^{q} (-1)^{q+j} A(j,q) \Big/ \binom{q}{j}.$$

From (2), we obtain

$$B(q,q) = B_1 + (q+1)^{-1} \sum_{j=0}^{q} (-1)^{q+j+1} N_q(j) \Big/ \binom{q}{j},$$

where B_1 only contains N_r terms with $r \le q-1$. From (4), the equation above can be expressed as

$$\begin{aligned} B(q,q) &= B_2 + (q+1)^{-1} \sum_{j=0}^{q} (-1)^{q+j+1} (-1)^j \binom{q}{j} N_q(0) \Big/ \binom{q}{j} \\ &= B_2 + (-1)^{q+1} N_q(0), \end{aligned}$$

where B_2 only contains N_r terms with $r \le q-1$. This completes the proof. $\square$

Note that in Theorems 1 and 2 we implicitly assume $q + 1 \leq \min(n, f)$. The result in (8) is similar to the MacWilliams identities in coding theory [see Pless (1982)]. In coding theory the design s is treated as a linear code. MacWilliams identities relate the A_q values of s to those of the dual code of s while ours relate to the A_q values of the complementary design $\bar{s}$. The proof of MacWilliams identities exploits the linearity of both codes.

We now discuss the application of Theorem 2 to characterize minimum aberration designs. By (8) and (9), it is immediate that

$$A_3(s) = \text{constant} - A_3(\bar{s}),$$
$$A_4(s) = \text{constant} + A_3(\bar{s}) + A_4(\bar{s}).$$

With some elementary algebra, we can show that

$$A_5(s) = \text{constant} - (2^{m-1} - n)A_3(\bar{s}) - A_4(\bar{s}) - A_5(\bar{s}).$$

In a recent paper, H. Chen and Hedayat (1994) independently obtained the first two equations above. By employing combinatorial enumerations, they gave separate proofs for each of the two equations. Our proof follows a different approach and works for general q.

Using these three equations, we can give the following rules for identifying minimum aberration designs.

RULE 1. A design s^* has minimum aberration if:

(i) $A_3(\bar{s}^*) = \max A_3(\bar{s})$ over all $|\bar{s}| = f$ and
(ii) $\bar{s}^*$ is the unique set (up to isomorphism) satisfying (i).

RULE 2. A design s^* has minimum aberration if:

(i) $A_3(\bar{s}^*) = \max A_3(\bar{s})$ over all $|\bar{s}| = f$,
(ii) $A_4(\bar{s}^*) = \min\{A_4(\bar{s})\colon A_3(\bar{s}) = A_3(\bar{s}^*)\}$ and
(iii) $\bar{s}^*$ is the unique set (up to isomorphism) satisfying (ii).

RULE 3. A design s^* has minimum aberration if:

(i) $A_3(\bar{s}^*) = \max A_3(\bar{s})$ over all $|\bar{s}| = f$,
(ii) $A_4(\bar{s}^*) = \min\{A_4(\bar{s})\colon A_3(\bar{s}) = A_3(\bar{s}^*)\}$,
(iii) $A_5(\bar{s}^*) = \max\{A_5(\bar{s})\colon A_3(\bar{s}) = A_3(\bar{s}^*)$ and $A_4(\bar{s}) = A_4(\bar{s}^*)\}$ and
(iv) $\bar{s}^*$ is the unique set (up to isomorphism) satisfying (iii).

If words of length q (≥ 6) are of interest, one can develop similar rules using Theorem 2. We now present an example to illustrate the use of Rule 1.

EXAMPLE 1. Let $f = 2^i - 1$. Clearly, if $\{I\} \cup \bar{s}$ is a subgroup of $\{I\} \cup H_m$, then $A_3(\bar{s}) = \binom{f}{2}/3$ is maximized. Since this group is unique (up to isomorphism), we thus obtain a sequence of minimum aberration 2^{n-k} designs with $n = 2^m - 2^i$ columns, $i = 1, \dots, m-1$ and $m = n - k$.

Application of Rules 1–3 allows us to construct many minimum aberration designs, some of which will be reported in the next section.

4. Minimum aberration 2^{n-k} designs with $f \leq 11$. In this section we obtain minimum aberration 2^{n-k} designs for general n and k with $f = 2^{n-k} - 1 - n = 3, 4, \ldots, 9$ by applying Rule 1 and with $f = 10, 11$ by applying Rule 2. (Note that the cases $f = 1$ and $f = 2$ are covered by Corollary 1.)

(i) $f = 3$. This case is covered by Example 1.

(ii) $f = 4$. There are three nonisomorphic choices for $\bar{s}$:

$$\bar{s}_1 = \{a, b, c, ab\}, \qquad \bar{s}_2 = \{a, b, c, abc\}, \qquad \bar{s}_3 = \{a, b, c, d\}.$$

Since $\bar{s}_1$ is the only one with positive $A_3(\bar{s}_1) = 1$, any design whose $\bar{s}$ is of the type $\bar{s}_1$ has minimum aberration. For the sake of brevity, we will say that $\bar{s}_1$ gives minimum aberration designs.

(iii) $f = 5$. First consider the complementary set, $\bar{s}_1 = \{a, b, c, ab, ac\}$. Because $\bar{s}_1$ can be viewed as a 2^{5-2} design and from Corollary 1 all 2^{5-2} designs are isomorphic, other $\bar{s}$ sets must have four or five independent generators, say, a, b, c, d, e. Among them, the following three are nonisomorphic:

$$\bar{s}_2 = \{a, b, c, d, ab\}, \qquad \bar{s}_3 = \{a, b, c, d, abc\},$$
$$\bar{s}_4 = \{a, b, c, d, abcd\}, \qquad \bar{s}_5 = \{a, b, c, d, e\}.$$

Since $A_3(\bar{s}_1) = 2 > A_3(\bar{s}_2) = 1 > A_3(\bar{s}_i) = 0$ for $i = 3, 4, 5$, $\bar{s}_1$ gives minimum aberration designs.

(iv) $f = 6$. Since the set $\bar{s}_1 = \{a, b, c, ab, ac, bc\}$ can be viewed as a 2^{6-3} design and all such designs are isomorphic, $\bar{s}_1$ is the only set with three independent generators. All the other sets must have at least four independent generators and their A_3 values can be shown to be at most 2, which is smaller than $A_3(\bar{s}_1) = 4$. Therefore $\bar{s}_1$ gives minimum aberration designs.

(v) $f = 7$. The most "dependent" $\bar{s}$ set is of the form

$$\bar{s}_1 = \{a, b, c, ab, ac, bc, abc\}, \tag{10}$$

which can be viewed as the Hamming code H_3. As noted in Example 1, $\bar{s}_1$ gives minimum aberration designs. For future applications, we will compute the A_3 values for the other sets. They must have at least four independent generators and are given below:

$$\bar{s}_2 = \{a, b, c, d, x_1, x_2, x_3\}, \qquad \bar{s}_3 = \{a, b, c, d, e, x_1, x_2\},$$
$$\bar{s}_4 = \{a, b, c, d, e, f, x\}, \qquad \bar{s}_5 = \{a, b, c, d, e, f, g\},$$

where a, b, c, d, e, f, g denote independent generators and x_1, x_2, x_3, x denote products of the independent generators in the corresponding set. It is easy to show that the maximum A_3 value for $\bar{s}_2$ is 4 and is uniquely attained by choosing $x_1 = ab$, $x_2 = ac$, $x_3 = bc$. It is also easy to show that

$$\max A_3(\bar{s}_3) = 2, \qquad \max A_3(\bar{s}_4) = 1, \qquad A_3(\bar{s}_5) = 0.$$

So the second largest A_3 value for $f = 7$ is 4.

(vi) $f = 8$. We will prove that the set

$$\bar{s}^* = \{a, b, c, d, ab, ac, bc, abc\}$$

is the unique set attaining the maximum A_3 value 7, and thus gives minimum aberration designs. To this end, we write any set of eight elements as $\bar{s} = t \cup \{x_8\}$, where $t = \{x_1, \dots, x_7\}$. It is known from the case of $f = 7$ that $\max A_3(t) = 7$.

(a) If $A_3(t) = 7$, t must be of the form $\{a, b, c, ab, ac, bc, abc\}$ as proved before. With this choice of t, $x_8 = d$.

(b) As proved in the case of $f = 7$, the next largest A_3 value for t is 4 and is uniquely attained by the set $t^* = \{a, b, c, d, ab, bc, ac\}$. Additional relations involving three elements in $\bar{s}$ must involve x_8 and take the form

$$x_i x_j = x_8, \qquad i \neq j \leq 7. \tag{11}$$

If t is chosen to be t^*, there is at most one pair of x_i and x_j to satisfy (11). Therefore

$$\max A_3(t^* \cup \{x_8\}) = 5.$$

For any other choice of t, $A_3(t) \leq 3$ as shown in the case of $f = 7$. Noting that (11) has at most three solutions, we have shown that the maximum A_3 value for $t \cup \{x_8\}$ with $A_3(t) \leq 3$ is 6, thus completing the proof.

(vii) $f = 9$. The proof is very similar to that of $f = 8$. Write $\bar{s} = t \cup \{x_8, x_9\}$, where t has seven elements. The maximum of $A_3(t)$ is 7 and is uniquely attained by the set given in (10). Call this set t_1. With this choice of t_1, x_8, x_9 might take the form $\{d, xd\}$, $x \in t_1$. The combined set $t_1 \cup \{d, xd\}$ is isomorphic for any choice of x in t_1. So we may take

$$\bar{s}_1^* = \{a, b, c, ab, ac, bc, abc, d, ad\} \tag{12}$$

as the set with $A_3(\bar{s}_1^*) = 8$. It remains to prove that any other nonisomorphic choice will have a smaller A_3 value. To this end, we note, as in part (b) of the case of $f = 8$, that the next largest A_3 value for t is 4 and is uniquely attained by the set $\{a, b, c, d, ab, bc, ac\}$. For this set the choice of $\bar{s}$ must be of the type

$$\{a, b, c, d, ab, bc, ac, x_1 d, x_2 d\}, \tag{13}$$

where x_1 and x_2 are words formed by a, b, c. For the set in (13), there are at most three additional relations involving three elements in the set, that is,

$$x_1 d = x_1 d, \qquad x_2 d = x_2 d, \qquad (x_1 d)(x_2 d) = x_1 x_2.$$

So the A_3 value for the set in (13) is at most 7. For other choice of t, $A_3(t) \leq 3$ and it is easy to show that no matter how x_8 and x_9 are chosen, the A_3 value of the resulting set cannot exceed 7, thus completing the proof.

It is interesting to point out that, for all the cases to which Rule 1 applies so far, the elements in the optimum sets $\bar{s}^*$ can be arranged in the Yates order. See Table 2. However, as we shall see in the cases of $f = 10$ and 11, this arrangement does not apply to designs obtained by applying Rule 2.

TABLE 2
The $\bar{s}$ set for minimum aberration design with $f = 1$ to 9
(for each f, the optimum $\bar{s}^*$ consists of the first f words in the second row)

f	1	2	3	4	5	6	7	8	9
$\bar{s}^*$	a	b	ab	c	ac	bc	abc	d	ad

(viii) $f = 11$. We want to find the unique $\bar{s}^*$ set that satisfies Rule 2. By complete enumerations it can be shown that $\bar{s}$ must be a subset of H_4. Since the size of $\bar{s}$ is larger than that of $H_4 \setminus \bar{s}$, it is easier to work with the smaller set $H_4 \setminus \bar{s}$. Since $H_4 \setminus \bar{s}$ has only four elements, according to the case of $f = 4$, there are only three nonisomorphic choices denoted by

$$\bar{\bar{s}}_1 = \{x, y, z, u\}, \qquad \bar{\bar{s}}_2 = \{x, y, z, xyz\}, \qquad \bar{\bar{s}}_3 = \{x, y, z, xy\},$$

which have $A_3(\bar{\bar{s}}_1) = A_4(\bar{\bar{s}}_1) = 0$, $A_3(\bar{\bar{s}}_2) = 0$, $A_4(\bar{\bar{s}}_2) = 1$, $A_3(\bar{\bar{s}}_3) = 1$, $A_4(\bar{\bar{s}}_3) = 0$. Clearly, $\bar{s}_1 = H_4 \setminus \bar{\bar{s}}_1$ satisfies Rule 2. We can write

$$\begin{aligned} \bar{s}_1 &= \{a, b, ab, c, ac, bc, abc, d, ad, bd, cd\}, \\ \bar{s}_2 &= \{a, b, ab, c, ac, bc, abc, d, ad, bd, abd\}. \end{aligned} \tag{14}$$

It is seen that $A_3(\bar{s}_1) = A_3(\bar{s}_2) = 13$ and $A_4(\bar{s}_1) = 25 < 26 = A_4(\bar{s}_2)$. Therefore any design whose complementary set $\bar{s}$ is of the type $\bar{s}_1$ in (14) has minimum aberration.

(ix) $f = 10$. As in the case of $f = 11$, by working with the complementary set $\bar{\bar{s}}$ of $\bar{s}$ within H_4 and noting that $\bar{\bar{s}}$ has five elements, we can use the result for $f = 5$ that there are only four nonisomorphic choices for $\bar{\bar{s}}$, that is,

$$\begin{aligned} \bar{\bar{s}}_1 &= \{x, y, z, u, xyzu\}, \qquad & \bar{\bar{s}}_2 &= \{x, y, z, u, xyz\}, \\ \bar{\bar{s}}_3 &= \{x, y, z, u, xy\}, \qquad & \bar{\bar{s}}_4 &= \{x, y, z, xy, xz\}. \end{aligned}$$

Among these four sets, $A_3(\bar{\bar{s}}_1) = A_3(\bar{\bar{s}}_2) = 0$, $A_3(\bar{\bar{s}}_3) = 1$ and $A_3(\bar{\bar{s}}_4) = 2$. So we can rule out $\bar{\bar{s}}_3$ and $\bar{\bar{s}}_4$. Since $A_4(\bar{\bar{s}}_1) = 0 < A_4(\bar{\bar{s}}_2) = 1$, $\bar{s}_1$ satisfies Rule 2 and its corresponding design has minimum aberration. We can write

$$\begin{aligned} \bar{s}_1 &= \{a, b, ab, c, ac, bc, d, ad, bd, cd\}, \\ \bar{s}_2 &= \{a, b, ab, c, ac, bc, abc, d, ad, bd\}, \end{aligned}$$

with $A_3(\bar{s}_1) = A_3(\bar{s}_2) = 10$, $A_4(\bar{s}_1) = 15 < 16 = A_4(\bar{s}_2)$.

For $f = 10$ and 11, it is interesting to note that the elements in $\bar{s}_2$ are arranged in the Yates order while the elements in the optimum $\bar{s}_1$ are not.

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