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CHARACTERIZATION OF RECURSIVELY ENUMERABLE SETS WITH SUPERSETS EFFECTIVELY ISOMORPHIC TO ALL RECURSIVELY ENUMERABLE SETS

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ABSTRACT. We show that the lattice of supersets of a recursively enumerable (r.e.) set A is effectively isomorphic to the lattice of all r.e. sets if and only if the complement $\bar{A}$ of A is infinite and $\{e \mid W_e \cap \bar{A} \text{ finite}\} \leq_1 \varnothing''$ (i.e. $\bar{A}$ is semilow$_{1.5}$). It is obvious that the condition "$\bar{A}$ semilow$_{1.5}$" is necessary. For the other direction a certain uniform splitting property (the "outer splitting property") is derived from semilow$_{1.5}$ and this property is used in an extension of Soare's automorphism machinery for the construction of the effective isomorphism. Since this automorphism machinery is quite complicated we give a simplified proof of Soare's Extension Theorem before we add new features to this argument.

1. Introduction. For any subset S of the natural numbers N let $\mathcal{E}(S)$ be the lattice of sets $\{W \cap S \mid W$ recursively enumerable$\}$ under inclusion and let $\mathcal{E}^*(S)$ be the quotient lattice of $\mathcal{E}(S)$ modulo the ideal of finite subsets of S. One writes D^* for the equivalence class in $\mathcal{E}^*(S)$ containing $D \in \mathcal{E}(S)$. $\mathcal{E}$, $\mathcal{E}^*$ are abbreviations for the lattices of all recursively enumerable (r.e.) sets $\mathcal{E}(N)$, respectively $\mathcal{E}^*(N)$.

An isomorphism Φ: $\mathcal{E}^*(S_1) \to \mathcal{E}^*(S_2)$ is called effective if there is a recursive permutation h of N such that $\forall e \in N(\Phi((W_e \cap S_1)^*) = (W_{h(e)} \cap S_2)^*)$. This is obviously equivalent to the existence of total recursive functions f, g such that

$$\forall e \in N\big(\Phi((W_e \cap S_1)^*) = (W_{f(e)} \cap S_2)^* \wedge \Phi^{-1}((W_e \cap S_2)^*) = (W_{g(e)} \cap S_1)^*\big)$$

(Soare [**6**]).

The notion semilow$_{1.5}$ was introduced by Soare [7] in the context of computational complexity. A simple definition in terms of "information content" can be given as follows.

DEFINITION 1.1 (SOARE [7]). $S \subseteq N$ is semilow$_{1.5}$: $\Leftrightarrow \{e \mid W_e \cap S \text{ finite}\} \leq_1 \varnothing''$.

We write $\bar{A}$ for $N - A$. If A is r.e. then $\bar{A}$ semilow$_{1.5}$ turns out to be an essential dynamical property of A. We will survey these properties in §2.

The following result will be proved in this paper.

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THEOREM 1.2. *If A is r.e. then $\mathfrak{E}^*(\bar{A})$ is effectively isomorphic to $\mathfrak{E}^*$ if and only if A is infinite and $\bar{A}$ is semilow$_{1.5}$.*

PROOF OF DIRECTION " $\Rightarrow$ " OF THEOREM 1.2 (SOARE [7]). Assume A is r.e. and Φ is an isomorphism from $\mathfrak{E}^*(\bar{A})$ onto $\mathfrak{E}^*$ and h is a recursive permutation of N such that $\forall e \in N(\Phi((W_e \cap \bar{A})^*) = W^*_{h(e)})$. Then we have

$$\forall e \in N\big(W_e \cap \bar{A} \text{ finite} \Leftrightarrow W_{h(e)} \text{ finite}\big).$$

It is obvious that $\{i \mid W_i \text{ finite}\} \equiv_1 \varnothing''$ (see Rogers [4]). Thus $\bar{A}$ is semilow$_{1.5}$. Further $\bar{A}$ is infinite because $\mathfrak{E}^*$ is infinite.

In §2 we introduce the outer splitting property. All r.e. sets A with $\bar{A}$ semilow$_{1.5}$ possess this property. This fact is crucial for the isomorphism construction in §4.

In §3 we give a simplified proof of Soare's Extension Theorem. The Extension Theorem is the key step in the proof of Soare's famous result that all maximal sets are automorphic in $\mathfrak{E}$. Soare [6] introduced the automorphism machinery in order to prove this Extension Theorem. It is still the easiest understandable application of this technique. All difficulties in this basic construction are multiplied when one tries to apply it to more difficult situations like semilow$_{1.5}$ sets.

In §4 we prove the missing direction of Theorem 1.2. We extend the automorphism machinery and add the special tools for semilow$_{1.5}$ sets from §2. This construction generalizes the previous applications of the automorphism machinery to maximal set (Soare [6]) and semilow$_1$ sets (Soare [9]).

We attempt to give complete proofs and to supply some motivation for the automorphism machinery in order to make this paper self-contained.

This paper is part of the general program to characterize all lattices which arise as lattices of supersets of r.e. sets and in the long run to characterize all orbits of r.e. sets under automorpism of $\mathfrak{E}^*$ together with the degrees of the involved r.e. sets. A survey and references are given in Soare [8].

The first major results in this program are due to Lachlan, Martin and Soare. Lachlan has shown that exactly the $\exists\forall\exists$-Boolean algebras occur as lattices of supersets of hyperhypersimple r.e. sets. Martin has shown that the degrees of hyperhypersimple r.e sets are exactly the high r.e. degrees. Later Soare introduced the automorphism machinery in order to prove that for every finite Boolean algebra $\mathfrak{L}$ the r.e. sets A with $\mathfrak{E}^*(\bar{A}) \cong \mathfrak{L}$ are automorphic in $\mathfrak{E}^*$.

The only other lattice of supersets of an r.e. set which has been explictly described so far is $\mathfrak{E}^*$ itself. Observe that the study of r.e. sets A with $\mathfrak{E}^*(\bar{A}) \cong \mathfrak{E}^*$ is a natural next step in the general program after Lachlan's result for hyperhypersimple sets. If an r.e. set A is not hyperhypersimple then $\mathfrak{E}^*$ is effectively embeddable into $\mathfrak{E}^*(\bar{A})$ (obvious).

If A is recursive and $\bar{A}$ is infinite then $\mathfrak{E}^*(\bar{A})$ is trivially isomorphic to $\mathfrak{E}^*$. Soare [9] has shown (using the full automorphism machinery) that for every r.e. set A with $\bar{A}$ infinite and semilow$_1$ (i.e. $\{e \mid W_e \cap \bar{A} \neq \varnothing\} \leq_T \varnothing'$) $\mathfrak{E}^*(\bar{A})$ is effectively isomorphic to $\mathfrak{E}^*$. By definition if A is of low degree then $\bar{A}$ is semilow$_1$. In addition every r.e. degree contains an r.e. set A with $\bar{A}$ semilow$_1$. Besides r.e. sets A with $\bar{A}$ semilow$_1$ no other r.e. sets have been found with $\mathfrak{E}^*(\bar{A})$ isomorphic to $\mathfrak{E}^*$ (effectively or not).

The question of an exact characterization of r.e. sets A with $\mathcal{E}^*(\bar{A})$ effectively isomorphic to $\mathcal{E}^*$ has been discussed in papers by Soare [**7, 9**], Bennison and Soare [**1**] and Shore [**5**]. Soare [**7**] noticed that $\bar{A}$ is necessarily semilow$_{1.5}$. Bennison and Soare [**1**] constructed for every r.e. set A with $\bar{A}$ infinite and semilow$_{1.5}$ and any given $\exists\forall\exists$-Boolean algebra $\mathfrak{L}$ an r.e. set $B \supseteq A$ with $\mathcal{E}^*(\bar{B}) \cong \mathfrak{L}$. Nevertheless they expected that not for all these sets A, $\mathcal{E}^*(\bar{A})$ is effectively isomorphic to $\mathcal{E}^*$. An example for an r.e. set A where $\bar{A}$ is semilow$_{1.5}$ but not semilow$_1$ is given in Bennison and Soare [**1**].

One can now use Theorem 1.2 to get a much larger $\mathcal{E}^*$-definable class of r.e. sets than the recursive sets so that for all sets A in this class $\mathcal{E}^*(\bar{A})$ is effectively isomorphic to $\mathcal{E}^*$. Shore [**5**] defined: An r.e. set A is effectively nowhere simple if there is a total recursive function f such that

$$\forall e \in N\left(W_{f(e)} \subseteq W_e \cap \bar{A} \wedge \left(W_e \cap \bar{A} \text{ infinite} \Rightarrow W_{f(e)} \text{ infinite}\right)\right).$$

Trivially if A is effectively nowhere simple then $\bar{A}$ is semilow$_{1.5}$. It is easy to construct an effectively nowhere simple set A such that $\bar{A}$ is not semilow$_1$. David Miller was the first who noticed that for $\bar{A}$ infinite the preceding definition is equivalent to the following definition over $\mathcal{E}^*$: $\exists$ r.e. S (S is infinite $\wedge S \cap A =^* \varnothing$ $\wedge \forall$ r.e. W ($W \cap \bar{A}$ infinite $\Rightarrow W \cap S$ infinite)) (define $S := \bigcup_{e \in N} W_{f(e)}$, for the other direction set $W_{f(e)} := S \cap W_e$). Thus the class of effectively nowhere simple sets A with $\bar{A}$ infinite has the desired properties.

Finally we would like to mention one point of technical interest. The construction in §4 is the first example of an application of the automorphism machinery where not every stream in one machine is covered by some stream in the other machine.

2. Semilow$_{1.5}$ and the outer splitting property. For the proof of Theorem 1.2 we need only Lemma 2.3 from this section.

In the following lemma we survey some characteristic properties of r.e. sets A with $\bar{A}$ semilow$_{1.5}$. The equivalences (a), (b), (c), (d) are due to Bennison and Soare [**1**]. Property (e) characterizes the dynamical properties of the considered sets in standard recursion theoretic terms. Observe that if one writes in (e) $W_e = \tilde{W}_e$ instead of $W_e =^* \tilde{W}_e$ this property becomes equivalent to $\bar{A}$ semilow$_1$.

For fixed enumerations of r.e. sets W_e, $W_{\tilde{e}}$ we define as usual the r.e. set

$$W_e \backslash W_{\tilde{e}} := \left\{x \mid \exists s\left(x \in W_{e,s} \wedge x \notin W_{\tilde{e},s}\right)\right\}.$$

LEMMA 2.1. *For an r.e. set A the following are equivalent*:

(a) *$\bar{A}$ is semilow$_{1.5}$.*

(b) *A has a type 1 r.e. complexity sequence.*

(c) *A has an effective type 1 r.e. complexity sequence.*

(d) *There exists a total 0-1 valued recursive function g such that for every $e \in N$, $\lim_s g(s, e)$ exists and $\lim_s g(s, e) = 1 \Rightarrow W_e \cap \bar{A}$ finite and $\lim_s g(s, e) = 0 \Rightarrow W_e \cap \bar{A} \neq \varnothing$.*

(e) *There is a simultaneous enumeration of A and r.e. sets $(\tilde{W}_e)_{e \in N}$ such that for every $e \in N$, $W_e =^* \tilde{W}_e$ and $\tilde{W}_e \backslash A$ infinite $\Leftrightarrow \tilde{W}_e - A$ infinite.*

PROOF. The equivalence of the first four properties is shown in Bennison and Soare [1].

(a) $\Rightarrow$ (e). Let f be a total recursive function such that for all $e \in N$, $W_e \cap \bar{A}$ infinite $\Leftrightarrow W_{f(e)}$ infinite. Fix an enumeration of $(W_e)_{e\in N}$ and A. Enumerate the sets $(\tilde{W}_e)_{e\in N}$ as follows.

Stage $s+1$. For every e enumerate all elements of $W_{e,s} \cap A_s$ into $\tilde{W}_e$. Further if some new element is enumerated in $W_{f(e)}$ at stage $s+1$ enumerate as well all the other elements of $W_{e,s}$ into $\tilde{W}_e$.

(e) $\Rightarrow$ (a). For an enumeration of $(\tilde{W}_e)_{e\in N}$ and A as in (e) we have $W_e \cap \bar{A}$ infinite $\Leftrightarrow \tilde{W}_e \setminus A$ infinite.

DEFINITION 2.2. An r.e. set A has the outer splitting property if there are total recursive functions f_0, f_1 such that for every $e \in N$

$$W_{f_0(e)} \cap W_{f_1(e)} = \varnothing, \qquad W_{f_0(e)} \cup W_{f_1(e)} = W_e, \qquad W_{f_1(e)} \cap \bar{A} \text{ finite},$$

and

$$\left(W_e \cap \bar{A} \text{ infinite} \Rightarrow W_{f_1(e)} \cap \bar{A} \neq \varnothing\right).$$

The outer splitting property is slightly stronger than saying that A is effectively nowhere hyperhypersimple.

This splitting property is a counterpart to some inner splitting properties which arose earlier. If one demands e.g. from the uniform split $W_{f_0(e)}, W_{f_1(e)}$ instead that $W_{f_0(e)} \subseteq A$ and (W_e infinite $\Rightarrow W_{f_0(e)}$ infinite) then this is equivalent to A promptly simple (Maass [2]). Observe that for both definitions the strength lies solely in the uniformity (unlike the splitting property in Maass, Shore and Stob [3]).

LEMMA 2.3. *Assume A is r.e. and $\bar{A}$ is semilow$_{1.5}$. Then A has the outer splitting property.*

PROOF. We use an idea from Bennison and Soare [1]. Let f be a recursive function s.t.

$$\forall e \in N\left(W_e \cap \bar{A} \text{ infinite} \Leftrightarrow W_{f(e)} \text{ infinite}\right).$$

By the recursion theorem we can assume that we have already indices i and j for the recursive functions f_0 and f_1 which are going to construct.

Finally there is trivially a recursive function h s.t.

$$\forall e \in N\left(W_e \cap \bar{A} = \varnothing \Leftrightarrow W_{h(e)} \text{ infinite}\right).$$

Construction: *Stage* $s+1$. Place $x \in W_{e,s+1} - W_{e,s}$ in

$$\begin{cases} W_{f_1(e)}, & \text{if } |W_{h(\{j\}(e)),s}| \geqslant |W_{f(\{j\}(e)),s}|, \\ W_{f_0(e)}, & \text{otherwise.} \end{cases}$$

End of the construction.

The clause $|W_{h(\{j\}(e)),s}| \geqslant |W_{f(\{j\}(e)),s}|$ tells us that at stage $s+1$ it looks more likely that $W_{f_1(e)} \cap \bar{A}$ becomes empty rather than that it becomes infinite.

In the end we cannot have $W_{f_1(e)} \cap \bar{A}$ infinite because we would then have $|W_{h(\{j\}(e)),s}| < |W_{f(\{j\}(e)),s}|$ for almost all s and therefore we would place almost all elements of W_e in $W_{f_0(e)}$. Similarly if $W_{f_1(e)} \cap \bar{A}$ becomes empty then we would place almost all elements of W_e in $W_{f_1(e)}$ and therefore $W_e \cap \bar{A}$ must be finite.

THEOREM 2.4. *There exists an r.e. set A such that A has the outer splitting property but $\bar{A}$ is not semilow$_2$ (i.e. $\{e \mid W_e \cap \bar{A}$ finite$\} \not\leq_T 0''$).*

PROOF. Fix a pairing function $\langle\ ,\ \rangle$ which maps $N \times N$ one-one onto N. Set $N_i := \{\langle x, i\rangle \mid x \in N\}$. Let h be a total recursive function such that for all $i \in N$, $N_i = W_{h(i)}$.

If $\bar{A}$ is semilow$_2$ then there is a total recursive function ϕ_i of two arguments such that for all $e \in N$

$$(1) \qquad W_e \cap \bar{A} \text{ infinite} \Leftrightarrow \exists k \in N\big(W_{\phi_i(e,k)} \text{ infinite}\big).$$

We will construct A in such a way that in case that ϕ_i is total (1) fails for $e := h(i)$.

Let $(\phi_i)_{i \in N}$ be a recursive enumeration of all two place partial recursive functions. Fix a simultaneous enumeration of the r.e. sets $(W_e)_{e \in N}$ without repetitions.

Construction of an r.e. set A and r.e. sets W_e', W_e'' for every $e \in N$.

Stage s. Assume $\langle x, i\rangle$ is enumerated in W_e at stage s. If $\langle x, i\rangle$ is already in A we put $\langle x, i\rangle$ in W_e'. Otherwise consider the following two cases.

(a) $i \geq e$. If W_e'' contains already an element of some N_j, $j \geq e$, we put $\langle x, i\rangle$ in W_e'. Otherwise we enumerate $\langle x, i\rangle$ in W_e''.

(b) $i < e$. If W_e'' contains already an element of N_i which is not yet in A we put $\langle x, i\rangle$ in W_e'. Otherwise we enumerate $\langle x, i\rangle$ in W_e''.

At the end of stage s we enumerate for every j, k such that $\phi_j(h(j), k)$ converges by stage s and has value e the first element of $N_j - A_{s-1}$ into A which is not among the first k elements of $N_j - A_{s-1}$ and which is not yet in some $W_{\tilde{e},s}''$ for $\tilde{e} \leq \max(j, k)$. End of the construction.

It is obvious that there are total recursive functions f_0, f_1 such that for every $e \in N$, $W_e' = W_{f_0(e)}$ and $W_e'' = W_{f_1(e)}$. Further

$$W_e' \cup W_e'' = W_e \quad \text{and} \quad W_e' \cap W_e'' = \varnothing$$

for every $e \in N$.

By construction we have all $k, i \in N$

$$(2) \qquad |W_k'' \cap N_i \cap \bar{A}| \leq 1.$$

Assume that ϕ_i is total. We show that (1) fails for $e := h(i)$. If $W_{\phi_i(h(i),k)}$ is infinite for some $k \in N$ then because of (2) at most $k + \max(i, k)$ elements remain in $N_i \cap \bar{A} = W_{h(i)} \cap \bar{A}$.

On the other hand if $W_{\phi_i(h(i),k)}$ is finite for every $k \in N$ then we get $|N_i \cap \bar{A}| \geq n$ for any given $n \in N$. We just go to a stage s where the finite sets $W_{\phi_i(h(i),k)}$, $k \leq n$, are completely enumerated. Then the first n elements $N_i \cap \bar{A}_s$ stay outside of A. Thus $\bar{A}$ is not semilow$_2$.

Concerning the outer splitting property it is obvious from (2) and case (a) in the construction that $W_e'' \cap \bar{A}$ is finite for every $e \in N$.

Assume that $W_e \cap \bar{A}$ is infinite. If $W_e \cap \bar{A} \cap N_j \neq \varnothing$ for some $j \geqslant e$ we put an element of W_e into W_e'' according to case (a) in the construction. This element is never enumerated into A. Otherwise $W_e \cap \bar{A} \cap N_j$ is infinite for some $j < e$. This implies as before that there is no $k \in N$ such that $\phi_j(h(j), k)$ converges and $W_{\phi_j(h(j),k)}$ is infinite. Therefore only finitely many elements of $W_e'' \cap N_j$ are enumerated into A. Thus one of the infinitely many elements of $W_e \cap \bar{A} \cap N_j$ is placed in W_e'' according to case (b) and stays in $\bar{A}$.

3. A simplified proof of Soare's Extension Theorem. In this section we prove Soare's Extension Theorem in such a way that the construction can easily be generalized to more difficult situations like semilow$_{1.5}$ sets. Whenever possible we use the notation from Soare [**6, 9**].

For fixed enumerations of r.e. sets W_e, $W_{\tilde{e}}$ one defines

$$W_e \backslash W_{\tilde{e}} := \{x \mid \exists s (x \in W_{e,s} \wedge x \notin W_{\tilde{e},s})\}$$

and

$$W_e \searrow W_{\tilde{e}} := (W_e \backslash W_{\tilde{e}}) \cap W_{\tilde{e}}.$$

We fix two copies of the natural numbers, N and $\hat{N}$. We use the variables $x, y, \ldots$ $(\hat{x}, \hat{y}, \ldots)$ for elements of N $(\hat{N})$. We will construct a simultaneous enumeration of r.e. sets $(U_e)_{e\in N}$, $(\hat{V}_e)_{e\in N}$, $(\hat{U}_e)_{e\in N}$, $(V_e)_{e\in N}$. The sets U_e, $\hat{V}_e$ are subsets of N, the sets $\hat{U}_e$, V_e are subsets of $\hat{N}$. Further we will consider an r.e. set $A \subseteq N$ and an r.e. set $B \subseteq \hat{N}$. We write $U_{e,s}$ for the set of elements which are enumerated in U_e by the end of stage s, analogously for the other sets.

For any $x \in N$, any stage s and any number e with $0 \leqslant e \leqslant x$ we define

$$\nu(s, e, x) := \left\langle e, \{i \leqslant e \mid x \in U_{i,s}\}, \{i \leqslant e \mid x \in \hat{V}_{i,s}\} \right\rangle.$$

Similarly for $\hat{x} \in \hat{N}$ we set

$$\nu(s, e, \hat{x}) := \left\langle e, \{i \leqslant e \mid x \in \hat{U}_{i,s}\}, \{i \leqslant e \mid x \in V_{i,s}\} \right\rangle.$$

We use the symbol ν as variable for triples $\langle e, \sigma, \tau \rangle$ where $e \geqslant 0$ is a natural number and σ, τ are subsets of $\{0, \ldots, e\}$. We call these triples states and we call $|\nu| := e$ the length of state $\nu = \langle e, \sigma, \tau \rangle$. For states $\nu = \langle e, \sigma, \tau \rangle$ and $\nu' = \langle e', \sigma', \tau' \rangle$ we define

$$\begin{aligned} \nu \preccurlyeq \nu' \ (\nu \text{ is an initial segment of } \nu')\colon \Leftrightarrow e \leqslant e' \wedge \sigma \\ = \sigma' \cap \{0, \ldots, e\} \wedge \tau = \tau' \cap \{0, \ldots, e\}. \end{aligned}$$

We say that x $(\hat{x})$ has state ν at the end of stage s if $\nu \preccurlyeq \nu(s, x, x)$ $(\nu \preccurlyeq \nu(s, \hat{x}, \hat{x}))$. We say that x $(\hat{x})$ has final state ν if $\nu \preccurlyeq \lim_s \nu(s, x, x)$ $(\nu \preccurlyeq \lim_s \nu(s, \hat{x}, \hat{x}))$.

The following is a key definition for all isomorphism constructions in $\mathfrak{E}^*$. For states $\nu = \langle e, \sigma, \tau \rangle$, $\nu' = \langle e, \sigma', \tau' \rangle$ we define

$$\begin{aligned} &\nu \geqslant \nu' \ (\nu \text{ covers } \nu')\colon \Leftrightarrow \sigma \supseteq \sigma' \wedge \tau \subseteq \tau', \\ &\nu \geqslant_\sigma \nu' \ (\nu\ \sigma\text{-exactly covers } \nu')\colon \Leftrightarrow \sigma = \sigma' \wedge \tau \subseteq \tau', \\ &\nu \geqslant_\tau \nu' \ (\nu\ \tau\text{-exactly covers } \nu')\colon \Leftrightarrow \sigma \supseteq \sigma' \wedge \tau = \tau'. \end{aligned}$$

These relations are crucial for the following reason. It will be our goal to find for elements $x \in N$ that have final state ν w.r.t. $(U_e)_{e\in N}$, $(\hat{V}_e)_{e\in N}$ some matching elements $\hat{x} \in \hat{N}$ that have final state ν w.r.t. $(\hat{U}_e)_{e\in N}$, $(V_e)_{e\in N}$; and vice versa. This will be not so easy, since we have only control over the sets $(\hat{V}_e)_{e\in N}$, $(\hat{U}_e)_{e\in N}$, whereas the "opponent" enumerates all the other sets. Now if we see an element $x \in N$ with $\nu = \nu(s, e, x)$ and and element $\hat{x} \in \hat{N}$ with $\nu' = \nu(s, e, \hat{x})$ (see the definitions above) then it is in *our* power to bring x and $\hat{x}$ into the same state of length e (by enumerating x into some sets $\hat{V}_i$ and $\hat{x}$ into some sets $\hat{U}_i$) if and only if $\nu \geqslant \nu'$. Further we can do this by only enumerating x into some sets $\hat{V}_i$ iff $\nu \geqslant_\sigma \nu'$ and we can do this by only enumerating $\hat{x}$ into some sets $\hat{U}_i$ iff $\nu \geqslant_\tau \nu'$.

Once we have brought elements in N and elements in $\hat{N}$ into matching states (as e.g. in the conclusion of the following Extension Theorem) it is relatively easy (see Soare **[6]**) to construct an actual isomorphism in $\mathcal{E}^*$ that maps U_i^* on $\hat{U}_i^*$ and whose inverse maps V_i^* on $\hat{V}_i^*$. We will give this construction in detail in the proof of direction " $\Leftarrow$ " of Theorem 1.2 at the very end of §4.

The Extension Theorem form Soare **[6]** is used for the construction of nontrivial automorphisms as follows. If A and B are maximal (Soare **[6]**) or if A and B are promptly simple and have semilow complement (Soare **[9]**, Maass **[2]**) then one can construct arrays $(U_n)_{n\in N}$, $(V_n)_{n\in N}$ s.t. for all n $U_n = {}^*W_n = {}^*V_n$ and simultaneously arrays $(\tilde{U}_n)_{n\in N}$, $(\tilde{V}_n)_{n\in N}$ that serve as images of the given arrays in $\bar{B} \subseteq \hat{N}$, resp. $\bar{A} \subseteq N$, in such a way that the assumption of the Extension Theorem is satisfied. This assumption is obviously necessary in order to extend the sets $\tilde{U}_n$ to sets $\hat{U}_n$ and the sets $\tilde{V}_n$ to sets $\hat{V}_n$ that serve as images of U_n, resp. V_n, in all of $\hat{N}$, resp. all of N. The Extension Theorem asserts that this necessary condition is also sufficient. Thus we get an automorphism of $\mathcal{E}^*$ that maps U_n^* on $\hat{U}_n^*$, whose inverse maps V_n^* on $\hat{V}_n^*$, and which maps in addition (by construction) A^* on B^*.

THEOREM 3.1 (SOARE'S EXTENSION THEOREM **[6]**). *Assume A and B are infinite r.e. sets and $(U_n)_{n\in N}$, $(\tilde{V}_n)_{n\in N}$, $(\tilde{U}_n)_{n\in N}$, $(V_n)_{n\in N}$ are recursive arrays of r.e. sets. Suppose there is a simultaneous enumeration of a recursive array including all the sets above such that*

$$A \searrow \tilde{V}_n = \varnothing = B \searrow \tilde{U}_n \quad \text{for all } n.$$

Further assume that for every state $\nu = \langle e, \sigma, \tau \rangle$ such that

$$D_\nu^A := \left\{x \mid \exists s\left(x \in A_{s+1} - A_s \wedge \sigma = \{i \leqslant e \mid x \in U_{i,s}\} \wedge \tau = \left\{i \leqslant e \mid x \in \tilde{V}_{i,s}\right\}\right)\right\}$$

is infinite there is some state $\nu' = \langle e, \sigma', \tau' \rangle \leqslant \nu$ such that

$$D_{\nu'}^B := \left\{\hat{x} \mid \exists s\left(\hat{x} \in B_{s+1} - B_s \wedge \sigma' = \left\{i \leqslant e \mid \hat{x} \in \tilde{U}_{i,s}\right\} \wedge \tau' = \left\{i \leqslant e \mid \hat{x} \in V_{i,s}\right\}\right)\right\}$$

is infinite and that for every state ν' such the $D_{\nu'}^B$ is infinite there is some state $\nu \geqslant \nu'$ such that D_ν^A is infinite.

Then one can extend the r.e. sets $\tilde{V}_n$ to r.e. sets $\hat{V}_n$ and the r.e. sets $\tilde{U}_n$ to r.e. sets $\hat{U}_n$ such that for every state ν infinitely many elements of A have final state ν iff infinitely many elements of B have final state ν.

The rest of this section is devoted to a *proof* of Theorem 3.1. The construction of the extending sets $(\hat{U}_n)_{n\in N}$, $(\hat{V}_n)_{n\in N}$ takes place on two identical pinball machines M and $\hat{M}$. The numbers $x \in N$ ($\hat{x} \in \hat{N}$) are played on pinball machine M ($\hat{M}$) and we consider their state w.r.t.

$$(U_n)_{n\in N},\quad (\hat{V}_n)_{n\in N}\quad \big((\hat{U}_n)_{n\in N}, (V_n)_{n\in N}\big).$$

A number $x \in A$ ($\hat{x} \in B$) is placed in machine M ($\hat{M}$) as soon as it is enumerated in A (B). x moves finitely often around in machine M ($\hat{M}$) until it comes to rest in one of its two pockets P or Q ($\hat{P}$ or $\hat{Q}$).

Since M and $\hat{M}$ are identical (except that everybody wears a hat in machine $\hat{M}$) we just describe one side.

Pinball machine M consists of hole H, tracks C and D, pockets P and Q and for every state ν a box B_ν in pocket P. There are three rules R_2, R_3, R_4 that govern the movement and the enumeration into sets $\hat{V}_n$ of numbers $x \in N$ in M (there is no rule R_1, the rules got their numbers from their predecessors in Soare [**6**]). The cooperation of both machines M and $\hat{M}$ will be essential and therefore the rules R_2, R_3, R_4 take into account what is happening simultaneously in machine $\hat{M}$.

We start now the exact description of the construction. We fix a recursive function g which enumerates simultaneously sets A, B, $(U_n)_{n\in N}$, $(\tilde{V}_n)_{n\in N}$, $(\tilde{U}_n)_{n\in N}$ and $(V_n)_{n\in N}$ as in the assumption of the Extension Theorem. We assume w.l.o.g. that g enumerates every element of sets U_n, V_n infinitely often into these sets. In the following we construct a simultaneous enumeration of A, B, $(U_n)_{n\in N}$, $(\hat{V}_n)_{n\in N}$, $(\hat{U}_n)_{n\in N}$ and $(V_n)_{n\in N}$ that satisfies the claim of the Extension Theorem. If one is very exact, one will see that from time to time we fail to enumerate a number into U_n or V_n that g has enumerated into U_n, resp. V_n. But since this happens only finitely often for every n, this does not make any difference as far as the claim is concerned.

Construction (the rules R_2, R_3, R_4, $\hat{R}_2$, $\hat{R}_3$, $\hat{R}_4$ will be described subsequently).

Stage $s = 0$. Do nothing.

Stage $s + 1$. Adopt the first case which holds.

Case 1. Some element is on track C or D ($\hat{C}$ or $\hat{D}$). Apply R_3 ($\hat{R}_3$) if it is on track C ($\hat{C}$). Apply R_2 ($\hat{R}_2$) if it is on track D ($\hat{D}$).

Case 2. Some element is above hole H or $\hat{H}$. Take the least such element (if this is not unique take the one above H) and put it on track C ($\hat{C}$) if it was above hole H ($\hat{H}$).

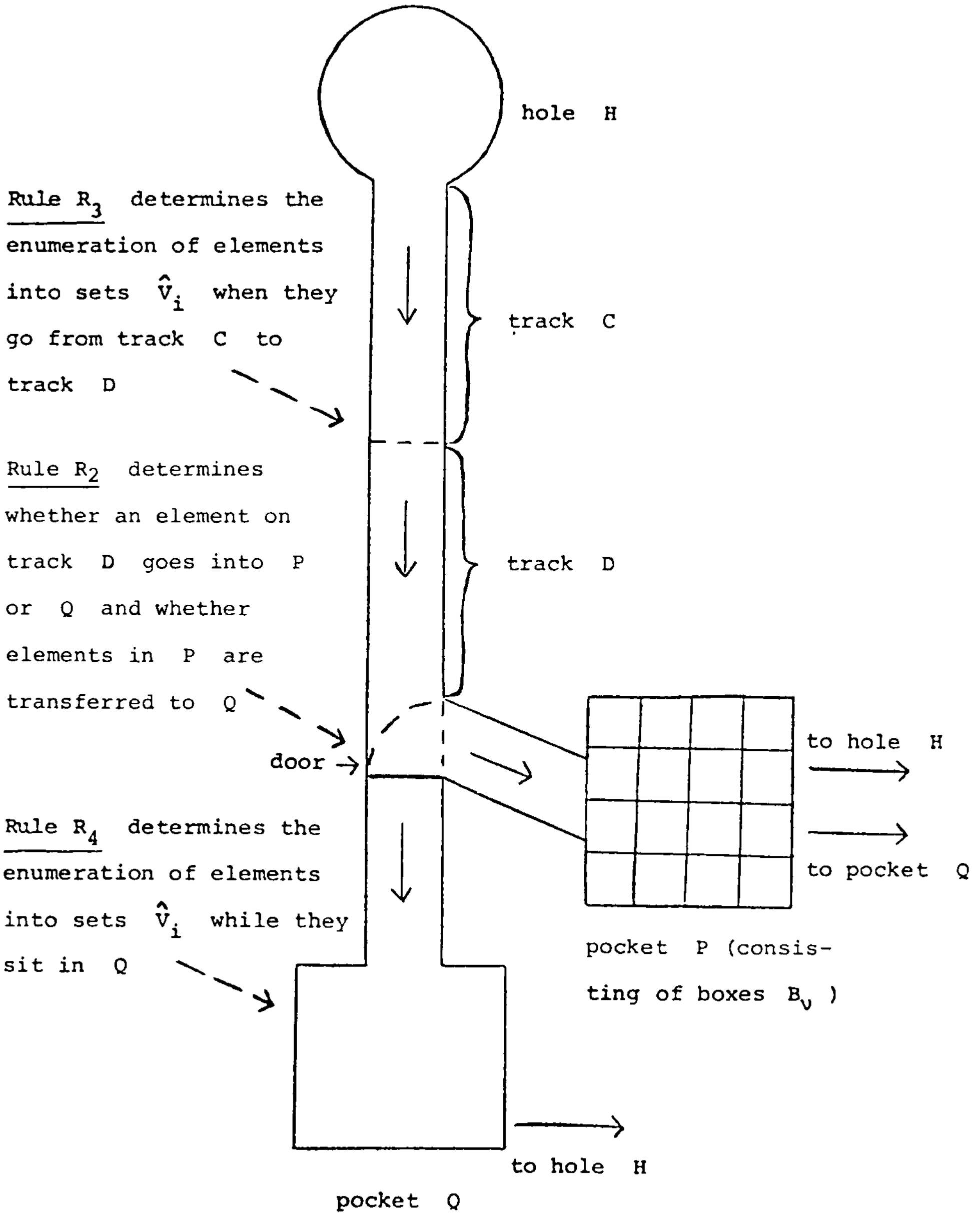
Case 3. Otherwise. We consider one more value of the fixed enumeration g.

(a) If g enumerates a number into U_n or $\tilde{V}_n$ ($n \in N$) which is not yet in A (a number into $\tilde{U}_n$ or V_n ($n \in N$) which is not yet in B), then we enumerate this number into the corresponding set U_n, respectively $\hat{V}_n$ ($\hat{U}_n$, respectively V_n).

(b) If g enumerates a new number into A (B) we enumerate this number into A (B) and place it above hole H ($\hat{H}$) (we say that this number is now in machine M ($\hat{M}$)).

(c) If g enumerates a number $x \geqslant n$ ($\hat{x} \geqslant n$) into U_n (V_n) which we have not yet enumerated in U_n (V_n) and which sits at the moment in pocket Q ($\hat{Q}$) or in a box B_ν

DIAGRAM OF MACHINE M



$(\hat{B}_\nu)$ with $|\nu| \geqslant n$, then we remove this number from its present position, place it above hole H $(\hat{H})$ and enumerate it into U_n (V_n).

At the end of stage $s+1$ we apply Rule R_4 $(\hat{R}_4)$ to every number which is now in pocket Q $(\hat{Q})$. End of the construction.

We write $B_{\nu,s}$ for the set of elements which are in box B_ν at the end of stage s.

Before we give the exact description of the rules, some motivation might be helpful. The ultimate goal is the extension of the sets $\tilde{V}_n$ to sets $\hat{V}_n$ and of the sets $\tilde{U}_n$ to sets $\hat{U}_n$ such that

$$\circledast \qquad \begin{cases} \text{for every state } \nu \\ \text{infinitely many elements of } A \text{ have final state } \nu \text{ iff} \\ \text{infinitely many elements of } B \text{ have final state } \nu. \end{cases}$$

In order to satisfy $\circledast$ it is enough to satisfy for all states ν the following requirement.

R_ν: if infinitely many elements of $B(A)$ have final state ν then there is at least one element $\geqslant |\nu|$ of $A(B)$ which has final state ν.

We use here the fact that if infinitely many elements of B have final state ν, then there is for every $k > |\nu|$ a state ν_k of length k s.t. infinitely many of these elements have final state ν_k. Thus by R_{ν_k} there is an element $\geqslant k$ in A which has final state $\nu_k \succcurlyeq \nu$.

All elements of $B(A)$ come to rest either in pocket $\hat{P}$ or in pocket $\hat{Q}$ (in P or Q). In the pockets $\hat{Q}$ and Q, which collect those elements for which it is more difficult to find matching partners in the other machine, we position the elements on a tree of states ν s.t. at most one element sits on each node ν. We write $\hat{q}(s, \nu)$ for the element on node ν in pocket $\hat{Q}$ at the end of the stage s. If it exists (i.e. $\hat{q}(s, \nu)\downarrow$), the element $\hat{q}(s, \nu)$ is in state ν at the end of stage s. During the play on the pinball machines we try to catch a partner for $\hat{q}(s, \nu)$ in box B_ν in pocket P. A priori we make sure that

(1) if $\hat{q}(s, \nu)\uparrow$, we remove at stage s all elements from B_ν and place them into pocket Q, the "garbage dump" of machine M (see Step 1 of Rule R_2 below) and,

(2) at any time box B_ν holds only elements in state ν and only finitely many elements reside permanently (i.e. from some stage on) in box B_ν (see case 3(c)) of the construction above and the definition of the sets $S_{\nu,t}$ and Steps 2 and 3 of Rule R_2 below).

Because of these properties (1) and (2), in order to satisfy all requirements R_ν it is sufficient to satisfy for all states ν the following requirement.

$\tilde{R}_\nu$: if infinitely many elements remain permanently in pocket $\hat{Q}$ (Q) in final state ν, then all boxes $B_{\nu'}$ $(\hat{B}_{\nu'})$ with $\nu' \preccurlyeq \nu$ get permanent residents.

Obviously there are two possible strategies to satisfy requirement $\tilde{R}_\nu$. Either we try to get permanent residents for all boxes $B_{\nu'}$, $\nu' \preccurlyeq \nu$, or we can try to drive almost all elements that come to rest in pocket $\hat{Q}$ out of state ν. Since the "opponent" has control over the sets U_n, we may never be able to find elements in final state ν' which we can use as permanent residents for the boxes $B_{\nu'}$, $\nu' \preccurlyeq \nu$. Therefore the first strategy will not always succeed. But if we want to satisfy $\tilde{R}_\nu$ according to the second strategy by enumerating elements that rest in $\hat{Q}$ into additional sets $\hat{U}_n$ so that their state becomes some $\nu' \geqslant_\tau \nu$, we have to make sure that at least for this ν' we are able to satisfy $\tilde{R}_{\nu'}$ by the first strategy (notice that it is impossible to satisfy all $\tilde{R}_{\nu'}$ with $|\nu'| = |\nu|$ via the second strategy). The decision which of the strategies we choose to satisfy $\tilde{R}_\nu$ and in case of the second strategy into which state $\nu' \geqslant_\tau \nu$ we lift elements in state ν in $\hat{Q}$ is the main problem of the whole construction. It is solved as follows.

Based on our previous experience we write down at every stage s a list $\mathcal{M}_s$ of those states ν' for which at the moment it makes sense to try to satisfy $\tilde{R}_{\nu'}$ via the first strategy. Thus we can assume at stage s that for all states ν in $\mathcal{P}_s := \{\nu \mid \exists \nu' \in \mathcal{M}_s (\nu' \geqslant_\tau \nu)\}$ one of the two strategies works for $\tilde{R}_\nu$. If $\nu \in \mathcal{M}_s$ we try to satisfy $\tilde{R}_\nu$ via the first strategy. In this situation it is largely the duty of Rule R_2 to catch permanent residents for boxes $B_{\nu'}$, $\nu' \preccurlyeq \nu$, in the respective states. If $\nu \notin \mathcal{M}_s$ but $\nu' \geqslant_\tau \nu$ for some $\nu' \in \mathcal{M}_s$ we try to satisfy $\tilde{R}_\nu$ by the second stategy (we lift elements in state ν in $\hat{Q}$ into this state ν'). This will be executed by Rule $\hat{R}_4$.

If there occurs at stage s in machine $\hat{M}$ an element in state ν s.t. $\nu \notin \mathcal{P}_s$, then we have at the moment no reasonable strategy to handle this element in such a way that $\tilde{R}_\nu$ gets satisfied. In this case $\tilde{R}_\nu$ demands attention at stage s. If $\tilde{R}_\nu$ demands attention at stage s we drop all previous plans concerning the satisfaction of requirement $\tilde{R}_{\tilde{\nu}}$ with $|\tilde{\nu}|>|\nu|$ in order to be able to concentrate on the satisfaction of $\tilde{R}_\nu$ (see condition (a) in the definition of $\mathcal{M}_{s+1}$ below). Thus all requirements $\tilde{R}_{\tilde{\nu}}$ with $|\tilde{\nu}|>|\nu|$ are injured at stage s.

It is largely due to the work of Rule R_3 that every requirement $\tilde{R}_{\tilde{\nu}}$ is only finitely often injured. Each time a requirement $\tilde{R}_\nu$ demands attention, Rule R_3 records this in a list $\mathcal{H}$. For each such entry into list $\mathcal{H}$ Rule R_3 makes as soon as possible a new attempt to solve the respective problem by producing a state $\nu' \geqslant_\tau \nu$ in $\mathcal{M}_s$ which then puts ν in $\mathcal{P}_s$ and provides a reasonable strategy for $\tilde{R}_\nu$. If one assumes for a contradiction (in Lemma 3.5) that a requirement $\tilde{R}_\nu$ demands infinitely often attention, one can show that after a while the corresponding infinitely many attempts of Rule R_3 are successful. Thus we get for some stage s_0, $\forall s \geqslant s_0$ $(\nu \in \mathcal{P}_s)$, and requirement $\tilde{R}_\nu$ no longer demands attention after stage s_0.

We define below a function $d(s, \hat{x})$ for elements $\hat{x}$ in machine $\hat{M}$ at stage s. Essentially $d(s, \hat{x})$ is the maximal length $e \leqslant \hat{x}$ s.t. there exist at stage s reasonable strategies that tell us how to process $\hat{x}$ in order to satisfy requirements $\tilde{R}_\nu$ with $|\nu| \leqslant e$.

The exact description of the rules follows. We need a few auxiliary definitions first.

If x is on track C at the end of stage s we define $\mathbb{S}_s(C)$ as the sequence of all states $\nu \preccurlyeq \nu(s, x, x)$ (we say then that x causes $\nu \in \mathbb{S}_s(C)$). $\mathbb{S}_s(C)$ is empty if there is no element on track C at the end of stage s. $\mathbb{S}(C)$ is defined as the concatenation of all sequences $\mathbb{S}_s(C)$, $s \in N$.

Sequences $\mathbb{S}_s(D)$, $\mathbb{S}(D)$ are defined analogously.

In order to define $\mathbb{S}_s(Q)$ we consider all x which are in pocket Q at the end of stage s and either x was not yet in Q at the end of stage $s-1$ or $\nu(s, x, x) \neq \nu(s-1, x, x)$. $\mathbb{S}_s(Q)$ is the sequence of all states ν such that $\nu \preccurlyeq \nu(s, x, x)$ for one of these x (we say then that x causes $\nu \in \mathbb{S}_s(Q)$). $\mathbb{S}(Q)$ is the concatenation of all sequences $\mathbb{S}_s(Q)$, $s \in N$.

We use X as a variable for tracks C, D and pocket Q.

For machine $\hat{M}$ we define $\mathbb{S}(\hat{C})$, $\mathbb{S}(\hat{D})$, $\mathbb{S}(\hat{Q})$ analogously and we use $\hat{X}$ as a variable for $\hat{C}$, $\hat{D}$, $\hat{Q}$.

We say that x $(\hat{x})$ causes $\nu \in \mathbb{S}(X)$ $(\mathbb{S}(\hat{X}))$ if there is some $s \in N$ such that x $(\hat{x})$ causes $\nu \in \mathbb{S}_s(X)$ $(\mathbb{S}_s(\hat{X}))$.

Further define (in increasing order of $\prec$) a function q as follows. $q(s, \nu)$ is the least $y \in Q_s$ such that $\nu \preccurlyeq \nu(s, y, y)$ and $q(s, \nu') \neq y$ for every $\nu' \prec \nu$. $q(s, \nu)$ is undefined if such y does not exist.

Observe that this definition implies that for every $y \in Q_s$ there is a unique state ν with $y = q(s, \nu)$. We have $\nu \preccurlyeq \nu(s, y, y)$ for this state ν.

Define for states ν and stages $t \in N$, $S_{\nu,t} := \{y \mid \exists t' > t \ (y \text{ causes } \nu \in \mathcal{S}_{t'}(D))\}$. We say that z is the *critical element of* $S_{\nu,t}$ if z is the first element that appears in $S_{\nu,t}$ (i.e. z causes $\nu \in \mathcal{S}_s(D)$ for some $s > t$ and there are no t', y such that $t < t' < s$ and y causes $\nu \in \mathcal{S}_{t'}(D)$).

Rule R_2. Suppose x is on track D at the end of stage s. Let $s' < s$ be the last stage before s such that some element was on track D at stage s' (if no such s' exists, let $s' := 0$).

Step 1. For each ν such that $\hat{q}(\cdot, \nu)$ has not had a constant value since stage s' put every element of $B_{\nu,s}$ into pocket Q.

Step 2. For each ν such that $B_{\nu,s} = \varnothing$ B_ν subscribes to all sets $S_{\nu',s}$ with $\nu \preccurlyeq \nu'$ and $|\nu'| \leqslant s$.

Step 3. Check whether there are ν and ν' such that $\nu \preccurlyeq \nu' \preccurlyeq \nu(s, x, x)$ and a stage $t < s$ such that B_ν has subscribed to the set $S_{\nu',t}$ and x is the critical element of $S_{\nu',t}$. If such exist, choose ν of minimal length and put x in B_ν. If not, put x in pocket Q. End of Rule R_2.

We define for every stage s a set $\mathfrak{M}_s$ of states by induction on s. We set

$\mathcal{P}_s := \{\nu \mid \exists \nu' \in \mathfrak{M}_s(\nu' \geqslant_T \nu)\}$.

$\mathfrak{M}_0 := \varnothing$.

$\nu \in \mathfrak{M}_{s+1}$: $\Leftrightarrow (\nu \in \mathfrak{M}_s$ and ν is not excluded from $\mathfrak{M}_{s+1}) \vee (\nu \notin \mathfrak{M}_s$ and $\nu \in \mathcal{S}_{s+1}(D))$.

We say that ν is excluded from $\mathfrak{M}_{s+1}$ if $\nu \in \mathfrak{M}_s$ and one of the following two conditions holds.

Condition (a). $\exists \nu' \exists \hat{X}(|\nu'| < |\nu|$ and $\nu' \in \mathcal{S}_s(\hat{X}) - \mathcal{P}_s)$.

Condition (b). $\exists \nu' \preccurlyeq \nu(B_{\nu',s} = \varnothing$ and for every $\nu'' \preccurlyeq \nu'$, $\hat{q}(\cdot, \nu'')$ has had a constant value since stage $|\nu|)$.

Further we define $\mathfrak{M}_\omega := \{\nu \mid \nu \in \mathfrak{M}_s$ for almost all $s\}$, $\mathcal{P}_\omega := \{\nu \mid \nu \in \mathcal{P}_s$ for almost all $s\}$ and by recursion over s for elements $\hat{x}$ in machine $\hat{M}$ at stage s:

$$d(s, \hat{x}) := \max\Big(\{-1\} \cup \big\{e \geqslant 0 \mid \nu(s, e, \hat{x}) \in \mathcal{P}_s \wedge \big(d(t, \hat{x}) \geqslant e \text{ at all stages } t < s \text{ where } \hat{x} \text{ was already in machine } \hat{M}\big)\big\}\Big).$$

Observe that $\lim_s d(s, \hat{x})$ exists for every $\hat{x} \in B$ because of the last clause in the definition.

For Rule R_3 define sequences $\mathcal{H}_t$ of pairs $\langle \nu, i \rangle$ with $i \in \{0, 1\}$ as follows.

$\mathcal{H}_t$ consists of all pairs $\langle \nu(t', e, \hat{x}), 0 \rangle$ such that for some e, $\hat{x}$, $\hat{X}$, t' we have $e \leqslant \hat{x}$, $t' \leqslant t$, $\hat{x}$ is in machine $\hat{M}$ at stage t' and $\nu(t, e, \hat{x}) \in \mathcal{S}_t(\hat{X}) - \mathcal{P}_t$ together with all pairs $\langle \nu, 1 \rangle$ such that $\nu \in \mathcal{S}_t(C)$.

We write $\mathcal{H}$ for the concatenation of the $\mathcal{H}_t$, $t \in N$, and $\mathcal{H}_{\leqslant s}$ for the initial segment $\mathcal{H}_0 \frown \cdots \frown \mathcal{H}_s$ of $\mathcal{H}$.

Rule R_3. Suppose x in on track C at the end of the stage s. Let $\langle \nu, i \rangle$ be the first pair in the sequences $\mathcal{H}_{\leqslant s}$ which has not yet been checked and *either* $i = 0$ and for $\langle e, \sigma, \tau \rangle := \nu$, $\langle e, \sigma', \tau', \rangle := \nu(s, e, x)$ we have $\langle e, \sigma', \tau' \rangle \geqslant \langle e, \sigma, \tau \rangle$ *or* $i = 1$ and $\nu \leqslant \nu(s, x, x)$ (if no such pair exists, put x immediately on track D). In the first case we enumerate x in all sets $\hat{V}_i$ with $i \in \tau - \tau'$ and place x on track D. In the second case we put x immediately on track D. Finally we check the considered pair $\langle \nu, i \rangle$ in the sequence $\mathcal{H}$.

Rule $\hat{R}_4$. If $\hat{x}$ is in the pocket $\hat{Q}$ at the end of stage $s + 1$, $e := d(s, \hat{x}) \geqslant 0$ and $\langle e, \sigma, \tau \rangle := \nu(s, e, \hat{x}) \notin \mathcal{M}_s$, then we choose among all states $\langle e, \sigma', \tau \rangle \in \mathcal{M}_s$ with $\sigma' \supseteq \sigma$ that one which has occurred most frequently in $\mathcal{S}(D)$. For this σ' we enumerate $\hat{x}$ in $\hat{U}_i$ for $i \in \sigma' - \sigma$.

The description of the construction is now complete.

In our previous explanations we had not mentioned that Rule R_3 has to leave as well the state of many elements that pass from track C to track D unchanged. This is accomplished by the pairs $\langle \nu, 1 \rangle$ in the list $\mathcal{H}$ and it is verified in Lemma 3.2. We need Lemma 3.2 for the proof of Lemma 3.3.

As we had described earlier, the main duty of Rule R_3 is to insure that if state ν occurs infinitely often in some stream $\mathcal{S}(\hat{X})$ in machine $\hat{\mathcal{M}}$ then we find after a while a reasonable strategy to satisfy requirement $\tilde{R}_\nu$, i.e. we get $\nu \in \mathcal{P}_\omega$. We verify in the proof of Claim 1 in the proof of Lemma 3.5 that Rule R_3 does this job. We further show in Case 1 of Claim 2 in the proof of Lemma 3.5 that Rule R_3 does not overreact, i.e. does not enumerate too many elements into sets $\hat{V}_n$.

Rule $\hat{R}_4$ executes the second strategy to satisfy requirements $\tilde{R}_\nu$ by making sure that infinitely many elements settle down in state ν in pocket $\hat{Q}$ only if we can expect to satisfy $\tilde{R}_\nu$ via the first strategy, i.e. if $\nu \in \mathcal{M}_\omega$. We verify in Lemma 3.6 that Rule $\hat{R}_4$ does its job. We show in addition in Case 2 of Claim 2 in the proof of Lemma 3.5 that Rule $\hat{R}_4$ does not enumerate too many elements into sets $\hat{U}_n$.

All rules are carefully balanced to make sure that every stream in one machine is covered by a stream in the other machine, i.e. if ν occurs infinitely often in a stream $\mathcal{S}(\hat{X})$ in machine $\hat{M}$ then some $\nu' \geqslant \nu$ occurs infinitely often in some stream $\mathcal{S}(X)$ in machine M and vice versa.

Concerning the list $\mathcal{M}_s$ we use at the beginning of the proof of Lemma 3.5 that states are excluded from $\mathcal{M}_{s+1}$ via Condition (a). Exclusion from $\mathcal{M}_{s+1}$ via Condition (b) is essential for the proof of Lemma 3.7.

We start now with the *exact analysis of the construction*. A trivial proof by induction on the enumeration given by the function g shows the following. Every $x \in A$ is placed above hole H at a certain stage. No number remains forever above hole H. At every stage there is at most one element on one of the tracks C, D, $\hat{C}$, $\hat{D}$. This number is moved downwards at the next stage. Further $x \in A$ can move upwards in machine M (i.e. from P or Q to hole H) only if x is enumerated in some new U_i with $i \leqslant x$. No number x jumps directly from one box in P to another (although x may be recycled to H and get into a different box when it enters P the next time). Therefore every $x \in A$ moves only finitely often in machine M and remains from some stage on permanently in Q or in a box B_ν in P. The analogous facts hold for elements $\hat{x} \in B$ in machine $\hat{M}$.

LEMMA 3.1. *For every permanent resident x of Q there exists a unique state ν such that $x = \lim q(s, \nu)$. This state ν satisfies $\nu \preccurlyeq \lim_s \nu(s, x, x)$. Further if for any ν $\lim_s q(s, \nu)$ exists and $\nu' \preccurlyeq \nu$ then $\lim_s q(s, \nu')$ exists as well.*

PROOF. Assume that there is a permanent resident x of Q such that for no ν $x = \lim_s q(s, \nu)$. Let x be minimal with this property. Let s_0 be a stage such that for every $y < x$ with $y \in Q_s$ for some $s \geqslant s_0$ there is a state ν_y with $\forall s \geqslant s_0\, (y = q(s, \nu_y))$. Further we assume that $\nu(s, x, x)$ is constant after stage s_0. Let ν_0 be of minimal length so that $x = q(s, \nu_0)$ for some $s > s_0$. Since by assumption we have not $x = \lim_x q(s, \nu_0)$ there is some $s_1 > s_0$ such that $x = q(s_1, \nu_0) \neq q(s_1 + 1, \nu_0) = y$ for some y. Then $y < x$ according to the definition of q and this contradicts the choice of s_0.

In order to prove the last part assume that $x = q(s, \nu)$ for all $s \geqslant s_0$. By the preceding there is some $s_1 > s_0$ such that for every $y < x$ with $\exists s \geqslant s_1\ (y \in Q_s)$ there is some state ν_y with $\forall s \geqslant s_1\ (y = q(s, \nu_y))$. Consider some $\nu' \prec \nu$. For every $s \geqslant s_1$ $q(s, \nu')$ is defined and less than x by the definition of q. Therefore $\nu' = \nu_y$ for some $y < x$.

LEMMA 3.2. *Every state which occurs infinitely often in $\mathcal{S}(C)$ occurs as well infinitely often in $\mathcal{S}(D)$.*

PROOF. Assume ν occurs infinitely often in $\mathcal{S}(C)$. Then the pair $\langle \nu, 1\rangle$ occurs infinitely often in the sequence $\mathcal{H}$ for Rule R_3. Therefore infinitely many of the elements that cause $\nu \in \mathcal{S}(C)$ are placed in the same state on track D.

LEMMA 3.3. *Assume ν is a state such that only finitely many states $\nu' >_\tau \nu$ occur in $\mathcal{S}(D)$. Then there are only finitely many stages s where ν is excluded from $\mathfrak{M}_{s+1}$ according to Condition* (b).

PROOF. Assume the claim is false for state ν. By definition this implies that ν occurs infinitely often in $\mathcal{S}(D)$. Further it implies that there is some $\nu_0 \preccurlyeq \nu$ of minimal length such that box B_{ν_0} has no permanent resident. B_{ν_0} subscribes then for infinitely many s to the set $S_{\nu,s}$. Each of these sets is nonempty and the critical elements of these sets $S_{\nu,s}$ together form an infinite set. Almost all of these elements are placed (in state ν) in the box B_{ν_0}, because the boxes $B_{\tilde\nu}$ with $\tilde\nu \prec \nu_0$ subscribe only to finitely many sets (by the minimality of ν_0).

By assumption there is some state $\tilde\nu$ such that $\nu_0 \preccurlyeq \tilde\nu \preccurlyeq \nu$ and $B_{\tilde\nu}$ causes infinitely often the exclusion of ν from $\mathfrak{M}_{s+1}$ via Condition (b). This implies that $\lim_s \hat q(s, \nu_0)$ exists. Therefore almost all of the infinitely many elements of sets $S_{\nu,s}$ which enter box B_{ν_0} are later placed above hole H because they are enumerated in some new U_e with $e \leqslant |\nu_0|$. All these elements run afterwards in some state $\nu' >_\tau \nu$ over track C. Therefore some $\nu' >_\tau \nu$ occurs infinitely often in $\mathcal{S}(C)$ and by Lemma 3.2 as well in $\mathcal{S}(D)$, a contradiction.

LEMMA 3.4. *Assume that there are only finitely many stages s such that states of length e are excluded from $\mathfrak{M}_{s+1}$ via Condition* (a). *Then for every state ν of length e $\nu \in \mathcal{P}_s$ for infinitely many $s \Rightarrow \nu \in \mathcal{P}_\omega$.*

PROOF. The claim is obvious if there is some $\tilde{\nu} \in \mathfrak{M}_\omega$ with $\nu \geqslant_\tau \nu$. Otherwise there is some $\tilde{\nu} \geqslant_\tau \nu$ which is infinitely often added to $\mathfrak{M}_{s+1}$ and later excluded. This implies that $\tilde{\nu}$ occurs infinitely often in $\mathbb{S}(D)$. Choose then $\nu' \geqslant_\tau \nu$ maximal w.r.t. $\geqslant_\tau$ such that ν' is excluded only finitely often from $\mathfrak{M}_{s+1}$ via Condition (b) according to Lemma 3.3. By the assumption of this lemma ν' is only finitely often excluded from $\mathfrak{M}_{s+1}$ via Condition (a). Thus $\nu' \in \mathfrak{M}_\omega$ and therefore $\nu \in \mathfrak{P}_\omega$.

LEMMA 3.5. (i) *If ν occurs infinitely often in $\mathbb{S}(\hat{X})$ for some $\hat{X}$ then $\nu \in \mathfrak{P}_\omega$.*
(ii) *If ν occurs infinitely often in $\mathbb{S}(X)$ for some X then $\nu \in \hat{\mathfrak{P}}_\omega$.*

PROOF. One proves (i) and (ii) simultaneously by induction on $|\nu|$. Assume (i) and (ii) hold for all ν with $|\nu| < e$. Then every state of length e is only finitely often excluded from $\mathfrak{M}_{s+1}$ or $\hat{\mathfrak{M}}_{s+1}$ via Condition (a).

Assume for a contradiction that (i) does not hold for state ν_1 of length e. Fix some $\hat{X}$ such that ν_1 occurs infinitely often in $\mathbb{S}(\hat{X})$. Because of Lemma 3.4 we have then $\nu_1 \in \mathbb{S}_s(\hat{X}) - \mathfrak{P}_s$ for infinitely many s. This implies that $\nu \in \mathfrak{M}_\omega \Rightarrow |\nu| \leqslant e$ (via Condition (a)).

Fix infinitely many different numbers $\hat{y}_j$, $j \in N$, and stages t_j such that for all $j \in N$, $\hat{y}_j$ causes $\nu_1 \in \mathbb{S}_{t_j}(\hat{X}) - \mathfrak{P}_{t_j}$. Let $\mathcal{G}_j$ be the finite sequence of states ν such that

$$\exists s \leqslant t_j \left(\hat{y}_j \text{ is in machine } \hat{M} \text{ at stage } s \text{ and } \left(\nu(s, e, \hat{y}_j) = \nu\right)\right).$$

Let $\mathcal{G}$ be the concatenation of the sequences $\mathcal{G}_j, j \in N$.

CLAIM 1. *If ν occurs infinitely often in $\mathcal{G}$ and some $\nu' \geqslant \nu$ occurs infinitely often in $\mathbb{S}(C)$, then some $\nu'' \geqslant_\tau \nu$ occurs infinitely often in $\mathbb{S}(D)$ and $\nu \in \mathfrak{P}_\omega$.*

PROOF OF CLAIM 1. The pair $\langle \nu, 0 \rangle$ occurs then infinitely often in the list $\mathcal{K}$. Therefore infinitely many of the elements that cause $\nu' \in \mathbb{S}(C)$ for some $\nu' \geqslant \nu$ are lifted into some state $\nu'' \geqslant_\tau \nu$ according to Rule R_3. If $\nu'' \geqslant_\tau \nu$ occurs infinitely often in $\mathbb{S}(D)$ then we have for infinitely many s, $\nu'' \in \mathfrak{M}_s$ and therefore $\nu \in \mathfrak{P}_s$. Thus $\nu \in \mathfrak{P}_\omega$ by Lemma 3.4.

CLAIM 2. *If ν occurs infinitely often in $\mathcal{G}$ then some $\nu' \geqslant \nu$ occurs infinitely often in $\mathbb{S}(C)$.*

PROOF OF CLAIM 2. By contradiction. Fix $\nu_2 = \langle e, \sigma_2, \tau_2 \rangle$ so that σ_2 is minimal and τ_2 is minimal for σ_2 such that the claim fails for ν_2. Because of the assumption of the Extension Theorem (the "covering property") we cannot have then that infinitely many $\hat{y}_j$ are already in state ν_2 when they enter machine $\hat{M}$. Therefore there exists a state $\nu_3 = \langle e, \sigma_3, \tau_3 \rangle \neq \nu_2$, an infinite set $J \subseteq N$ and stages $s_j \leqslant t_j$ for $j \in J$ such that for every $j \in J$, $\hat{y}_j$ is in machine $\hat{M}$ at stage $s_j - 1$, $\nu(s_j - 1, e, \hat{y}_j) = \nu_3$ and $\nu(s_j, e, \hat{y}_j) = \nu_2$.

Assume first that $\sigma_3 = \sigma_2$ and $\tau_3 \subsetneq \tau_2$. Then $\nu_3 \geqslant \nu_2$. By the minimal choice of τ_2 the claim holds for ν_3 and thus as well for ν_2. We assume now that $\sigma_3 \subsetneq \sigma_2$. Then there is an infinite set $J' \subseteq J$ such that (Case 1) for every $j \in J'$, Rule $\hat{R}_3$ is applied to $\hat{y}_j$ at stage s_j or (Case 2) for every $j \in J'$, Rule $\hat{R}_4$ is applied to $\hat{y}_j$ at stage s_j.

Case 1. Because of the induction hypothesis only finitely many pairs $\langle \nu, 0 \rangle$ with $|\nu| < e$ occur in the list $\hat{\mathcal{K}}$ for Rule $\hat{R}_3$. Therefore for almost all $j \in J'$ one checks during the application of $\hat{R}_3$ at stage s_j some pair $\langle \langle \tilde{e}, \tilde{\sigma}, \tilde{\tau} \rangle, 0 \rangle$ in $\hat{\mathcal{K}}$ with $\tilde{e} \geqslant e$ and

$\langle e, \tilde{\sigma} \cap (e+1), \tilde{\tau} \cap (e+1)\rangle \geq \nu_2$. The element in machine M which caused the occurrence of $\langle\langle \tilde{e}, \tilde{\sigma}, \tilde{\tau}\rangle, 0\rangle$ in $\hat{\mathcal{K}}$ did therefore run in some state $\nu' \geq \nu_2$ over track C (consider e.g. the first time it comes over track C after it has reached state $\tilde{\sigma} \cap (e+1)$ w.r.t. $U_0, \ldots, U_e$).

Case 2. We have for almost all $j \in J'$, $d(s_j - 1, \hat{y}_j) \geq e$ (this follows from our minimal choice of σ_2 together with Claim 1). In case that $d(s_j - 1, \hat{y}_j) > e$ for infinitely many $j \in J'$, there is for each of these j some $\nu(j) \succ \nu_2$ in $\mathfrak{M}_{s_j - 1}$ with $|\nu(j)| > e$. As we have mentioned above, no state ν with $|\nu| > e$ is in $\mathfrak{M}_\omega$. Therefore infinitely often some state $\nu \succcurlyeq \nu_2$ is added to $\mathfrak{M}_{s+1}$. This implies that some $\nu \succcurlyeq \nu_2$ occurs infinitely often in $\mathcal{S}(D)$ and therefore some $\nu \geq \nu_2$ occurs infinitely often in $\mathcal{S}(C)$. In case that $d(s_j - 1, \hat{y}_j) = e$ for almost all $j \in J'$ we see that our claim holds for ν_3. If follows then from Claim 1 that some $\nu'' \geq_\tau \nu_3$ occurs infinitely often in $\mathcal{S}(D)$. If we choose ν'' with this property maximal w.r.t. $\geq_\tau$ then this $\nu'' \geq_\tau \nu_3$ occurs infinitely often in $\mathcal{S}(D)$ and $\nu'' \in \mathfrak{M}_\omega$ (by Lemma 3.3 and our induction hypothesis). Since one chooses in Rule $\hat{R}_4$ that state in $\mathfrak{M}_s$ which occurred most frequently in $\mathcal{S}(D)$, ν_2 occurs infinitely often in $\mathcal{S}(D)$.

The state ν_1 which was fixed before Claim 1 occurs infinitely often in $\mathcal{I}$. Therefore Claim 1 and Claim 2 together imply that $\nu_1 \in \mathcal{P}_\omega$, a contradiction.

Part (ii) of Lemma 3.5 is proved analogously.

LEMMA 3.6. *For every $e \in N$ there are only finitely many numbers $\hat{x} \in B$ with $d(\hat{x}) := \lim_s d(s, \hat{x}) < e$. Further if $\hat{x}$ finally remains in pocket $\hat{Q}$ and $d(\hat{x}) \geq 0$ then $\lim_s \nu(s, d(\hat{x}), \hat{x}) \in \mathfrak{M}_\omega$.*

PROOF. If $\hat{x}$ is in machine $\hat{M}$ at the end of stage s then $\hat{x}$ sits at the moment on track $\hat{C}$ or $\hat{D}$, in pocket $\hat{P}$ or $\hat{Q}$ or above hole $\hat{H}$. If $\hat{x}$ is in pocket $\hat{P}$, then $\hat{x}$ ran before in state $\nu(s, \hat{x}, \hat{x})$ over track $\hat{D}$. If $\hat{x}$ is above hole $\hat{H}$ then $\hat{x}$ runs afterwards in state $\nu(s, \hat{x}, \hat{x})$ over track $\hat{C}$. In any case for every $\nu \preccurlyeq \nu(s, \hat{x}, \hat{x})$, $\hat{x}$ causes $\nu \in \mathcal{S}(\hat{X})$ for some $\hat{X}$. Thus Lemma 3.5 implies that for every $e \in N$ $d(\hat{x}) < e$ for only finitely many $\hat{x} \in B$. If $\hat{x}$ with $d(\hat{x}) \geq 0$ remains in pocket $\hat{Q}$ after some stage s_0 and $d(s, \hat{x})$, $\nu(s, \hat{x}, \hat{x})$ remain constant after stage s_0 then for every $s \geq s_0$, $\lim_s(\nu(s, d(\hat{x}), \hat{x})) = \nu(s+1, d(\hat{x}), \hat{x}) \in \mathfrak{M}_s$ because of the action of Rule $\hat{R}_4$ at stage $s+1$.

LEMMA 3.7. *If infinitely many elements remain permanently in pocket $\hat{Q}$ in final state ν, then infinitely many elements remain permanently in pocket P in final state ν.*

PROOF. Assume S is an infinite set such that every element of S remains permanently in $\hat{Q}$ in final state ν. We show first that box B_ν gets a permanent resident in final state ν.

According to Lemma 3.1 there is for every $\hat{x} \in S$ some state $\nu_{\hat{x}}$ with $\hat{x} = \lim_s \hat{q}(s, \nu_{\hat{x}})$ and $\nu \preccurlyeq \nu_{\hat{x}}$ or $\nu \succcurlyeq \nu_{\hat{x}}$. Since no two $\hat{x} \in S$ have the same $\nu_{\hat{x}}$ there is some $\hat{x} \in S$ with $\nu \preccurlyeq \nu_{\hat{x}}$. Therefore for every $\nu'' \preccurlyeq \nu$, $\lim_s \hat{q}(s, \nu'')$ exists according to Lemma 3.1. Fix some stage s_0 such that

$$\forall \nu'' \preccurlyeq \nu, \forall s \geq s_0 \quad \hat{q}(s, \nu'') = \hat{q}(s_0, \nu'').$$

By Lemma 3.6 $T := \{\nu' \mid \nu' = \lim_s \nu(s, d(\hat{x}), \hat{x})$ for some $\hat{x} \in S\}$ is an infinite subset of $\mathfrak{M}_\omega$. Fix some $\nu_1 \in T$ with $\nu_1 \succcurlyeq \nu$ and $|\nu_1| \geqslant s_0$. Define

$$t_0 := \mu t, \forall t' \geqslant t \qquad (\nu_1 \in \mathfrak{M}_{t'}).$$

Because of Condition (b) in exclusion from $\mathfrak{M}_{s+1}$ we have $\forall t \geqslant t_0\ (B_{\nu,t} \neq \varnothing)$. Therefore B_ν subscribes only to finitely many sets and the critical element of one of these sets remains permanently in B_ν. According to Rule R_2 every element that enters box B_ν is in state ν. This state does not change as long as it remains in B_ν. Thus every permanent resident of B_ν has final state ν.

In order to show the existence of infinitely many permanent residents of pocket P in final state ν we observe that for every $e \geqslant |\nu|$ there is some state $\nu_e \succcurlyeq \nu$ such that infinitely many $\hat{x} \in S$ have final state ν_e. By the preceding there is a permanent resident of B_{ν_e} in final state ν_e.

LEMMA 3.8. *If pocket P has infinitely many permanent residents in final state ν then pocket $\hat{Q}$ has as well infinitely many permanent residents in final state ν.*

PROOF. Let S be the infinite set of permanent residents of P in final state ν. For every $x \in S$ there is a state ν_x such that $\nu_x \preccurlyeq \nu$ or $\nu \preccurlyeq \nu_x$ and x remains permanently in box B_{ν_x}. Because of Step 1 in Rule R_2, $\lim_s \hat{q}(s, \nu_x)$ exists for every $x \in S$. Finally only finitely many elements can stay permanently in a single box and so $\{\nu_x \mid x \in S\}$ is infinite. End of the proof of the Extension Theorem.

4. Construction of the isomorphism for semilow$_{1.5}$ sets. We fix for this section an r.e. set A with $\bar{A}$ infinite and semilow$_{1.5}$.

Let f be a total recursive function such that

$$\forall e \in N\big(W_e \cap \bar{A} \text{ infinite} \Leftrightarrow W_{f(e)} \text{ infinite}\big).$$

We fix a simultaneous enumeration of $(W_e)_{e \in N}$ without repetitions where only one element is enumerated at every stage. We say that *W_e is verified at stage s* if in this enumeration some number is enumerated in $W_{f(e)}$ at stage s.

According to Lemma 2.3 A has the outer splitting property. We fix recursive functions f_0, f_1 as in Definition 2.2. For an r.e. set W_e we call *$W_{f_1(e)}$ the critical part of W_e*.

We use the recursion theorem in order to compute during the construction r.e. indices for various r.e. sets which are constructed on the side. We can then apply the preceding tools to these sets during the construction.

The construction uses the same pinball machines M and $\hat{M}$ as in §3, but the rules are slightly different. We fix again two copies N and $\hat{N}$ of the natural numbers. All numbers $x \in N$ ($\hat{x} \in \hat{N}$) are fed into machine M ($\hat{M}$) at some point of the construction. But, different from §3, numbers may leave a machine: as soon as $x \in N$ is enumerated into A we remove x forever from machine M. Thus only the numbers in $\bar{A}$ remain in M whereas all elements of $\hat{N}$ remain in $\hat{M}$.

As before we construct on the side of machine M arrays $(U_n)_{n\in N}$, $(\hat{V}_n)_{n\in N}$ and on the side of machine $\hat{M}$ arrays $(\hat{U}_n)_{n\in N}$, $(V_n)_{n\in N}$. The notations concerning states, covering etc. are the same as in §3. The goal of this construction is to satisfy

(1) $\forall n \in N(U_n =^* W_n =^* V_n)$ and

(2) for every state ν infinitely many $x \in \bar{A}$ have final state ν if and only if infinitely many $\hat{x} \in \hat{N}$ have final state ν.

The overall strategy to satisfy these goals is the same as in §3 and we have formulated this earlier construction in such a way that it can easily be adapted to the present situation.

The main difference is that now it would be fatal to maintain during the construction an analogous covering of streams as before. If some state ν occurs infinitely often in a stream $\mathbb{S}(X)$ in machine M, it may be the case that all numbers $x \in N$ that cause $\nu \in \mathbb{S}(X)$ are fakes which drop later into A and which should not be taken seriously. It we attempt to cover these x by elements $\hat{x}$ in machine M, which we lift for this purpose into some state $\tilde{\nu} \leqslant \nu$, these $\hat{x}$ would in the end remain without covering elements in machine M. This problem is serious because we can not tell whether an element x is a fake or not until it is actually enumerated into A, which is usually too late.

The good point is of course that the opponent has promised to make $\bar{A}$ semilow$_{1.5}$. Therefore by using the recursion theorem we can force the opponent to reveal step by step which r.e. sets W_e are going to have an infinite intersection with $\bar{A}$. Using our previous terminology the opponent has to verify such W_e infinitely often. Therefore the rules of machine $\hat{M}$ do not react anymore directly to events in machine M but instead react to verifications by the opponent concerning events in machine M.

In addition we have to make sure that the work of Rule R_3 is not in vain. This may happen if it only raises states of elements that later drop into A. Here the outer splitting property comes to our rescue. It provides us with a sieve to catch from any set W_e with infinitely many real elements (i.e. $W_e \cap \bar{A}$ infinite) sufficiently many but not too many of these real elements (the elements in the critical part of W_e intersected with $\bar{A}$). Furthermore because of the uniformity of this splitting procedure we can iterate it and use a nested sequence of sieves. Thus we can e.g. catch as well real elements in the critical part of the uncritical part W_e (i.e. $W_{f_1(f_0(e))}$) etc.

Rule R_2 nearly remains the same. But if box B_ν subscribes now to a set $S_{\nu',s}$, then not only the critical element of $S_{\nu',s}$ but all elements in the critical part of $S_{\nu',s}$ (these may be infinitely many) are delivered to box B_ν. Accordingly R_2 has to be aware that it is not enough anymore to make $B_{\nu,s} \neq \varnothing$ for almost all stages s because this may be caused by infinitely many fake residents.

These precautions allow us to maintain a more adequate covering of streams during the construction. If infinitely many elements of $\bar{A}$ cause $\nu \in \mathbb{S}(X)$ in machine M then there is some $\tilde{\nu} \leqslant \nu$ s.t. infinitely many elements of $\hat{N}$ cause $\tilde{\nu} \in \mathbb{S}(\hat{X})$ for some $\hat{X}$ in machine $\hat{M}$ and vice versa.

We give now the exact description of the construction. We use the same definitions as in §3 unless we say otherwise. After the construction we verify in Lemmas 4.1–4.8 that it satisfies the above-mentioned goals (1) and (2). We show after Lemma 4.8 how one derives the missing direction of Theorem 1.2 from (1) and (2).

We fix a recursive function g which enumerates simultaneously $(W_e)_{e\in N}$ and $(\hat{W}_e)_{e\in N}$ where $(W_e)_{e\in N}$ $((\hat{W}_e)_{e\in N})$ are standard indexings of the r.e. subsets of N $(\hat{N})$. We assume that g enumerates every element of these sets infinitely often. Further we assume that $\hat{W}_0 = \hat{N}$, $W_0 = N$, $W_1 = A$ and every number comes first into W_0 $(\hat{W}_0)$ before it comes into any other W_e $(\hat{W}_e)$.

Construction (we use the rules R_2, R_3, R_4, $\hat{R}_2$, $\hat{R}_3$, $\hat{R}_4$ in the construction which will be described subsequently).

Stage $s = 0$. Do nothing.

Stage $s + 1$. Adopt the first case which holds.

Case 1. Some element is on track C or D $(\hat{C}$ or $\hat{D})$. Apply R_3 $(\hat{R}_3)$ if it is on track C $(\hat{C})$. Apply R_2 $(\hat{R}_2)$ if it is on track D $(\hat{D})$.

Case 2. Some element is above hole H or $\hat{H}$. Take the least such element (if this is not unique take the one above H) and put it on track C $(\hat{C})$ if it was above hole H $(\hat{H})$.

Case 3. Otherwise. We consider then one more value of the fixed enumeration function g.

(a) If g enumerates a new number into W_0 $(\hat{W}_0)$ we enumerate this number into U_0 (V_0) and place it above hole H $(\hat{H})$ (we say that this number now enters machine M $(\hat{M})$).

(b) If g enumerates a new number into W_1 then we remove this number from machine M and enumerate it into U_1.

(c) If g enumerates a number $x \geqslant e$ into W_e where $e > 1$ (a number $\hat{x} \geqslant e$ into $\hat{W}_e$ where $e \geqslant 1$) which is not yet in U_e (V_e) and which sits at the moment in pocket Q $(\hat{Q})$ or in some box B_ν $(\hat{B}_\nu)$ with $|\nu| \geqslant e$, then we remove this number from its present position, place it above hole H $(\hat{H})$ and enumerate it into U_e (V_e).

Rule R_2. Suppose x is on track D at the end of stage s. Let $s' < s$ be the last stage before s such that some element was on track D at stage s' (if no such s' exists, let $s' := 0$).

Step 1. For each ν such that $\hat{q}(\cdot, \nu)$ has not had a constant value since stage s', put every element of $B_{\nu,s}$ into pocket Q.

Step 2. For each ν such that $B_{\nu,s} = \varnothing$ or the least number in box B_ν has changed since stage s', box B_ν subscribes to all sets $S_{\nu',s}$ with $\nu \preccurlyeq \nu'$ and $|\nu'| \leqslant s$.

Step 3. Check whether there are ν and ν' such that $\nu \preccurlyeq \nu' \preccurlyeq \nu(s, x, x)$ and a stage $t < s$ such that B_ν has subscribed to the set $S_{\nu',t}$ and x is in the critical part of $S_{\nu',t}$. If such exist, choose ν of minimal length and put x in B_ν. If not, put x in pocket Q.

We define for every stage s a set $\mathfrak{M}_s$ of states by induction on s. We set

$\mathcal{P}_s := \{\nu \mid \exists \nu' \in \mathfrak{M}_s(\nu' \geqslant_\tau \nu)\}$.

$\mathfrak{M}_0 := \varnothing$.

$\nu \in \mathfrak{M}_{s+1} :\Leftrightarrow (\nu \in \mathfrak{M}_s$ and ν is not excluded from $\mathfrak{M}_{s+1}) \vee (\nu \notin \mathfrak{M}_s$ and there is some $t \leqslant s+1$ with $\nu \in \mathbb{S}_t(D)$ and $s+1$ is the first stage where for every $\nu' \preccurlyeq \nu$ the set $\{y \mid y$ causes $\nu' \in \mathbb{S}(D)\}$ has t times been verified).

We say that ν is excluded from $\mathfrak{M}_{s+1}$ if $\nu \in \mathfrak{M}_s$ and one of the following two conditions holds.

Condition (a). $\exists\nu'\exists\hat{x}(|\nu'|<|\nu| \wedge \nu' \in \mathcal{S}_s(\hat{X}) - \mathcal{P}_s)$.

Condition (b). For $t_0 := \max\{t \leqslant s \mid \nu \in \mathfrak{M}_t - \mathfrak{M}_{t-1}\}$ there is some $\nu' \preccurlyeq \nu$ such that no element constantly remained in $B_{\nu'}$ since the end of stage t_0 and for every $\nu'' \preccurlyeq \nu'$, $\hat{q}(\cdot, \nu'')$ has had a constant value since stage $|\nu|$.

Further we define $\mathfrak{M}_\omega := \{\nu \mid \nu \in \mathfrak{M}_s$ for almost all $s\}$, $\mathcal{P}_\omega := \{\nu \mid \nu \in \mathcal{P}_s$ for almost all $s\}$ and by recursion over s for elements $\hat{x}$ in machine $\hat{M}$ at stage s $d(s, \hat{x}) := \max(\{-1\} \cup \{e \geqslant 0 \mid \nu(s, e, \hat{x}) \in \mathcal{P}_s \wedge (d(t, \hat{x}) \geqslant e$ at all stages $t < s$ where $\hat{x}$ was already in machine $\hat{M})\})$.

Observe that $\lim_s d(s, \hat{x})$ always exists because of the last clause in the definition of $d(s, \hat{x})$.

For Rule R_3 we define sets $\mathcal{K}_t$ of pairs $\langle \nu, i\rangle$ with $i \in \{0, 1\}$ as follows.

$\mathcal{K}_t$ consists of all pairs $\langle \nu(t', e, \hat{x}), 0\rangle$ such that for some $\hat{x}$, $\hat{X}$, t' we have $t' \leqslant t$ and $\nu(t, e, \hat{x}) \in \mathcal{S}_t(\hat{X}) - \mathcal{P}_t$ together with all pairs $\langle \nu, 1\rangle$ such that $\{y \mid y$ causes $\nu \in \mathcal{S}(C)\}$ is verified at stage t.

We well order every $\mathcal{K}_t$ in some canonical way and we write $\mathcal{K}$ for the concatenation of all the well-ordered sets $\mathcal{K}_t$, $t \in N$. Further we write $\mathcal{G}_t$ for the set of triples $\langle \nu, i, n\rangle$ such that $\langle \nu, i\rangle$ appears as the nth pair in the sequence $\mathcal{K}$ and this occurrence comes from $\langle \nu, i\rangle \in \mathcal{K}_t$. $\mathcal{G} := \bigcup\{\mathcal{G}_t \mid t \in N\}$.

By recursion over n we define for every $\langle \nu, i, n\rangle \in \mathcal{G}$ a recursive set $C_{\nu,i,n}$ and an associated r.e. set $T_{\nu,i,n} := \{x \mid \exists s(\langle x, s\rangle \in C_{\nu,i,n})\}$. For $\langle \nu, 0, n\rangle \in \mathcal{G}_t$ we define $C_{\nu,0,n} := \{\langle x, s\rangle \mid s > t \wedge (x$ causes $\nu' \in \mathcal{S}_s(C)$ for some $\nu' \geqslant \nu) \wedge$ (there is no $\langle \tilde{\nu}, \tilde{i}, \tilde{n}\rangle$ with $\tilde{n} < n$ such that $\langle x, s\rangle \in C_{\tilde{\nu},\tilde{i},\tilde{n}}$ and x is in the critical part of $T_{\tilde{\nu},\tilde{i},\tilde{n}})\}$.

For $\langle \nu, 1, n\rangle \in \mathcal{G}_s$ we define $C_{\nu,1,n} := \{\langle x, s\rangle \mid s > t \wedge (x$ causes $\nu \in \mathcal{S}_s(C)) \wedge$ (there is no $\langle \tilde{\nu}, \tilde{i}, \tilde{n}\rangle$ with $\tilde{n} < n$ such that $\langle x, s\rangle \in C_{\tilde{\nu},\tilde{i},\tilde{n}}$ and x is in the critical part of $T_{\tilde{\nu},\tilde{i},\tilde{n}})\}$.

The sets $C_{\nu,0,n}$ and $C_{\nu,1,n}$ are recursive because if $\langle x, s\rangle \in C_{\tilde{\nu},\tilde{i},\tilde{n}}$ for some $\tilde{n} < n$ then x is enumerated in $T_{\tilde{\nu},\tilde{i},\tilde{n}}$ at stage s and we can check immediately whether x is in the critical or in the noncritical part of $T_{\tilde{\nu},\tilde{i},\tilde{n}}$.

It is obvious from these definitions that for every $\langle x, s\rangle$ there is at most one $\langle \nu, i, n\rangle \in \mathcal{G}$ such that $\langle x, s\rangle \in C_{\nu,i,n}$ and x is in the critical part of $T_{\nu,i,n}$.

Rule R_3. Suppose x is on track C at the end of stage s. If there is no $\langle \nu, i, n\rangle \in \bigcup\{\mathcal{G}_t \mid t \geqslant s\}$ such that $\langle x, s\rangle \in C_{\nu,i,n}$ and x is in the critical part of $T_{\nu,i,n}$, we put x immediately on track D. Otherwise we consider the unique triple $\langle \nu, i, n\rangle$ with this property. If $i = 0$ we have $\langle e, \sigma', \tau'\rangle := \nu(s, e, x) \geqslant \nu =: \langle e, \sigma, \tau\rangle$. We enumerate x in all $\hat{V}_i$ with $i \in \tau - \tau'$ and then place x on track D. If $i = 1$ we put x immediately on track D.

Rule $\hat{R}_4$. For $\hat{x}$ in pocket $\hat{Q}$ we choose in the case that $e := d(s, \hat{x}) \geqslant 0$ and $\langle e, \sigma, \tau\rangle := \nu(s, e, \hat{x}) \notin \mathfrak{M}_s$ among all states $\langle e, \sigma', \tau\rangle \in \mathfrak{M}_s$ with $\sigma' \supseteq \sigma$ that one for which $\{y \mid y$ causes $\langle e, \sigma', \tau\rangle \in \mathcal{S}(D)\}$ has most recently been verified. We enumerate $\hat{x}$ in $\hat{U}_i$ for every $i \in \sigma' - \sigma$.

The following list $\hat{\mathfrak{M}}$, which will be used in the rules $\hat{R}_3$ and R_4, is slightly different from the previous list $\mathfrak{M}$.

We define for every stage s the set $\hat{\mathfrak{M}}_s$ by induction on s. We set

$$\hat{\mathcal{P}}_s := \left\{\nu \mid \exists \nu' \in \hat{\mathfrak{M}}_s\,(\nu' \leqslant_\sigma \nu)\right\}.$$

Well order $\hat{\mathcal{K}}_t$ in some canonical way. Let $\hat{\mathcal{K}}$ be the concatenation of the sequences $\hat{\mathcal{K}}_t$, $t \in N$.

Rule $\hat{R}_3$. Suppose $\hat{x}$ is on track $\hat{C}$ at the end of stage s. We look for some $t \leq s$ such that an unchecked pair $\langle\langle e, \sigma, \tau\rangle, i\rangle \in \hat{\mathcal{K}}_t$ exists with either $i = 0$ and $\langle e, \sigma, \tau\rangle \geq \nu(s, e, \hat{x}) =: \langle e, \sigma', \tau'\rangle$ and for every $\tilde{e} \leq e$ the set $\{y \mid y$ causes $\tilde{\nu} \in \mathbb{S}(C)$ for some $\tilde{\nu} \geq \langle \tilde{e}, \sigma \cap \{0, \ldots, \tilde{e}\}, \tau' \cap \{0, \ldots, \tilde{e}\}\rangle\}$ has at least t times been verified or $i = 1$ and $\langle e, \sigma, \tau\rangle = \nu(s, e, \hat{x})$. If this does not exist we put $\hat{x}$ immediately on track $\hat{D}$. Otherwise we choose the first pair $\langle\langle e, \sigma, \tau\rangle, i\rangle$ in $\hat{\mathcal{K}}$ with the preceding properties and check it. If $i = 0$ we first enumerate $\hat{x}$ in $\hat{U}_i$ for every $i \in \sigma - \sigma'$ and then put $\hat{x}$ on track $\hat{D}$. If $i = 1$ we place $\hat{x}$ immediately on track $\hat{D}$.

Rule $\hat{R}_2$ is analogous to Rule R_2 and Rule R_4 is analogous to Rule $\hat{R}_4$.

This completes the description of the construction.

One verifies the immediate properties of the movement of elements in the machine as in §3. But now exactly the elements of $\bar{A}$ remain in machine M and the elements of $\hat{N}$ remain in machine $\hat{M}$.

LEMMA 4.1. (a) *For every permanent resident x of Q there exists a unique state ν such that $x = \lim_s q(s, \nu)$. This state ν satisfies $\nu \preccurlyeq \lim_s \nu(s, x, x)$. Further if for any ν $\lim_s q(s, \nu)$ exists and $\nu' \prec \nu$ then $\lim_s q(s, \nu')$ exists as well.*

(b) *The same holds for $\hat{Q}$.*

PROOF. See Lemma 3.1.

LEMMA 4.2. *For every state ν*

(a) *$\{y \mid y$ causes $\nu \in \mathbb{S}(C)\} \cap \bar{A}$ infinite $\Rightarrow \{y \mid y$ causes $\nu \in \mathbb{S}(D)\} \cap \bar{A}$ infinite and*

(b) *$\{\hat{y} \mid \hat{y}$ causes $\nu \in \mathbb{S}(\hat{C})\}$ infinite $\Rightarrow \{\hat{y} \mid \hat{y}$ causes $\nu \in \mathbb{S}(\hat{D})\}$ infinite.*

PROOF. (a) If $\{y \mid y$ causes $\nu \in \mathbb{S}(C)\} \cap \bar{A}$ is infinite then $\{n \mid \langle \nu, 1, n\rangle \in \mathcal{G}\}$ is as well infinite, where $\mathcal{G}$ is the list in Rule R_3. Fix some n with $\langle \nu, 1, n\rangle \in \mathcal{G}$. The critical part of every set $T_{\tilde{\nu}, \tilde{i}, \tilde{n}}$ with $\tilde{n} < n$ contains only finitely many elements of $\bar{A}$. Therefore $x \in T_{\nu,1,n}$ for almost all of the infinitely many elements of $\bar{A}$ that cause $\nu \in \mathbb{S}(C)$. Therefore the critical part of $T_{\nu,1,n}$ contains some $x_0 \in \bar{A}$. Consider a stage s such that $\langle x_0, s\rangle \in C_{\nu,1,n}$. According to Rule R_3 x_0 is placed in state ν on track D at stage $s + 1$. Observe that this x_0 cannot be the same for infinitely many n with $\langle \nu, 1, n\rangle \in \mathcal{G}$ for the following reason. If $\langle \nu, 1, n\rangle \in \mathcal{G}_t$ then $\langle x_0, s\rangle \in C_{\nu,1,n}$ implies that x_0 causes $\nu \in \mathbb{S}_s(C)$ for some $s > t$ and x_0 can do this for only finitely many s.

(b) If $\{\hat{y} \mid \hat{y}$ causes $\nu \in \mathbb{S}(\hat{C})\}$ is infinite, then $\langle \nu, 1\rangle \in \hat{\mathcal{K}}_t$ for infinitely many t. Each of these pairs is checked at a stage $s + 1 > t$ where some $\hat{x}$ is placed on track $\hat{D}$ in state ν according to Rule $\hat{R}_3$.

LEMMA 4.3. (a) *Assume that only finitely many elements of $\bar{A}$ cause $\nu' \in \mathbb{S}(D)$ for some $\nu' >_\tau \nu$. Then there are only finitely many s such that ν is excluded from $\mathfrak{M}_{s+1}$ via Condition* (b).

(b) *The analogous fact holds for $\hat{\mathfrak{M}}$.*

PROOF. (a) Assume the claim is false for state ν. By definition this implies that infinitely many elements of $\bar{A}$ cause $\nu \in \mathbb{S}(D)$. Further it implies that there is some

$\nu_0 \preccurlyeq \nu$ of minimal length such that box B_{ν_0} has no permanent resident. B_{ν_0} subscribes then for infinitely many s to the set $S_{\nu,s}$. By the remark above each of the sets $S_{\nu,s}$ has an infinite intersection with $\bar{A}$. Therefore there are infinitely many elements of $\bar{A}$ which are in the critical part of some set $S_{\nu,s}$ to which B_{ν_0} has subscribed. Almost all of these elements are placed (in state ν) in the box B_{ν_0} because the boxes $B_{\tilde{\nu}}$ with $\tilde{\nu} \prec \nu_0$ subscribe only to finitely many sets (by the minimality of ν_0).

By assumption there is some state $\tilde{\nu}$ such that $\nu_0 \preccurlyeq \tilde{\nu} \preccurlyeq \nu$ and $B_{\tilde{\nu}}$ causes infinitely often the exclusion of ν from $\mathfrak{M}_{s+1}$ via Condition (b). This implies that $\lim_s \hat{q}(s, \nu_0)$ exists. Therefore almost all of the infinitely many elements of $\bar{A}$ which enter box B_{ν_0} in state ν are later placed above hole H because they are enumerated in some new U_e with $e \leqslant |\nu_0|$. All these elements run afterwards in some state $\nu' >_\tau \nu$ over track C. According to Lemma 3.2 this implies that $\{y \mid y \text{ causes } \nu' \in \mathbb{S}(D)\} \cap \bar{A}$ is infinite for some $\nu' >_\tau \nu$, a contradiction.

Part (b) is proved in the same way.

LEMMA 4.4. (a) *Assume that every state ν of length e is excluded from $\mathfrak{M}_{s+1}$ by Condition* (a) *for only finitely many s. Then we have for every ν of length e, $\nu \in \mathcal{P}_s$ for infinitely many $s \Rightarrow \nu \in \mathcal{P}_\omega$.*

(b) *The same holds for $\hat{\mathcal{P}}_\omega$.*

PROOF. (a) Consider some ν of length e such that $\nu \in \mathcal{P}_s$ for infinitely many s. The claim is obvious if there is some $\tilde{\nu} \in \mathfrak{M}_\omega$ with $\tilde{\nu} \geqslant_\tau \nu$. Otherwise there is some $\tilde{\nu} \geqslant_\tau \nu$ which is infinitely often added to $\mathfrak{M}_{s+1}$ and later excluded. This implies that $\{y \mid y$ causes $\tilde{\nu} \in \mathbb{S}(D)\} \cap \bar{A}$ is infinite. Choose $\nu' \geqslant_\tau \nu$ maximal w.r.t. $\geqslant_\tau$ such that $\{y \mid y$ causes $\nu' \in \mathbb{S}(D)\} \cap \bar{A}$ is infinite. By Lemma 4.3 and our assumption we have $\nu' \in \mathfrak{M}_\omega$. Thus $\nu \in \mathcal{P}_\omega$.

(b) is proved in the same way.

LEMMA 4.5. (i) $\{\hat{y} \mid \hat{y}$ *causes* $\nu \in \mathbb{S}(\hat{X})\}$ *infinite for some* $\hat{X} \Rightarrow \nu \in \mathcal{P}_\omega$.

(ii) $\{y \mid y$ *causes* $\nu \in \mathbb{S}(X)\} \cap \bar{A}$ *infinite for some* $X \Rightarrow \nu \in \hat{\mathcal{P}}_\omega$.

PROOF. One proves (i) and (ii) simultaneously by induction on $|\nu|$. Assume (i) and (ii) hold for all ν with $|\nu| < e$. Then every state of length e is only finitely often excluded from $\mathfrak{M}$ or $\hat{\mathfrak{M}}$ by Condition (a). We assume for a contradiction that (i) does not hold for some ν of length e. Fix ν_1 such that for some $\hat{X}$ $\{\hat{y} \mid \hat{y}$ causes $\nu_1 \in \mathbb{S}(\hat{X})\}$ is infinite and $\nu_1 \notin \mathcal{P}_\omega$. Fix such an $\hat{X}$. Because of Lemma 4.3 we have then $\nu_1 \in \mathbb{S}_s(\hat{X}) - \mathcal{P}_s$ for infinitely many s. This implies that $\nu \in \mathfrak{M}_\omega \Rightarrow |\nu| \leqslant e$ (by Condition (a)).

Fix infinitely many different numbers $\hat{y}_j$, $j \in N$, and stages t_j such that for all $j \in N$ $\hat{y}_j$ causes $\nu_1 \in \mathbb{S}_{t_j}(\hat{X}) - \mathcal{P}_{t_j}$. Let $\mathcal{G}_j$ be the finite sequence of states ν such that $\exists s \leqslant t_j$ ($\hat{y}_j$ is in machine $\hat{M}$ at stage s and $\nu(s, e, \hat{y}_j) = \nu$) and let $\mathcal{G}$ be the concatenation of all $\mathcal{G}_j, j \in N$.

CLAIM 1. *If ν occurs infinitely often in $\mathcal{G}$ and $\{y \mid y$ causes $\nu' \in \mathbb{S}(C)$ for some $\nu' \geqslant \nu\} \cap \bar{A}$ is infinite then $\{y \mid y$ causes $\nu' \in \mathbb{S}(D)$ for some $\nu' \geqslant_\tau \nu\} \cap \bar{A}$ is as well infinite and $\nu \in \mathcal{P}_\omega$.*

PROOF OF CLAIM 1. Assume that $\nu = \langle e, \sigma, \tau \rangle$ occurs infinitely often in $\mathcal{G}$ and for $\nu_0 = \langle e, \sigma_0, \tau_0 \rangle \geqslant \nu$ the set $\{y \mid y$ causes $\nu_0 \in \mathbb{S}(C)\} \cap \overline{A}$ is infinite. Then $S := \{n \mid \langle \nu, 0, n \rangle \in \mathcal{G}\}$ is infinite. Fix some $n \in S$. The critical part of every set $T_{\tilde{\nu}, \tilde{i}, \tilde{n}}$ with $\tilde{n} < n$ contains only finitely many elements of $\overline{A}$. Therefore almost all of the infinitely many elements of $\{y \mid y$ causes $\nu_0 \in \mathbb{S}(C)\} \cap \overline{A}$ are enumerated into $T_{\nu,0,n}$. Thus the critical part of $T_{\nu,0,n}$ contains some $x_n \in \overline{A}$. According to Rule R_3 this x_n is placed on track D in some state $\langle e, \sigma', \tau \rangle$ with $\sigma' \supseteq \sigma$. Only finitely many $n \in S$ can have the same x_n, because every element comes only finitely often over track C. Therefore there exists some $\nu' \geqslant_\tau \nu$ such that $\{y \mid y$ causes $\nu' \in \mathbb{S}(D)\} \cap \overline{A}$ is infinite. Choose ν' with this property maximal w.r.t. $\geqslant_\tau$. Then $\nu' \in \mathfrak{M}_\omega$ by Lemma 4.3 and the induction hypothesis. Thus $\nu \in \mathcal{P}_\omega$.

CLAIM 2. *If ν occurs infinitely often in $\mathcal{G}$ then $\{y \mid y$ causes $\nu' \in \mathbb{S}(C)$ for some $\nu' \geqslant \nu\} \cap \overline{A}$ is infinite.*

PROOF OF CLAIM 2. By contradiction. Fix $\nu_2 = \langle e, \sigma_2, \tau_2 \rangle$ so that σ_2 is minimal and τ_2 is minimal for σ_2 such that the claim fails for ν_2. There is some σ such that infinitely many elements of $\overline{A}$ cause $\langle e, \sigma, \varnothing \rangle \in \mathbb{S}(C)$. Therefore $\sigma_2 \neq \varnothing$. Fix some infinite set $J \subseteq N$, a state $\nu_3 = \langle e, \sigma_3, \tau_3 \rangle$ and stages $s_j \leqslant t_j$ for $j \in J$ such that for every $j \in J$ $\nu(s_j - 1, e, \hat{y}_j) = \langle e, \sigma_3, \tau_3 \rangle \neq \langle e, \sigma_2, \tau_2 \rangle = \nu(s_j, e, \hat{y}_j)$. Assume first that $\sigma_3 = \sigma_2$ and $\tau_3 \subsetneq \tau_2$. Since $\langle e, \sigma_3, \tau_3 \rangle$ occurs infinitely often in $\mathcal{G}$ the set $\{y \mid y$ causes $\nu' \in \mathbb{S}(C)$ for some $\nu' \geqslant \langle e, \sigma_3, \tau_3 \rangle\} \cap \overline{A}$ is infinite by the minimal choice of τ_2. But this implies the claim for ν_2 since $\langle e, \sigma_3, \tau_3 \rangle \geqslant \nu_2$. Thus we can assume that $\sigma_3 \subsetneq \sigma_2$. Then there is an infinite set $J' \subseteq J$ such that (Case 1) for every $j \in J'$ Rule $\hat{R}_3$ is applies to $\hat{y}_j$ at stage s_j or (Case 2) for every $j \in J'$ Rule $\hat{R}_4$ is applied to $\hat{y}_j$ at stage s_j.

Case 1. Because of the induction hypothesis only finitely many pairs $\langle \nu, 0 \rangle$ with $|\nu| < e$ occur in the list $\hat{\mathcal{K}}$ for Rule $\hat{R}_3$. Therefore for almost all $j \in J'$ one checks at stage s_j a pair $\langle \langle \tilde{e}, \tilde{\sigma}, \tilde{\tau} \rangle, 0 \rangle$ in $\hat{\mathcal{K}}$ with $\tilde{e} \geqslant e$. If this pair comes from $\hat{\mathcal{K}}_t$ then $\{y \mid y$ causes $\tilde{\nu} \in \mathbb{S}(C)$ for some $\tilde{\nu} \geqslant \langle e, \sigma_2, \tau_2 \rangle\}$ is verified t times according to Rule $\hat{R}_3$. Altogether this set is verified infinitely often and the claim holds for ν_2, a contradiction.

Case 2. If ν is a state such that for infinitely many $j \in J'$ $\hat{y}_j$ is in $\hat{M}$ at stage r_j and $\nu(r_j, e, \hat{y}_j) = \nu$ for some $r_j \leqslant s_j - 1$ then ν occurs infinitely often in $\mathcal{G}$ and Claim 2 holds for ν by our minimal choice of σ_2. We get then from Claim 1 that $\nu \in \mathcal{P}_\omega$. This implies that we have for almost all $j \in J'$ $\forall s \leqslant s_j - 1$ ($\hat{y}_j$ is in machine $\hat{M}$ at stage $s \Rightarrow \nu(s, e, \hat{y}_j) \in \mathcal{P}_s$). Therefore $d(s_j - 1, \hat{y}_j) \geqslant e$ for almost all $j \in J'$.

In the case that $d(s_j - 1, \hat{y}_j) > e$ for infinitely many $j \in J'$, there is for each of these j some $\nu(j) \succ \langle e, \sigma_2, \tau_2 \rangle$ in $\mathfrak{M}_{s_j - 1}$. None of these $\nu(j)$ is in $\mathfrak{M}_\omega$ as we have shown before Claim 1. Therefore infinitely often some state $\nu \succcurlyeq \langle e, \sigma_2, \tau_2 \rangle$ is added to $\mathfrak{M}_{s+1}$. By the definition of $\mathfrak{M}_{s+1}$ $\{y \mid y$ causes $\langle e, \sigma_2, \tau_2 \rangle \in \mathbb{S}(D)\}$ is then verified infinitely often. Thus $\{y \mid y$ causes $\nu' \in \mathbb{S}(C)$ for some $\nu' \geqslant \langle e, \sigma_2, \tau_2 \rangle\} \cap \overline{A}$ is infinite, a contradiction.

The case remains where $d(s_j - 1, \hat{y}_j) = e$ for almost all $j \in J'$. Since Claim 2 holds for $\langle e, \sigma_3, \tau_3 \rangle$ we get from Claim 1 that $\{y \mid y$ causes $\tilde{\nu} \in \mathbb{S}(D)$ for some $\tilde{\nu} \geqslant_\tau \langle e, \sigma_3, \tau_3 \rangle\} \cap \overline{A}$ is infinite. We show that then $\{y \mid y$ causes $\nu_2 \in \mathbb{S}(D)\} \cap \overline{A}$ is as well infinite, which implies that Claim 2 holds for ν_2—a contradiction. Consider some $\tilde{\nu} \geqslant_\tau \langle e, \sigma_3, \tau_3\rangle$ which is maximal w.r.t. $\geqslant_\tau$ such that $\{y \mid y$ causes $\tilde{\nu} \in \mathcal{S}(D)\}$ is verified infinitely often. As before we have $\tilde{\nu} \in \mathfrak{M}_\omega$. But then $\{y \mid y$ causes $\nu_2 \in \mathcal{S}(D)\}$ is as well verified infinitely often because otherwise $\nu_2 \neq \tilde{\nu}$ and in the considered applications of $\hat{R}_4$ we would almost always raise $\hat{x}$ to some state different from ν_2.

We can now finish part (i) of the induction step. The state ν_1 which was fixed before Claim 1 occurs infinitely often in $\mathcal{G}$. Therefore Claim 1 and Claim 2 together imply that $\nu_1 \in \mathcal{P}_\omega$, a contradiction.

Assume now for a contradiction that (ii) does not hold for some ν of length e. Fix ν_1 such that for some X $\{y \mid y$ causes $\nu_1 \in \mathcal{S}(X)\} \cap \bar{A}$ is infinite and $\nu_1 \notin \hat{\mathcal{P}}_\omega$. Fix such an X. By Lemma 4.4 $\{y \mid y$ causes $\nu_1 \in \mathcal{S}_s(X) - \hat{\mathcal{P}}_s$ for some $s\} \cap \bar{A}$ is as well infinite. Thus $\nu \in \hat{\mathfrak{M}}_\omega \Rightarrow |\nu| \leqslant e$.

Fix infinitely many different numbers $y_j \in \bar{A}$, $j \in N$, and stages t_j such that for every $j \in N$ y_j causes $\nu_1 \in \mathcal{S}_{t_j}(X) - \hat{\mathcal{P}}_{t_j}$. Let $\mathcal{G}_j$ be the finite sequence of states ν such that $\exists s \leqslant t_j$ (y_j is in machine M at stage s and $\nu(s, e, y_j) = \nu$) and let $\mathcal{G}$ be the concatenation of all $\mathcal{G}_j$, $j \in N$.

Claim 1′. *If ν occurs infinitely often in $\mathcal{G}$ and $\{\hat{y} \mid \hat{y}$ causes $\nu' \in \mathcal{S}(\hat{C})$ for some $\nu' \leqslant \nu\}$ is infinite then $\{\hat{y} \mid \hat{y}$ causes $\nu' \in \mathcal{S}(\hat{D})$ for some $\nu' \leqslant_\sigma \nu\}$ is as well infinite and $\nu \in \hat{\mathcal{P}}_\omega$.*

Proof of Claim 1′. If $\nu = \langle e, \sigma, \tau\rangle$ occurs infinitely often in $\mathcal{G}$ then $\langle \nu, 0\rangle$ occurs infinitely often in $\hat{\mathcal{K}}$. Fix some $\nu' = \langle e, \sigma', \tau'\rangle \leqslant \nu$ such that $\{\hat{y} \mid \hat{y}$ causes $\nu' \in \mathcal{S}(\hat{C})\}$ is infinite. Because of Rule $\hat{R}_3$ it is enough to show that $\{y \mid y$ causes $\tilde{\nu} \in \mathcal{S}(C)$ for some $\tilde{\nu} \geqslant \langle e, \sigma, \tau'\rangle\} \cap \bar{A}$ is infinite. But every y_j that causes an occurrence of ν in $\mathcal{G}$ at some stage $r_j \leqslant t_j$ is either above hole H at the end of stage r_j and runs afterwards in state ν over track C or is not above hole H at the end of stage r_j and did at some stage $s \leqslant r_j$ run over C in some state $\tilde{\nu} \geqslant_\sigma \nu$. In any case y_j causes $\tilde{\nu} \in \mathcal{S}(C)$ for some $\tilde{\nu} \geqslant \langle e, \sigma, \tau'\rangle$. Thus $\{y \mid y$ causes $\nu' \in \mathcal{S}(\hat{D})$ for some $\nu' \leqslant_\sigma \nu\}$ is infinite because of $\hat{R}_3$. This implies $\nu \in \hat{\mathcal{P}}_\omega$ analogously as in Claim 1.

*Claim*2′. *If ν occurs infinitely often in $\mathcal{G}$ then infinitely often some $\nu' \leqslant \nu$ occurs in $\mathcal{S}(\hat{C})$.*

Proof of Claim 2′. By contradiction. Fix $\nu_2 = \langle e, \sigma_2, \tau_2\rangle$ so that τ_2 is minimal and σ_2 is minimal for τ_2 such that the claim fails for ν_2. There is some τ such that $\langle e, \varnothing, \tau\rangle$ occurs infinitely often in $\mathcal{S}(\hat{C})$. Therefore $\tau_2 \neq \varnothing$. Fix some infinite set $J \subseteq N$, a state $\nu_3 = \langle e, \sigma_3, \tau_3\rangle$ and stages $s_j \leqslant t_j$ for $j \in J$ such that for every $j \in J$ $\nu(s_j - 1, e, y_j) = \langle e, \sigma_3, \tau_3\rangle \neq \langle e, \sigma_2, \tau_2\rangle = \nu(s_j, e, y_j)$. By the minimal choice of σ_2 we have analogously as before $\tau_3 \subsetneq \tau_2$. Then there is an infinite set $J' \subseteq J$ such that either (Case 1) for every $j \in J'$ Rule R_3 is applied to y_j at stage s_j or (Case 2) for every $j \in J'$ Rule R_4 is applied to y_j at stage s_j.

Case 1. By the induction hypothesis only finitely many triples $\langle \nu, 0, n\rangle$ with $|\nu| < e$ occur in the list $\mathcal{G}$. The critical part of every set contains only finitely many elements of $\bar{A}$ and $y_j \in \bar{A}$ for every $j \in J'$. Thus for almost all $j \in J'$ there is some $\langle \nu, 0, n\rangle \in \mathcal{G}$ such that $\langle y_j, s_j - 1\rangle \in C_{\nu,0,n}$ and y_j is in the critical part of $T_{\nu,0,n}$

with ν of the form $\nu = \langle e', \sigma, \tau \rangle$ such that $e' \geqslant e$, $\nu(s_j - 1, e, y_j) = \langle e, \sigma_3, \tau_3 \rangle \geqslant \langle e, \sigma \cap \{0, \ldots, e\}, \tau \cap \{0, \ldots, e\} \rangle$, $\tau_2 = \tau \cap \{0, \ldots, e\}$ and therefore $\langle e, \sigma_2, \tau_2 \rangle \geqslant \langle e, \sigma \cap \{0, \ldots, e\}, \tau \cap \{0, \ldots, e\} \rangle$.

Since the critical part of every set contains only finitely many elements of $\bar{A}$ the set of these triples $\langle \nu, 0, n \rangle$ in $\mathfrak{G}$ is infinite. Therefore there is some $\nu' \leqslant \langle e, \sigma_2, \tau_2 \rangle$ such that there are infinitely many numbers $\hat{x}_i$ and stages r_i, $i \in N$, such that for every $i \in N$ $\nu(r_i, e, \hat{x}_i) = \nu'$. If $\hat{x}_i$ is at the end of stage r_i not yet in $\hat{M}$ or it sits above hole $\hat{H}$, then $\hat{x}_i$ runs after stage r_i in some state $\nu'' \leqslant \nu'$ over track $\hat{C}$. Otherwise $\hat{x}_i$ did already at some stage $s \leqslant r_i$ run over track $\hat{C}$ in some state $\nu'' \leqslant \nu'$. Therefore some $\nu'' \leqslant \langle e, \sigma_2, \tau_2 \rangle$ occurs infinitely often in $\mathfrak{S}(\hat{C})$, a contradiction.

Case 2. This case is treated in the same way as in the proof of Claim 2.

Since ν_1 occurs infinitely often in $\mathfrak{G}$ we get the contradiction $\nu_1 \in \hat{\mathfrak{P}}_\omega$ from Claims 1′ and 2′.

LEMMA 4.6. *For every $e \in N$ there are only finitely many $\hat{x} \in \hat{N}$ with $d(\hat{x}) := \lim_s d(s, \hat{x}) < e$. Further if $\hat{x}$ remains permanently in pocket $\hat{Q}$ and $d(\hat{x}) \geqslant 0$ then* $\lim_s \nu(s, d(\hat{x}), \hat{x}) \in \mathfrak{M}_\omega$.

PROOF. See Lemma 3.6.

LEMMA 4.7. *If infinitely many elements remain finally in pocket $\hat{Q}$ (Q) in final state ν, then infinitely many elements remain finally in pocket P $(\hat{P})$ in final state ν.*

PROOF. See Lemma 3.7.

LEMMA 4.8. *If infinitely many elements remain finally in pocket P $(\hat{P})$ in final state ν, then infinitely many elements remain finally in pocket $\hat{Q}$ (Q) in final state ν.*

PROOF. See Lemma 3.8.

PROOF OF DIRECTION "$\Leftarrow$" OF THEOREM 1.2. We have constructed a simultaneous enumeration of r.e. sets $(U_e)_{e \in N}$, $(\hat{V}_e)_{e \in N}$, $(\hat{U}_e)_{e \in N}$, $(V_e)_{e \in N}$. It is obvious from the construction that $U_e \subseteq W_e$ and $V_e \subseteq W_e$. Further, besides numbers less than e only permanent residents of boxes B_ν $(\hat{B}_\nu)$ with $|\nu| < e$ can be in the difference $W_e \cap \bar{A} - U_e$ $(W_e - V_e)$. Thus since every box has only finitely many permanent residents we have $W_e \cap \bar{A} =^* U_e \cap \bar{A}$ and $W_e =^* V_e$ for every $e \in N$.

Every number $x \in \bar{A}$ remains finally in pocket P or Q in machine M and every number $\hat{x} \in \hat{N}$ remains finally in pocket $\hat{P}$ or $\hat{Q}$ in machine $\hat{M}$. Therefore Lemmas 4.7 and 4.8 together imply that for every ν infinitely many $x \in \bar{A}$ have final state ν if and only if infinitely many $\hat{x} \in \hat{N}$ have final state ν. According to Soare **[6]** one can then easily define a function $p := \bigcup_{n \in N} p_n$ which maps $\bar{A}$ one-one onto $\hat{N}$ and which induces an isomorphism between $\mathfrak{E}(\bar{A})$ and $\mathfrak{E}$ as well as between $\mathfrak{E}^*(\bar{A})$ and $\mathfrak{E}^*$. Every p_n is a finite function from a subset of $\bar{A}$ into $\hat{N}$.

Set $p_0 := \varnothing$. For n odd set

$$x_n := \mu x \in \bar{A} \, (x \notin \operatorname{domain} p_{n-1}),$$

$$n := \max\Big(\{-1\} \cup \Big\{ e \geqslant 0 \,|\, \exists \hat{x} \notin \operatorname{range} p_{n-1} \Big(\lim_s \nu(s, e, x_n) = \lim_s \nu(s, e, \hat{x}) \Big) \Big\} \Big)$$

and

$$\hat{x}_n := \mu\hat{x}(e_n = -1 \wedge \hat{x} \notin \operatorname{range} p_{n-1}) \\ \vee \left(e_n \geqslant 0 \wedge \hat{x} \notin \operatorname{range} p_{n-1} \wedge \lim_s \nu(s, e_n, x_n) = \lim_s \nu(s, e_n, \hat{x}_n)\right)\Big).$$

Define $p_n := p_{n-1} \cup \{\langle x_n, \hat{x}_n \rangle\}$. For n even we start with some $\hat{x}_n \in \hat{N}$ and associate analogously some $x_n \in \bar{A}$.

We get from the preceding that for every $e \in N$ $p[W_e \cap \bar{A}] =^* p[U_e \cap \bar{A}] =^* \hat{U}_e$ and $p^{-1}[W_e] =^* p^{-1}[V_e] =^* \hat{V}_e \cap \bar{A}$. Therefore $p[W_e \cap \bar{A}] \in \mathcal{E}$ and $p^{-1}[W_e] \in \mathcal{E}(\bar{A})$ for every $e \in N$. Thus we can define a map Ψ: $\mathcal{E}(\bar{A}) \to \mathcal{E}$ by $\Psi(W_e \cap \bar{A}) := p[W_e \cap \bar{A}]$, and a map Φ: $\mathcal{E}^*(\bar{A}) \to \mathcal{E}^*$ by $\Phi((W_e \cap \bar{A})^*) := (p[W_e \cap \bar{A}])^*$. By the preceding both maps are isomorphisms.

For every $e \in N$ let $f(e)$ ($g(e)$) be a canonically chosen index for $\hat{U}_e$ ($\hat{V}_e$). Then f and g are total recursive functions which witness that Φ is an effective isomorphism from $\mathcal{E}^*(\bar{A})$ onto $\mathcal{E}^*$.

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