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Characterization of Ricci Almost Soliton on Lorentzian Manifolds

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Abstract: Ricci solitons (*RS*) have an extensive background in modern physics and are extensively used in cosmology and general relativity. The focus of this work is to investigate Ricci almost solitons (*RAS*) on Lorentzian manifolds with a special metric connection called a semi-symmetric metric **u**-connection ($\mathcal{SSM}$-connection). First, we show that any quasi-Einstein Lorentzian manifold having a $\mathcal{SSM}$-connection, whose metric is *RS*, is Einstein manifold. A similar conclusion also holds for a Lorentzian manifold with $\mathcal{SSM}$-connection admitting *RS* whose soliton vector Z is parallel to the vector $\mathbf{u}$. Finally, we examine the gradient Ricci almost soliton (*GRAS*) on Lorentzian manifold admitting $\mathcal{SSM}$-connection.

Keywords: Lorentzian manifolds; symmetric spaces; semi-symmetric metric connection; Ricci soliton; gradient Ricci almost soliton

MSC: 53C25; 53C20; 53C21; 53C65

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1. Introduction

Let M be an n-dimensional pseudo-Riemannian manifold. A vanishing of torsion tensor $\widehat{\tau}$ associated to a linear connection $\widehat{D}$ on M, that is, $\widehat{\tau}(X_1, X_2) = \widehat{D}_{X_1}X_2 - \widehat{D}_{X_2}X_1 - [X_1, X_2] = 0$, then $\widehat{D}$ is called a symmetric connection. If it is not, then it is called non-symmetric connection. Many geometers classify the linear connection $\widehat{D}$ into various classes based on the different forms, for instance, a semi-symmetric ($\mathcal{SS}$) if the following condition holds:

$$\widehat{\tau}(X_1, X_2) = u^\sharp(X_2)X_1 - u^\sharp(X_1)X_2, \quad \forall\ X_1, X_2 \in \Gamma(TM) \tag{1}$$

where the one-form $u^\sharp$ and the associated vector field $\mathbf{u}$ are connected through a pseudo-Riemannian metric g by

$$u^\sharp(X_1) = g(X_1, \mathbf{u}). \tag{2}$$

If we replace X_1 by φX_1 and X_2 by φX_2, where φ is a $(1,1)$ tensor field, in the RHS of Equation (1),then $\widehat{D}$ becomes a quarter-symmetric connection. If $\widehat{D}g = 0$, then the connection $\widehat{D}$ is known as a metric connection, otherwise, it is non-metric [1]. A linear connection is symmetric and metric if and only if it is the Levi–Civita connection. Hayden's [2] introduced a metric connection $\widehat{D}$ with a non-vanishing torsion on a Riemannian manifold, which was later renamed as a Hayden connection. After Pak [3] proved that it is a $\mathcal{SSM}$ connection, many questions remain about it. Yano [4] started studying a Riemannian manifold with $\mathcal{SSM}$ connection $\widehat{D}$ and shown that it is conformally flat when curvature tensor vanish. Recently, Chaubey et al. [5] initiate to study of the concept of $\mathcal{SSM}$ P-connection on a Riemannian manifold, including its geometric properties. Further, this idea has been examined in [6]. After that, this notion was introduced on a Lorentzian manifold by Chaubey et al. [7] and studied its geometrical and physical properties under some classification.

Hamilton [8] introduced the concept of *Ricci soliton (RS)* in his seminal work, which is a generalization of the Einstein metric. A pseudo-Riemannian manifold (M, g) is said to be *RS* if there is a smooth vector field Z on M satisfying

$$\frac{1}{2}\mathcal{L}_Z g + Ric_g = \lambda g \tag{3}$$

where $\mathcal{L}$ indicates the Lie derivative operator, Ric_g is Ricci tensor and λ is a real number ($\lambda = \frac{1}{n}(divZ + s)$, where $n = dimM$ and s indicates the scalar curvature of (M, g)). If $\lambda < 0$, the *RS* is referred to as expanding. Conversely, if $\lambda > 0$, it is referred to as shrinking. In the case $\lambda = 0$, we obtain a steady *RS*. A Ricci soliton (*RS*) is known as a *Gradient Ricci soliton (GRS)* if there is a potential function h that satisfies $Z = \text{grad}h$. In such a case Equation (3) becomes

$$Hes_h + Ric_g = \lambda g, \tag{4}$$

where Hes_h denotes the Hessian of h. The mathematical community has taken a great interest in the work of Pigola et al. [9] as they have expanded the concept of Ricci solitons by adding the condition on λ in Equation (3) to be a smooth function on M. In this setting, we refer to Equations (3) and (4) as being the fundamental equations of a *Ricci almost soliton (RAS)* and *gradient Ricci almost soliton (GRAS)*, respectively. The (gradient) *RAS* structure links geometric details regarding the curvature through Ricci tensor and the geometry of the potential function level sets by means of their second fundamental form. Therefore, it is a natural problem to examine the (gradient) *RAS* structure under certain curvature conditions. Many geometers have examined different examples, rigidity outcomes, and characterizations related to (gradient) *RS* structure; for instance, Hamilton [10] and Ivey [11] obtained several classification outcomes for compact case. Further, in [12], certain results were established for the solitons of the Ricci flow on contact Riemannian manifolds. Batat et al. [13] examined the existence of locally conformally flat Lorentzian steady *GRS* structure of non Bryant type. Barros and Ribeiro [14] discussed the structure equations for *RAS* structure. As a result of these equations, they proved that if a compact non-trivial *RAS* with constant scalar curvature or a conformal associated vector field, then it is isometric to a sphere. In [15], the authors examined *GRS* structure on locally conformally flat Lorentzian manifolds by focusing on their local structure. Recently, Chaubey et al. [16] characterized the *RS* structure on Lorentzian manifolds having a semi-symmetric non-metric **u**-connection with $Z = \mathbf{u}$. Some authors presented some crucial results related to the special submanifolds in different spaces [17–29]. We can find more motivations of our paper from several articles (see [30–45]). In this work, the *RS* and *GRAS* structure on a Lorentzian manifold with a semi-symmetric metric **u**-connection are characterized and their geometrical and physical properties are studied, inspired by the studies mentioned above.

2. Lorentzian Manifolds Admitting Semi-Symmetric Metric u-Connection

The Lorentzian manifold is a significant sub-class of pseudo-Riemannian manifolds with a crucial role in mathematical physics, particularly in the theory of general relativity and cosmology. A *Lorentzian manifold* is a doublet of a smooth connected para-compact Hausdorff manifold M and Lorentzian metric g, which is a symmetric tensor of type (0, 2) that is non-degenerate and has signature $(-, +, +, \cdots, +)$ for each point $p \in M$. A non-zero vector field $\mathbf{u} \in T_pM$ is called timelike (resp. null, and spacelike) if $g_p(\mathbf{u}, \mathbf{u}) < 0$ (resp. $=, > 0$).

The Levi–Civita connection D that corresponds to metric g on M defines a linear connection $\widehat{D}$ on M and is given by

$$\widehat{D}_{X_1}X_2 = D_{X_1}X_2 + u^{\sharp}(X_2)X_1 - g(X_1, X_2)\mathbf{u}, \ \forall \ X_1, X_2 \in \Gamma(TM). \tag{5}$$

Mishra et al. [46] and Chaubey et al. [47], studied an almost contact metric manifold and characterized the case where $\mathbf{u} = \xi$ and $\widehat{D}\xi = 0$, which led to several significant geometrical results. Later this notion was extended by Chaubey et al. [5,7] on both Riemannian and Lorentzian manifolds, called *semi-symmetric metric* **u**-*connection* (shortly, $\mathcal{SSM}$ **u**-connection).

Consider $\widehat{D}\mathbf{u} = 0$, so that by virtue of Equation (5) one can obtain

$$D_{X_1}\mathbf{u} = X_1 + u^\sharp(X_1)\mathbf{u}, \tag{6}$$

where the associated vector field **u** defined by Equation (2) is a unit timelike vector field, that is, $g(\mathbf{u},\mathbf{u}) = u^\sharp(\mathbf{u}) = -1$. Utilizing this together with Equations (2) and (5), one can easily obtain

$$(\widehat{D}_{X_1}u^\sharp)(X_2) = (D_{X_1}u^\sharp)(X_2) - g(X_1,X_2) - u^\sharp(X_1)u^\sharp(X_2).$$

Since $(\widehat{D}_{X_1}u^\sharp)(X_2) = 0$, thus from the above relation we infer

$$(D_{X_1}u^\sharp)(X_2) = g(X_1,X_2) + u^\sharp(X_1)u^\sharp(X_2). \tag{7}$$

The restricted curvature with respect to the Levi–Civita connection D is stated as follows:

Lemma 1 (see [7])**.** *A Lorentzian manifold of dimension n admitting $\mathcal{SSM}$ **u**-connection $\widehat{D}$ satisfies*

$$R(X_1,X_2)\mathbf{u} = u^\sharp(X_2)X_1 - u^\sharp(X_1)X_2, \tag{8}$$
$$R(\mathbf{u},X_1)X_2 = g(X_1,X_2)\mathbf{u} - u^\sharp(X_2)X_1, \tag{9}$$
$$g(R(X_1,X_2)X_3,\mathbf{u}) = u^\sharp(X_1)g(X_2,X_3) - u^\sharp(X_2)g(X_1,X_3), \tag{10}$$

for $X_1, X_2, X_3 \in \Gamma(TM)$.

First, we prove the following lemmas, which we used to prove our main results:

Lemma 2. *An n-dimensional Lorentzian manifold M with $\mathcal{SSM}$ **u**-connection $\widehat{D}$ satisfies*

$$(D_{X_1}Q)\mathbf{u} = (n-1)X_1 - QX_1, \tag{11}$$
$$(D_{\mathbf{u}}Q)X_1 = -2QX_1 + 2(n-1)X_1, \tag{12}$$

where Q indicates the Ricci operator and is defined by $Ric_g(X_1,X_2) = g(QX_1,X_2)$.

Proof. Contraction of Equation (8) leads to

$$Ric_g(X_1,\mathbf{u}) = (n-1)u^\sharp(X_1), \tag{13}$$

which provides $Q\mathbf{u} = (n-1)\mathbf{u}$. Differentiating this along X_1 and adopting Equation (6) yields Equation (11). Now, differentiating Equation (8) along X_3, calling back Equation (7) entails

$$(D_{X_3}R)(X_1,X_2)\mathbf{u} = -R(X_1,X_2)X_3 + g(X_2,X_3)X_1 - g(X_1,X_3)X_2.$$

Consider $\{E_i\}_{i=1}^n$ as a local orthonormal basis on M. Replacing $X_1 = X_3 = E_i$ in the above equation and then summing over i leads to

$$\sum_{i=1}^{n} g((D_{E_i}R)(E_i,X_2)\mathbf{u},X_3) = Ric_g(X_2,X_3) - (n-1)g(X_2,X_3). \tag{14}$$

Applying the second Bianchi identity yields

$$\sum_{i=1}^{n} g(D_{E_i}R)(X_3, \mathbf{u})X_2, E_i) = g((D_{X_3}Q)\mathbf{u}, X_2) - g((D_{\mathbf{u}}Q)X_3, X_2).$$

The above equation together with Equation (14) provides

$$g((D_{\mathbf{u}}Q)X_3, X_2) = -2\{Ric_g(X_2, X_3) - (n-1)g(X_2, X_3)\},$$

which yields Equation (12). □

Lemma 3. *An n-dimensional Lorentzian manifold with $\mathcal{SSM}$ $\mathbf{u}$-connection $\widehat{D}$ satisfies*

$$\mathbf{u}(s) = 2\{n(n-1) - s\}.$$

Proof. Taking g-trace of Equation (11) gives the desired identity. □

Lemma 4. *The scalar curvature s of a Lorentzian manifold M with $\mathcal{SSM}$ $\mathbf{u}$-connection $\widehat{D}$ satisfies*

$$grads = -\mathbf{u}(s)\mathbf{u}. \tag{15}$$

Proof. Using Lemma 3, we may write $\mathcal{L}_{\mathbf{u}}s = 2\{n(n-1) - s\}$. After applying the exterior derivative d and taking note of the fact $\mathcal{L}_{\mathbf{u}}$ commutes with d to this equation, entails $\mathcal{L}_{\mathbf{u}}ds = -2ds$. This can also be expressed in terms of the gradient operator as $\mathcal{L}_{\mathbf{u}}\mathrm{grad}s = -2\mathrm{grad}s$. Employ Equation (6) to obtain

$$D_{\mathbf{u}}\mathrm{grad}s = -\mathrm{grad}s + \mathbf{u}(s)\mathbf{u}. \tag{16}$$

Further, it is worth mentioning that $\mathbf{u}(s) = g(\mathbf{u}, \mathrm{grad}s) = 2\{n(n-1) - s\}$. Differentiating this along X_1 and employing Equation (6) gives

$$g(D_{X_1}\mathrm{grad}s, \mathbf{u}) = -3X_1(s) - \mathbf{u}(s)u^\sharp(X_1).$$

For a smooth function ν, it is well known that $g(D_{X_1}\mathrm{grad}\nu, X_2) = g(D_{X_2}\mathrm{grad}\nu, X_1)$. By virtue of this fact, the above equation can exhibit as

$$D_{\mathbf{u}}\mathrm{grad}s = -3grads - \mathbf{u}(s)\mathbf{u}.$$

Employing the foregoing equation with Equation (16) gives the desired result. □

If a Lorentzian manifold M of dimension n with non-vanishing Ricci tensor Ric_g satisfy

$$Ric_g = \alpha g + \beta u^\sharp \otimes u^\sharp, \tag{17}$$

for smooth functions α and β, where $u^\sharp$ is a non-zero one-form, and the vector field corresponding to the one-form $u^\sharp$ is a unit timelike vector field, then it is referred to as a perfect fluid space–time. However, some geometers are calling M quasi-Einstein [48]. Particularly, if $\beta = 0$ and $\alpha = constant$, then M is called Einstein.

Lemma 5. *A Lorentzian manifold of dimension n admitting $\mathcal{SSM}$ $\mathbf{u}$-connection $\widehat{D}$ is a quasi-Einstein if and only if the Ricci tensor satisfy*

$$Ric_g = \left(\frac{s}{n-1} - 1\right)g + \left(\frac{s}{n-1} - n\right)u^\sharp \otimes u^\sharp. \tag{18}$$

Proof. We know that M is quasi-Einstein; g-trace of Equation (17) shows that the scalar curvature s takes the form

$$s = n\alpha - \beta. \tag{19}$$

In addition, in light of Equation (13) in Equation (17) we see that $\alpha - \beta = n - 1$. Combining this with the above equation gives $\alpha = \left(\frac{s}{n-1} - 1\right)$ and $\beta = \left(\frac{s}{n-1} - n\right)$. Therefore, the Equation (17) can be written as Equation (18). □

3. Ricci Solitons on Lorentzian Manifolds with $\mathcal{SSM}$ u-Connection

In this section, we analyze the geometric properties of RS on Lorentzian manifold carrying $\mathcal{SSM}$ **u**-connection, showing the following results.

Lemma 6. *If a Lorentzian manifold M with $\mathcal{SSM}$ **u**-connection $\widehat{D}$ has a RS structure (g, Z), then the soliton is shrinking and Ricci tensor satisfy*

$$(\mathcal{L}_Z Ric_g)(X_1, \mathbf{u}) = 0. \tag{20}$$

Proof. Taking covariant derivative of Equation (3) along X_3, we obtain

$$(D_{X_3}\mathcal{L}_Z g)(X_1, X_2) = -2(D_{X_3}Ric_g)(X_1, X_2). \tag{21}$$

By utilizing the symmetry of $\mathcal{L}_Z D$ in the commutation formula (see Yano [49]):

$$\begin{aligned}(\mathcal{L}_Z D_{X_1} g - D_{X_1}\mathcal{L}_Z g - D_{[V,X_1]} g)(X_2, X_3) = &-g((\mathcal{L}_Z D)(X_1, X_2), X_3)\\ &-g((\mathcal{L}_Z D)(X_1, X_3), X_2),\end{aligned}$$

and through a simple computation, we derive

$$\begin{aligned}2g((\mathcal{L}_Z D)(X_1, X_2), X_3) = \ &(D_{X_1}\mathcal{L}_Z g)(X_2, X_3) + (D_{X_2}\mathcal{L}_Z g)(X_3, X_1)\\ &-(D_{X_3}\mathcal{L}_Z g)(X_1, X_2).\end{aligned} \tag{22}$$

Utilizing Equation (21) in the expression Equation (22), we have

$$\begin{aligned}g((\mathcal{L}_Z D)(X_1, X_2), X_3) = \ &(D_{X_3}Ric_g)(X_1, X_2) - (D_{X_1}Ric_g)(X_2, X_3)\\ &-(D_{X_2}Ric_g)(X_3, X_1).\end{aligned}$$

Switching X_2 by $\mathbf{u}$ in the proceeding equation, calling back Equations (11) and (12) leads to

$$(\mathcal{L}_Z D)(X_1, \mathbf{u}) = 2\{QX_1 - (n-1)X_1\}. \tag{23}$$

Differentiate Equation (23) along X_2 and make use of Equation (6) in order to obtain

$$(D_{X_2}\mathcal{L}_Z D)(X_1, \mathbf{u}) = -(\mathcal{L}_Z D)(X_1, X_2) - 2u^\sharp(X_2)QX_1 + 2(n-1)u^\sharp(X_2)X_1 + 2(D_{X_2}Q)X_1.$$

Employing this in the following identity (see [49]):

$$(\mathcal{L}_Z R)(X_1, X_2)X_3 = (D_{X_1}\mathcal{L}_Z D)(X_2, X_3) + (D_{X_2}\mathcal{L}_Z D)(X_1, X_3),$$

we achieve

$$\begin{aligned}(\mathcal{L}_Z R)(X_1, X_2)\mathbf{u} \ = \ &2(n-1)(u^\sharp(X_1)X_2 - u^\sharp(X_2)X_1) + 2(u^\sharp(X_2)QX_1 - u^\sharp(X_1)QX_2\\ &+(D_{X_1}Q)X_2 - (D_{X_2}Q)X_1).\end{aligned} \tag{24}$$

Inserting $X_2 = \mathbf{u}$ in Equation (24), utilizing Equations (11) and (12) we acquire $(\mathcal{L}_Z R)(X_1, \mathbf{u})\mathbf{u} = 0$. On the other hand, taking Lie differentiation of $R(X_1, \mathbf{u})\mathbf{u} = -X_1 - u^\sharp(X_1)\mathbf{u}$ (obtained form Equation (8)) gives

$$(\mathcal{L}_Z R)(X_1, \mathbf{u})\mathbf{u} - g(X_1, \mathcal{L}_Z \mathbf{u})\mathbf{u} + 2u^\sharp(\mathcal{L}_Z \mathbf{u})X_1 + \{(\mathcal{L}_Z u^\sharp)X_1\}\mathbf{u} = 0, \tag{25}$$

which by virtue of $(\mathcal{L}_Z R)(X_1, \mathbf{u})\mathbf{u} = 0$ transforms

$$g(X_1, \mathcal{L}_Z \mathbf{u})\mathbf{u} - 2u^\sharp(\mathcal{L}_Z \mathbf{u})X_1 = \{(\mathcal{L}_Z u^\sharp)X_1\}\mathbf{u}. \tag{26}$$

With the aid of Equation (13), the soliton Equation (3) takes the form

$$(\mathcal{L}_Z g)(X_1, \mathbf{u}) = 2(\lambda - (n-1))u^\sharp(X_1). \tag{27}$$

Now, Lie differentiating $u^\sharp(X_1) = g(X_1, \mathbf{u})$ and $g(\mathbf{u}, \mathbf{u}) = -1$ along Z and calling Equation (27) obtains

$$\begin{aligned} (\mathcal{L}_Z u^\sharp)X_1 - g(X_1, \mathcal{L}_Z \mathbf{u}) &= 2(\lambda - (n-1))u^\sharp(X_1), \\ u^\sharp(\mathcal{L}_Z \mathbf{u}) &= \lambda - (n-1). \end{aligned}$$

Utilizing the above relations in Equation (25) we infer $\lambda - (n-1)(X_1 + \eta(X_1)\mathbf{u}) = 0$. Taking g-trace of this provides $\lambda = n-1$, which means the soliton is shrinking. Now g-trace of Equation (24) provides

$$(\mathcal{L}_Z Ric_g)(X_1, \mathbf{u}) = -\mathbf{u}(s)u^\sharp(X_1) - X_1(s).$$

where we take the well-known formulae $divQ = \frac{1}{2}\text{grad}s$ and $trace_g DQ = \text{grad}s$. The above equation together with Lemma 4, one can find $(\mathcal{L}_Z Ric_g)(X_1, \mathbf{u}) = 0$. □

Theorem 1. *If quasi-Einstein Lorentzian manifold M with $\mathcal{SSM}$ **u**-connection $\widehat{D}$ has a RS structure (g, Z), then M is Einstein.*

Proof. Lie differentiation of Equation (13) along Z, recalling Equation (27) infers

$$(\mathcal{L}_Z Ric_g)(X_1, \mathbf{u}) + Ric_g(X_1, \mathcal{L}_Z \mathbf{u}) = (n-1)\{2(\lambda - (n-1))u^\sharp(X_1) + g(X_1, \mathcal{L}_Z \mathbf{u})\}.$$

In light of Equations (20) and (18), $\lambda = n-1$ and $u^\sharp(\mathcal{L}_Z \mathbf{u}) = 0$ in the previous equation, we achieve

$$(s - n(n-1))\mathcal{L}_Z \mathbf{u} = 0. \tag{28}$$

Suppose that $s \neq n(n-1)$ in some open set $\mathcal{U}$ of M. Then on $\mathcal{U}$, $\mathcal{L}_Z \mathbf{u} = 0 = \mathcal{L}_Z u^\sharp$. Consider the following well known formula (see Yano [49]):

$$(\mathcal{L}_Z D)(X_1, X_2) = \mathcal{L}_Z D_{X_1} X_2 - D_{X_1} \mathcal{L}_Z X_2 - D_{[Z, X_1]} X_2.$$

Replacing X_2 by $\mathbf{u}$ in the foregoing equation, utilizing $\mathcal{L}_Z \mathbf{u} = 0 = \mathcal{L}_Z u^\sharp$ and Equation (6), we have

$$\begin{aligned} (\mathcal{L}_Z D)(X_1, \mathbf{u}) = \quad & \mathcal{L}_Z X_1 + \{(\mathcal{L}_Z u^\sharp)X_1\}\mathbf{u} + u^\sharp(\mathcal{L}_Z X_1)\mathbf{u} \\ & + u^\sharp(X_1)\mathcal{L}_Z \mathbf{u} - \mathcal{L}_Z X - u^\sharp(\mathcal{L}_Z X)\mathbf{u} = 0. \end{aligned}$$

Comparing of the above equation with Equation (23), one can see $QX_1 = (n-1)X_1$. Taking g-trace of this infers $s = n(n-1)$ on $\mathcal{U}$. Thus, we arrive at a contradiction on $\mathcal{U}$. Thus, Equation (28) gives $s = n(n-1)$, and so, we can from Equation (18) that M is Einstein. □

It is a known fact, as stated in [48], that any 3-dimensional Lorentzian manifold is quasi-Einstein, and in higher dimensions, there are Lorentzian manifolds which are not quasi-Einstein. Studying RS structure on Lorentzian 3-manifold admitting $\mathcal{SSM}$ **u**-connection becomes interested due to Theorem 1. Here, we prove the following outcome:

Theorem 2. *Let M Lorentzian 3-manifold admitting $\mathcal{SSM}$ **u**-connection $\widehat{D}$. If (g, Z) is a RS, then it is of constant curvature* 1.

Proof. In dimension 3, the Riemannian curvature tensor is given by

$$\begin{aligned} R(X_1, X_2)X_3 &= g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2 + g(QX_2, X_3)X_1 - g(QX_1, X_3)X_2 \\ &\quad -\frac{s}{2}\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\}. \end{aligned} \tag{29}$$

Setting $X_2 = X_3 = \mathbf{u}$ in Equation (29) and applying Equations (8) and (13), we obtain

$$QX_1 = \left(\frac{s}{2} - 1\right)X_1 + \left(\frac{s}{2} - 3\right)u^\sharp(X_1)\mathbf{u}. \tag{30}$$

By following the same steps as in the proof of Theorem 1 and utilizing the fact that $(\mathcal{L}_Z R)(X_1, \mathbf{u})\mathbf{u} = 0$, we can deduce that $s = 6$. Hence, from Equation (30) one can infer $QX_1 = 2X_1$. This together with Equation (30) provides

$$R(X_1, X_2)X_3 = g(X_2, X_3)X_1 - g(X_1, X_3)X_2,$$

which means M is of constant curvature 1. □

Theorem 3. *Let (M, g) be a Lorentzian manifold admitting $\mathcal{SSM}$ **u**-connection $\widehat{D}$. If (g, Z) is a RS structure with $Z = \sigma\mathbf{u}$, then (M, g) is Einstein.*

Proof. By our assumption: $Z = \sigma\mathbf{u}$ for smooth function σ on M. Differentiating this along X_1 provides

$$D_{X_1}Z = X_1(\sigma)\mathbf{u} + \mathbf{u}\{X_1 + u^\sharp(X_1)\mathbf{u}\},$$

where we applied Equation (6). As a result of this, the fundamental Equation (3) becomes

$$\begin{aligned} 2Ric_g(X_1, X_2) + X_1(\sigma)u^\sharp(X_2) + X_2(\sigma)u^\sharp(X_1) = \;& 2(\lambda - \sigma)g(X_1, X_2) \\ & -2\sigma u^\sharp(X_1)u^\sharp(X_2). \end{aligned} \tag{31}$$

Replacing X_1, X_2 by $\mathbf{u}$ in Equation (31) and recalling Equation (13) yields $\mathbf{u}(\sigma) = \lambda - (n-1)$. Taking into account of this, Equation (13) and putting $X_2 = \mathbf{u}$ in Equation (31) gives $X_1(\sigma) = -(\lambda - (n-1))u^\sharp(X_1)$. This together with Equation (31) extracts

$$Ric_g = (\lambda - \sigma)g + (\lambda - (n-1) - \sigma)u^\sharp \otimes u^\sharp.$$

The g-trace of the foregoing equation provides $\lambda - \sigma = \frac{r}{n-1} - 1$. Substitute this value in the above equation to obtain

$$Ric_g = \left(\frac{s}{n-1} - 1\right)g + \left(\frac{s}{n-1} - n\right)u^\sharp \otimes u^\sharp. \tag{32}$$

Consequently, g is quasi-Einstein. Employing Theorem 1 we conclude that g is Einstein. □

4. Gradient Ricci Almost Solitons on Lorentzian Manifolds with $\mathcal{SSM}$ u-Connection

We consider the $GRAS$ structure on Lorentzian manifolds with $\mathcal{SSM}$ **u**-connection and prove the following outcome:

Theorem 4. *Let (M, g) be a Lorentzian manifold admitting $\mathcal{SSM}$ $\mathbf{u}$-connection $\widehat{D}$. If (g, Z) is a GRAS structure, then either M is Einstein or the potential vector field is pointwise collinear with $\mathbf{u}$ on an open set $\mathcal{U}$ on M.*

Proof. The equation of *GRAS* structure Equation (4) can be exhibited as

$$D_{X_1}\mathrm{grad}h = -QX_1 + \lambda X_1. \tag{33}$$

Differentiating Equation (33) along X_2 we achieve

$$D_{X_2}D_{X_1}\mathrm{grad}h = -(D_{X_2}Q)X_1 - Q(D_{X_2}X_1) + X_2(\lambda)X_1 + \lambda D_{X_2}X_1. \tag{34}$$

Employing Equations (33) and (34) in the definition $R(X_1, X_2) = [D_{X_1}, D_{X_2}] - D_{[X_1,X_2]}$, one can obtain

$$R(X_1, X_2)\mathrm{grad}h = (D_{X_2}Q)X_1 - (D_{X_1}Q)X_2 + X_1(\lambda)X_2 - X_2(\lambda)X_1. \tag{35}$$

Take inner product of the foregoing equation with $\mathbf{u}$, call back Equations (8) and (11) to obtain

$$X_2(h)u^\sharp(X_1) - X_1(h)u^\sharp(X_2) = X_1(\lambda)u^\sharp(X_2) - X_2(\lambda)u^\sharp(X_1).$$

Replacing X_2 by $\mathbf{u}$ in the previous equation yields

$$X_1(\lambda + h) = -\mathbf{u}(\lambda + h)u^\sharp(X_1), \tag{36}$$

from which one can deduce

$$d(\lambda + h) = -\mathbf{u}(\lambda + h)u^\sharp. \tag{37}$$

Now, setting $X_1 = \mathbf{u}$ in Equation (35) and taking inner product with X_3 infers

$$g(R(\mathbf{u}, X_2)\mathrm{grad}h, X_3) = Ric_g(X_2, X_3) - (n-1)g(X_2, X_3) + \mathbf{u}(\lambda)g(X_2, X_3) - X_2(\lambda)u^\sharp(X_3).$$

Again, taking inner product of Equation (10) with $\mathrm{grad}h$ reveals

$$g(R(\mathbf{u}, X_2)\mathrm{grad}h, X_3) = u^\sharp(X_3)X_2(h) - g(X_2, X_3)\mathbf{u}(h).$$

A comparison of the last two equations provides

$$Ric_g(X_1, X_2) = \{(n-1) - \mathbf{u}(\lambda + h)\}g(X_1, X_2) - \mathbf{u}(\lambda + h)u^\sharp(X_1)u^\sharp(X_2). \tag{38}$$

Contraction of Equation (38) gives

$$\mathbf{u}(\lambda + h) = \left(n - \frac{s}{(n-1)}\right). \tag{39}$$

Employing Equation (39) in Equation (38), one can easily obtain Equation (18). On the other hand, the g-trace of Equation (35) yields

$$Ric_g(X_1, \mathrm{grad}h) = \frac{1}{2}X_1(s) - (n-1)X_1(\lambda).$$

The previous equation together with Equation (18) implies

$$\frac{1}{2}X_1(s) - (n-1)X_1(\lambda) = \left(\frac{s}{(n-1)} - 1\right)X_1(h) - \left(n - \frac{s}{(n-1)}\right)u^\sharp(X_1)\mathbf{u}(h). \tag{40}$$

Replacing X_1 by $X_1 + u^\sharp(X_1)\mathbf{u}$ in Equation (40), it follows from Lemma 3 and Equation (15) that

$$(n-1)(X_1(\lambda) + u^\sharp(X_1)\mathbf{u}(\lambda)) = \left(1 - \frac{s}{(n-1)}\right)\{X_1(h) + u^\sharp(X_1)\mathbf{u}(h)\}. \tag{41}$$

Consideration of Equation (36) in Equation (41) implies

$$\left(n - \frac{s}{(n-1)}\right)\{\mathrm{grad}h + \mathbf{u}(h)\mathbf{u}\} = 0.$$

If $s = n(n-1)$, which is together with Equation (18) gives that $Ric_g = (n-1)g$, and hence M is Einstein. Supposing $s \neq n(n-1)$ on some open set $\mathcal{U}$ of M, we obtain $\mathrm{grad}h = -\mathbf{u}(h)\mathbf{u}$, and this completes the proof. □

5. Geometrical and Physical Motivations

Quasi-Einstein manifolds were first conceptualized while examining exact solutions of the Einstein field equations as well as during investigation of quasi-umbilical hypersurfaces. For instance, the Robertson–Walker spacetimes are considered quasi-Einstein manifolds. In particular, every Ricci-flat pseudo-Riemannian manifold is quasi-Einstein (e.g., Schwarzschild spacetime). Quasi-Einstein spacetime is used as a model for a perfect fluid space–time in cosmology. Consequently, in the evolution of the universe it determines the final phase [50]. According to standard cosmological models, the matter content of the universe working as a perfect fluid, which includes both dust fluid and viscous fluid. Many geometers consider such space–time to investigate geometrical aspects in terms of *RS*, Yamabe soliton *(YS)*, etc., and characterize their importance in general relativity.

Geometric flows are now a crucial aspect in both pseudo-Riemannian geometry and general relativity. The study of the geometry of *RS* is highly pursued subject not only because of its elegant geometry but also because of its wide range of applications in various fields. Hamilton [8] provided a physical model of three distinct classes to study the kinetic and potential behavior of relativistic space–time in cosmology and general relativity. These classes give examples of ancient, eternal, and immortal solutions, namely, shrinking ($\lambda > 0$) which exists on minimal time interval $-\infty < t < a$ where $a < \infty$, steady ($\lambda = 0$) which exists for all time and expanding ($\lambda < 0$) which exists on maximal time interval $a < t < \infty$. In [51], Woolgar briefly explained how *RS* arises in the renormalization group (RG) flow of a nonlinear sigma model. Duggal [52,53] states a necessary condition for a vector field Z to be a curvature inheritance (CI) symmetry that $P_{ij}R^i_{kme} + P_{ik}R^i_{kme} = 0$ holds, where $P_{ij} = \mathcal{L}_Z g_{ij}$. The general solution of this identity is $\mathcal{L}_Z g_{ij} = 2\psi g_{ij} + \phi_{ij}$, where ϕ_{ij} is a second-order symmetric tensor and ψ is a smooth function on a pseudo-Riemannian manifold. Choosing $\phi_{ij} = -2Ric_{g_{ij}}$ results in Z being a *RAS*, which shows a relationship between the CI symmetry and a class of *RAS* structure. This supports us in discussing the many physical applications of a class of *RAS* space–time of relativity.

6. Conclusions

We use methods of local pseudo-Riemannian geometry to classify Einstein metrics in such broader classes of metrics as RAs structure on Lorentzian manifolds, finding special connections. Our main result (Theorem 1) reveals that any quasi-Einstein Lorentzian manifold having $\mathcal{SSM}$ $\mathbf{u}$-connection is Einstein, when g is *RS*. It is crucial to note that the examination of quasi-Einstein Lorentzian manifolds holds significant importance as they represent the third phase in the evolution of the universe. Therefore, the investigations of quasi-Einstein manifolds provide a deeper understanding of the universe's global nature, including the topology, because the nature of the singularities can be defined from a differential geometric viewpoint. Our investigation also paves the way for future research opportunities in this domain, particularly in exploring many physical applications within diverse spatial contexts like Lorentz and other space. Moving forward, we plan to delve into the applications of our main results, integrating concepts from singularity theory, submanifold theory, and related fields [54–75]. By doing so, we anticipate uncovering a plethora of novel findings and expanding the frontiers of knowledge.

Following this result, we can think about many physical applications. In our Theorem 3, we assume that soliton vector field $Z = \sigma\mathbf{u}$ means that Z is a material curve as it maps flow lines into flow lines, which plays a vital role in relativistic fluid dynamics. So our Theorem 3 gives the relation between material curves and Einstein manifolds. We delegate for further study the following questions:

- Do Theorems 1 and 3 hold true in the absence of assuming the quasi-Einstein condition or Z being collinear to the vector field $\mathbf{u}$?
- Are the findings of this paper applicable to generalized m-quasi-Einstein Lorentzian manifolds?

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