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CHARACTERIZATION OF SCALING FUNCTIONS IN A MULTIRESOLUTION ANALYSIS

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Abstract. We characterize the scaling functions of a multiresolution analysis in a general context, where instead of the dyadic dilation one considers the dilation given by a fixed linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and all (complex) eigenvalues of A have absolute value greater than 1. In the general case the conditions depend on the map A. We identify some maps for which the obtained condition is equivalent to the dyadic case, i.e., when A is a diagonal matrix with all numbers in the diagonal equal to 2. There are also easy examples of expanding maps for which the obtained condition is not compatible with the dyadic case. The complete characterization of the maps for which the obtained conditions are equivalent is out of the scope of the present note.

1.

A multiresolution analysis (MRA) is a general method introduced by Mallat [10] and Meyer [11] for constructing wavelets. On $\mathbb{R}^n$ $(n \geq 1)$ by an MRA we will mean a sequence of subspaces V_j, $j \in \mathbb{Z}$ of the Hilbert space $L^2(\mathbb{R}^n)$ that satisfies the following conditions:

(i) $\forall j \in \mathbb{Z}, \qquad V_j \subset V_{j+1}$;
(ii) $\forall j \in \mathbb{Z}, \qquad f(\mathbf{x}) \in V_j \Leftrightarrow f(2\mathbf{x}) \in V_{j+1}$;
(iii) $W = \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;
(iv) there exists a function $\phi \in V_0$, that is called a *scaling function*, such that $\{ \phi(\mathbf{x} - \mathbf{k}) \}_{\mathbf{k}\in\mathbb{Z}^n}$ is an orthonormal basis for V_0.

We will consider MRA in a general context, where instead of the dyadic dilation one considers the dilation given by a fixed linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and all (complex) eigenvalues of A have absolute value greater than 1. Given such a linear map A one defines an $A-$MRA as a sequence of subspaces V_j, $j \in \mathbb{Z}$ of the Hilbert space $L^2(\mathbb{R}^n)$ (see [9], [7], [13], [14]) that satisfies the conditions (i), (iii), (iv) and

(ii$_1$) $\forall j \in \mathbb{Z}, \qquad f(\mathbf{x}) \in V_j \Leftrightarrow f(A\mathbf{x}) \in V_{j+1}$.

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Remark 1. There is formally a more general definition of MRA, when one considers a discrete lattice $\Gamma \subset \mathbb{R}^n$ (a discrete subgroup given by integral linear combinations of a vector space basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and a map $M : \mathbb{R}^n \to \mathbb{R}^n$ such that $M(\Gamma) \subset \Gamma$ and all (complex) eigenvalues of M have absolute value greater than 1. Then the related MRA is defined as a sequence of subspaces V_j, $j \in \mathbb{Z}$ of the Hilbert space $L^2(\mathbb{R}^n)$ that satisfies the conditions (i), (iii) and

(ii$_2$) $\forall j \in \mathbb{Z}, \qquad f(\mathbf{x}) \in V_j \Leftrightarrow f(M\mathbf{x}) \in V_{j+1}$.

(iv$_2$) There exists a function $\phi \in V_0$, that is called a *scaling function*, such that $\{ \phi(\mathbf{x} - \mathbf{k}) \}_{\mathbf{k} \in \Gamma}$ is an orthonormal basis for V_0.

It is an easy exercise to observe that by the map $S : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$S\mathbf{e}_j = \mathbf{v}_j, \qquad 1 \le j \le n,$$

where $\{\mathbf{e}_j\}_{j=1}^n$ is the natural basis of $\mathbb{R}^n$, one obtains an associated A_1-MRA V_j^*, $j \in \mathbb{Z}$, where $V_j^* = S^{-1} V_j S$ with the dilation map $A_1 = S^{-1} M S$ (see [13] and [14], p. 108).

Remark 2. Usually in the definition of an MRA appears the following condition: $\bigcap_{j \in \mathbb{Z}} V_j = \{ \mathbf{0} \}$, which follows from the conditions (i), (ii) and (iv) (cf. for example [3], [8] for $n = 1$). We will give a proof for the general case (see Lemma 4 below).

A priori the condition (iv) appears to be independent from the rest of the conditions in the definitions of MRA and $A-$MRA. A key tool for the characterization of a function ϕ which satisfies the condition (iv) is the following well-known result (cf. [6], p. 132; [13], p. 34).

Lemma A. *The system* $\{ g(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n \}$, *where* $g \in L^2(\mathbb{R}^n)$, *is an orthonormal system if and only if*

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} |\widehat{g}(\mathbf{t} + \mathbf{k})|^2 = 1 \qquad \textit{for a.e.} \quad \mathbf{t} \in \mathbb{R}^n. \tag{1}$$

In this paper we adopt the convention that the Fourier transform of a function $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\mathbf{y}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{x} \cdot \mathbf{y}} d\mathbf{x}.$$

The main purpose of our paper is related to a result proved by E. Hernández and G. Weiss (cf. [8], p. 382).

Theorem B. *Let* V_j, $j \in \mathbb{Z}$ *be a sequence of closed subspaces of* $L^2(\mathbb{R})$ *satisfying* (i), (ii) *and* (iv) *for* $n = 1$. *Then the following two conditions are equivalent:*

$$\text{a)} \lim_{j \to \infty} |\widehat{\phi}(2^{-j} y)| = 1 \quad \textit{for a.e.} \quad y \in \mathbb{R}; \quad \text{b)} \quad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

Remark 3. The above theorem is formulated in a modified form in order to indicate the "essential" part of the result that we are interested in. However, Lemma 2 below permits us to give a formulation of our result in a similar style.

Our aim is to achieve a bit deeper understanding of the relation between the behaviour of the function $\widehat{\phi}$ in the neighborhood of the origin and the condition (iii). In particular our result permits us to get rid of the assumption that $|\widehat{\phi}|$ is continuous at the origin in Theorem 1.7 of E. Hernández and G. Weiss (cf. [8], pp. 46–48). We prefer to prove our result in the general case because it can be useful for the description of wavelet functions in the frame of the result of the article [5]. Moreover, the cost of the exposition in the general case is little. We identify some maps for which the obtained condition is equivalent to the dyadic case, i.e., when A is a diagonal matrix with all numbers in the diagonal equal to 2. There are also easy examples of expanding maps for which the obtained condition is not compatible with the dyadic case. The complete characterization of the maps for which the obtained conditions are equivalent is out of the scope of the present note.

Let us introduce some notation before formulating our result. Further in the paper for $\mathbf{m} = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$ and $\mathbf{l} = (\ell_1, \ell_2, \dots, \ell_n) \in \mathbb{Z}^n$ we define $2^{\mathbf{l}} = (2^{\ell_1}, 2^{\ell_2}, \dots, 2^{\ell_n}) \in \mathbb{Z}^n$ and $\mathbf{v} = [\mathbf{m}2^{\mathbf{l}}] = (m_1 2^{\ell_1}, m_2 2^{\ell_2}, \dots, m_n 2^{\ell_n})$. Furthermore, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and if we write $F \in L^2(\mathbb{T}^n)$ we will understand that F is defined on the whole space $\mathbb{R}^n$ as a 1-periodic function with respect to all variables. With some abuse of the notation we consider also that $\mathbb{T}^n$ is the unit cube $[0,1)^n$. The translation of a function $f \in L^2(\mathbb{R}^n)$ by $\mathbf{b} \in \mathbb{R}^n$ will be denoted by $\tau_{\mathbf{b}} f(\mathbf{t}) = f(\mathbf{t} - \mathbf{b})$. We denote by D_A the dilation operator $D_A f(\mathbf{x}) = d_A^{\frac{1}{2}} f(A\mathbf{x})$ in $L^2(\mathbb{R}^n)$, where $d_A = |\det A|$. Here and further we use the same notation for the linear map and its matrix with respect to the canonical base.

We will define $B_r(\mathbf{y}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\}$ and will write B_r if $\mathbf{y}$ is the origin. For a set $E \subset \mathbb{R}^n$ and a number $a \in \mathbb{R}$ we will denote $aE = \{\, \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = a\mathbf{t} \text{ for } \mathbf{t} \in E \,\}$. If $\mathbf{x} \in \mathbb{R}^n$ then $\mathbf{x} + E = \{\mathbf{x} + \mathbf{y} : \text{ for } \mathbf{y} \in E\}$.

The Lebesgue measure of a set $E \subset \mathbb{R}^n$ will be denoted as $|E|_n$. Letting $\mathbf{x} \in \mathbb{R}^n$, we will say that $\mathbf{x}$ is a point of density for a set $E \subset \mathbb{R}^n$, $|E|_n > 0$, if

$$\lim_{r \to 0} \frac{|E \cap B_r(\mathbf{x})|_n}{|B_r(\mathbf{x})|_n} = 1.$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a measurable function. We say that $\mathbf{x} \in \mathbb{R}^n$ is a point of approximate continuity of the function f if there exists $E \subset \mathbb{R}^n$, $|E|_n > 0$, such that $\mathbf{x}$ is a point of density for the set E and

$$\lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in E}} f(\mathbf{y}) = f(\mathbf{x}). \tag{2}$$

It can be shown that (cf. [12], [1]) for any finite measurable function almost all points are points of approximate continuity. Let us introduce

Definition 1. A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be *locally nonzero* at a point $\mathbf{x} \in \mathbb{R}^n$ if for any $\varepsilon > 0$, there exists r, $0 < r < 1$, such that

$$|\{\, \mathbf{y} \in B_r(\mathbf{x}) : f(\mathbf{y}) = 0 \,\}|_n < \varepsilon |B_r(\mathbf{x})|_n.$$

We will say that $\mathbf{x} \in \mathbb{R}^n$ is a point of $A-$density for a set $E \subset \mathbb{R}^n$, $|E|_n > 0$ if for any $r > 0$,

$$\lim_{j \to \infty} \frac{|E \cap (A^{-j} B_r + \mathbf{x})|_n}{|A^{-j} B_r + \mathbf{x}|_n} = 1.$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a measurable function. We say that $\mathbf{x} \in \mathbb{R}^n$ is a point of $A-$approximate continuity of the function f if there exists $E \subset \mathbb{R}^n$, $|E|_n > 0$, such that $\mathbf{x}$ is a point of $A-$density for the set E and (2) holds.

Definition 2. A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be $A-$*locally nonzero* at a point $\mathbf{x} \in \mathbb{R}^n$ if for any $\varepsilon > 0$ and $r > 0$ there exists $j \in \mathbb{N}$ such that

$$|\{\, \mathbf{y} \in A^{-j} B_r + \mathbf{x} : f(\mathbf{y}) = 0 \,\}|_n < \varepsilon |A^{-j} B_r + \mathbf{x}|_n.$$

We prove the following.

Theorem 1. *Let V_j be a sequence of closed subspaces in $L^2(\mathbb{R}^n)$ satisfying the conditions* (i), (ii$_1$) *and* (iv). *Then the following conditions are equivalent:*

- **A$_1$:** $W = \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$*;*
- **B$_1$:** $\widehat{\phi}$ *(the Fourier transform of the scaling function ϕ) is A^*-locally nonzero at the origin;*
- **C$_1$:** *the origin is a point of A^*-approximate continuity of the function $|\widehat{\phi}|$ if we set $|\widehat{\phi}(\mathbf{0})| = 1$.*

The following theorem is an immediate consequence of the above result.

Theorem 2. *Let V_j be a sequence of closed subspaces in $L^2(\mathbb{R}^n)$ satisfying the conditions* (i), (ii) *and* (iv). *Then the following conditions are equivalent:*

- **A:** $W = \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$*;*
- **B:** $\widehat{\phi}$ *(the Fourier transform of the scaling function ϕ) is locally nonzero at the origin;*
- **C:** *the origin is a point of approximate continuity of the function $|\widehat{\phi}|$ provided that $|\widehat{\phi}(\mathbf{0})| = 1$.*

Remark 4. If we consider the polar decomposition of a map A and have that all eigenvalues of the positive operator $\sqrt{A^*A}$ are equal, then it is easy to show that for those maps the conditions $\mathbf{B}_1$ and $\mathbf{C}_1$ are respectively equivalent to the conditions $\mathbf{B}$ and $\mathbf{C}$.

Remark 5. If the matrix of the map A is a diagonal matrix of real numbers and at least two elements on the diagonal have distinct absolute values, then there exist functions for which the conditions B_1 and C_1 hold but the conditions B and C are not true and vice versa.

2.

Different versions of the following Lemmas 1–2 have appeared in various publications (cf. [6], pp. 131–132; [13], pp. 28–29). We refer to [4] as a recent reference for the dyadic case when $n = 1$; for the general case see [14].

Lemma 1. *Let $\phi \in L^2(\mathbb{R}^n)$ and assume that $\{\tau_{\mathbf{k}}\phi\}_{\mathbf{k}\in\mathbb{Z}^n}$ is an orthonormal basis of V_0. Suppose that $V_j, j \in \mathbb{Z}^n$, is a sequence of closed subspaces in $L^2(\mathbb{R}^n)$ satisfying the condition* (ii$_1$). *Then a function f is in V_j if and only if there is a function $F_j \in L^2(\mathbb{T}^n)$ such that*

$$D^j_{A^*}\widehat{f}(\mathbf{t}) = F_j(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad a.e.\ on \quad \mathbb{R}^n \quad if \quad j \geq 0, \tag{3}$$

$$D^{|j|}_{A^{*-1}}\widehat{f}(\mathbf{t}) = F_j(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad a.e.\ on \quad \mathbb{R}^n \quad if \quad j < 0, \tag{4}$$

and

$$\|f\|_{L^2(\mathbb{R}^n)} = \|F_j\|_{L^2(\mathbb{T}^n)}. \tag{5}$$

Proof. Suppose $j \geq 0$ and let $f \in V_j$. Then $D^j_{A^{-1}}f \in V_0$ and by our hypothesis $D^j_{A^{-1}}f(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}}\tau_{\mathbf{k}}\phi(\mathbf{t})$ in $L^2(\mathbb{R}^n)$, where $\{c_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^n} \in l^2$. Taking the Fourier transform of this expression we get

$$D^j_{A^*}\widehat{f}(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_k e^{-2\pi i\mathbf{k}\cdot\mathbf{t}}\widehat{\phi}(\mathbf{t}).$$

Hence (3) is true with $F_j(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}} e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}$. The condition (5) holds because D_{A^*} is a unitary operator and the functions $D^j_{A^{-1}}f$, F_j are represented by orthonormal systems with the same coefficients.

On the other hand, assume that (3) holds for $f \in L^2(\mathbb{R}^n)$, where $F_j \in L^2(\mathbb{T}^n)$. If we write $F_j(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}} e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}$, then it follows that

$$D^j_{A^{-1}} f(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_k \tau_{\mathbf{k}} \phi(A^j\mathbf{t}) \quad \text{a.e. on} \quad \mathbb{R}^n,$$

and therefore $f \in V_j$. When $j < 0$ the proof is similar. □

Lemma 2. *Let $\phi \in L^2(\mathbb{R}^n)$ and assume that $\{\tau_{\mathbf{k}}\phi\}_{\mathbf{k}\in\mathbb{Z}^n}$ is an orthonormal basis of V_0. Suppose that $V_j, j \in \mathbb{Z}^n$, is a sequence of closed subspaces in $L^2(\mathbb{R}^n)$ satisfying the condition* (ii$_1$)*. Then the following conditions are equivalent:*

a): $\forall j \in \mathbb{Z}, \qquad V_j \subset V_{j+1}$*;*

b): *there exists $H \in L^\infty(\mathbb{T}^n)$, $\|H\|_\infty \le d^{1/2}_{A^*}$ such that*

$$D_{A^*}\widehat{\phi}(\mathbf{t}) = H(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad \textit{a.e. on} \quad \mathbb{R}^n. \tag{6}$$

Proof. If **a**) holds, then $\phi \in V_1$. By Lemma 1 there exists $H \in L^2(\mathbb{T}^n)$ such that $D_{A^*}\widehat{\phi}(\mathbf{t}) = H(\mathbf{t})\widehat{\phi}(\mathbf{t})$, $H \in L^2(\mathbb{T}^n)$. Let

$$\Phi(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} |\widehat{\phi}(\mathbf{t}+\mathbf{k})|^2.$$

Then

$$\begin{aligned} \Phi(A^*\mathbf{t}) &= \sum_{\mathbf{k}\in\mathbb{Z}^n} |\widehat{\phi}(A^*\mathbf{t}+\mathbf{k})|^2 = \sum_{\mathbf{k}\in A^*\mathbb{Z}^n} |\widehat{\phi}(A^*\mathbf{t}+\mathbf{k})|^2 \\ &+ \sum_{\mathbf{k}\notin A^*\mathbb{Z}^n} |\widehat{\phi}(A^*\mathbf{t}+\mathbf{k})|^2 = d^{-1}_{A^*} \sum_{\mathbf{k}\in\mathbb{Z}^n} |D_{A^*}\widehat{\phi}(\mathbf{t}+\mathbf{k})|^2 + R(\mathbf{t}) \\ &= d^{-1}_{A^*}|H(\mathbf{t})|^2\Phi(\mathbf{t}) + R(\mathbf{t}) \end{aligned}$$

where $R(\mathbf{t})$ is nonnegative. Now, Lemma A gives that for almost all $\mathbf{t} \in \mathbb{R}^n$,

$$1 \ge \Phi(A^*\mathbf{t}) \ge d^{-1}_{A^*}|H(\mathbf{t})|^2\Phi(\mathbf{t}) \ge d^{-1}_{A^*}|H(\mathbf{t})|^2$$

and therefore, $|H(\mathbf{t})| \le d^{1/2}_{A^*}$ a.e. on $\mathbb{R}^n$.

Finally, to prove **b**)⇒**a**) let $j \ge 0$ and $f \in V_j$. Then, by Lemma 1, there is a function $F \in L^2(\mathbb{T}^n)$ such that

$$D^j_{A^*}\widehat{f}(\mathbf{t}) = F(\mathbf{t})\widehat{\phi}(\mathbf{t}) = d^{-1/2}_{A^*} F(\mathbf{t}) H({A^*}^{-1}\mathbf{t})\widehat{\phi}({A^*}^{-1}\mathbf{t}).$$

Hence,

$$D^{j+1}_{A^*}\widehat{f}(\mathbf{t}) = F(A^*\mathbf{t})H(\mathbf{t})\widehat{\phi}(\mathbf{t}).$$

Noting that $A^* : \mathbb{Z}^n \to \mathbb{Z}^n$ we have that $D_{A^*}F \in L^2(\mathbb{T}^n)$; hence $HD_{A^*}F \in L^2(\mathbb{T}^n)$ and therefore by Lemma 1 we get $f \in V_{j+1}$. For the case $j < 0$ the proof is similar. □

We need the following lemma for the proof of Lemma 4.

Lemma 3. *Let $g \in L(\mathbb{T}^n)$, let A be a fixed linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$, $d_A \ne 0$, and let $\hat{A} : \mathbb{T}^n \to \mathbb{T}^n$ be the induced endomorphism. Then*

$$\int_{\mathbb{T}^n} g(\hat{A}\mathbf{t})d\mathbf{t} = \int_{\mathbb{T}^n} g(\mathbf{t})d\mathbf{t}.$$

Proof. We have that card $\mathbb{Z}^n/A(\mathbb{Z}^n) = d_A$ (see [14], p. 109). Let $\{\mathbf{m}_i\}_{i=1}^{d_A} \subset \mathbb{Z}^n$ be elements of $\mathbb{Z}^n$ that belong to distinct cosets of $\mathbb{Z}^n/A(\mathbb{Z}^n)$. Let $T_i : \mathbb{R}^n \to \mathbb{R}^n$ be defined by the equation $T_i(\mathbf{x}) = \mathbf{x} + \mathbf{m}_i$ and consider the induced endomorphisms $\hat{A}_i^{-1} : \mathbb{T}^n \to \mathbb{T}^n$ of the maps $A^{-1}T_i : \mathbb{R}^n \to \mathbb{R}^n$ $(1 \leq i \leq d_A)$, where A^{-1} is the map given by the matrix A^{-1}. It is easy to check that the images of those maps $\hat{A}_i^{-1}(\mathbb{T}^n), 1 \leq i \leq d_A$ are pairwise disjoint and that

$$\text{meas}(\hat{A}_i^{-1}(\mathbb{T}^n)) = d_A^{-1}, \quad 1 \leq i \leq d_A.$$

Hence, $\bigcup_{i=1}^{d_A} \hat{A}_i^{-1}(\mathbb{T}^n) = \mathbb{T}^n$. If we denote $E_i = \hat{A}_i^{-1}(\mathbb{T}^n), 1 \leq i \leq d_A$, then we will have $\hat{A}(E_i) = \mathbb{T}^n$ and therefore

$$\int_{\mathbb{T}^n} g(\hat{A}\mathbf{t})d\mathbf{t} = \sum_{i=1}^{d_A} \int_{E_i} g(\hat{A}\mathbf{t})d\mathbf{t}$$
$$= \sum_{i=1}^{d_A} d_A^{-1} \int_{\mathbb{T}^n} g(\mathbf{t})d\mathbf{t} = \int_{\mathbb{T}^n} g(\mathbf{t})d\mathbf{t}.$$

□

The following lemma is related to Remark 2.

Lemma 4. *Suppose that $V_j, j \in \mathbb{Z}^n$, is an $A-MRA$. Then $\bigcap_{j\in\mathbb{Z}} V_j = \{ \mathbf{0} \}$.*

Proof. Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$ and suppose that $\|f\|_2 = 1$. By Lemma 1 we have that for any $j < 0$,

$$D_{A^{*-1}}^{-j}\widehat{f}(\mathbf{t}) = F_j(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad \text{a.e. on} \quad \mathbb{R}^n, \quad F_j \in L^2(\mathbb{T}^n)$$

and

$$\|f\|_{L^2(\mathbb{R}^n)} = \|F_j\|_{L^2(\mathbb{T}^n)}. \tag{7}$$

Thus for any $j < 0$,

$$\widehat{f}(\mathbf{t}) = F_j((A^*)^{|j|}\mathbf{t})D_{A^*}^{|j|}\widehat{\phi}(\mathbf{t}) \quad \text{a.e. on} \quad \mathbb{R}^n. \tag{8}$$

If we show that for any closed ball $B \subset \mathbb{R}^n$ which does not contain the origin

$$\int_B |\widehat{f}(\mathbf{t})|d\mathbf{t} = 0, \tag{9}$$

then we will get a contradiction with the condition that the norm of the function is one and thus finish the proof of the lemma. Then by (8) we obtain that

$$\int_B |\widehat{f}(\mathbf{t})|d\mathbf{t} \quad = \quad d_A^{|j|/2} \int_B |F_j((A^*)^{|j|}\mathbf{t}))\widehat{\phi}((A^*)^{|j|}\mathbf{t})|d\mathbf{t}$$
$$\leq \left(d_A^{|j|} \int_B |F_j((A^*)^{|j|}\mathbf{t})|^2 d\mathbf{t}\right)^{1/2} \times \left(\int_B |\widehat{\phi}(A^{*|j|}\mathbf{x})|^2 d\mathbf{x}\right)^{1/2}$$
$$\leq \left(d_A^{-|j|} \int_{(A^*)^{|j|}B} |F_j(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2} \times \left(\int_{(A^*)^{|j|}B} |\widehat{\phi}(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2}.$$

Using the fact that the map A is expansive and that the closed ball B does not contain the origin we obtain immediately that

$$\int_{(A^*)^{|j|}B} |\widehat{\phi}(\mathbf{x})|^2 d\mathbf{x} \to 0 \quad \text{when} \quad j \to -\infty. \tag{10}$$

On the other hand, we can find a cube with sides parallel to the coordinate axes and with vertices with integer coordinates such that $B \subset Q$. Then by (7) and Lemma 3 we get that for any $j < 0$,

$$d_A^j \int_{(A^*)^{|j|}Q} |F_j(\mathbf{x})|^2 d\mathbf{x} \leq |Q|.$$

Hence for fixed B the integral

$$d_A^j \int_{(A^*)^{|j|}B} |F_j(\mathbf{x})|^2 d\mathbf{x} \leq |Q| \quad \text{for any} \quad j < 0.$$

Thus by (10) we obtain (9). □

For the proof of Theorem 1 we need the following.

Lemma 5. *Let V_j be a sequence of closed subspaces in $L^2(\mathbb{R}^n)$ satisfying the conditions* (i), (iii), (iv) *and* (ii$_1$). *Then for any bounded measurable set $E \subset \mathbb{R}^n$,*

$$\lim_{j\to\infty} \frac{1}{|(A^*)^{-j}E|_n} \int_{(A^*)^{-j}E} |\widehat{\phi}(\mathbf{t})|^2 d\mathbf{t} = 1. \tag{11}$$

Proof. We take $f \in L^2(\mathbb{R}^n)$ such that $\widehat{f} = \chi_E$. Then

$$\|f\|_2^2 = \|\widehat{f}\|_2^2 = |E|_n.$$

Let P_j be the orthogonal projection onto V_j. Then by property (iii$_1$) we have $\|f - P_j f\|_2 \to 0$ as $j \to \infty$. Hence, when $j \to \infty$,

$$\|P_j f\|_2^2 \to \|f\|_2^2 = |E|_n. \tag{12}$$

The system $\{ \phi_{j\mathbf{k}} \}_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n}$, where

$$\phi_{j\mathbf{k}}(\mathbf{x}) = d_A^{\frac{j}{2}} \phi(A^j\mathbf{x} - \mathbf{k}),$$

is an orthonormal basis of V_j according to the properties (i), (ii$_1$), (iv). Observe that

$$\begin{aligned}
\widehat{\phi_{j\mathbf{k}}}(\mathbf{t}) &= d_A^{\frac{j}{2}} \int_{\mathbb{R}^n} \phi(A^j\mathbf{x} - \mathbf{k}) e^{-2\pi i \mathbf{x}\cdot\mathbf{t}} d\mathbf{x} \\
&= d_A^{\frac{j}{2}} \int_{\mathbb{R}^n} \phi(A^j\mathbf{x} - \mathbf{k}) e^{-2\pi i (A^j\mathbf{x}-\mathbf{k})\cdot(A^*)^{-j}\mathbf{t}} e^{-2\pi i \mathbf{k}\cdot(A^*)^{-j}\mathbf{t}} d\mathbf{x} \\
&= d_A^{-\frac{j}{2}} e^{-2\pi i \mathbf{k}\cdot(A^*)^{-j}\mathbf{t}} \widehat{\phi}((A^*)^{-j}\mathbf{t}).
\end{aligned}$$

Thus $P_j f = \sum_{k\in\mathbb{Z}^n}\langle f, \phi_{j\mathbf{k}}\rangle \phi_{j\mathbf{k}}$ and

$$\begin{aligned}
\|P_j f\|_2{}^2 &= \sum_{\mathbf{k}\in\mathbb{Z}^n} |\langle f, \phi_{j\mathbf{k}}\rangle|^2 = \sum_{\mathbf{k}\in\mathbb{Z}^n} \left|\int f(\mathbf{x})\overline{\phi_{j\mathbf{k}}(\mathbf{x})}d\mathbf{x}\right|^2 \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^n} \left|\int \widehat{f}(\mathbf{t})\overline{\widehat{\phi_{j\mathbf{k}}}(\mathbf{t})}d\mathbf{t}\right|^2 \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^n} \left|d_A^{-\frac{j}{2}}\int \widehat{f}(\mathbf{t})e^{2\pi i\mathbf{k}\cdot(A^*)^{-j}\mathbf{t}}\overline{\widehat{\phi}((A^*)^{-j}\mathbf{t})}d\mathbf{t}\right|^2 \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^n} \left|d_A^{\frac{j}{2}}\int \widehat{f}(A^{*j}\mathbf{y})\overline{\widehat{\phi}(\mathbf{y})}e^{2\pi i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y}\right|^2 \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^n} \left|d_A^{\frac{j}{2}}\int_{(A^*)^{-j}E} \overline{\widehat{\phi}(\mathbf{y})}e^{2\pi i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y}\right|^2,
\end{aligned}$$

where $(A^*)^{-j}E = \{\ \mathbf{y}\in\mathbb{R}^n : A^{*j}\mathbf{y}\in E\ \}$.

Let j_1 be the minimal natural number such that $A^{*-j_1}E\subset[-1,1]^n$. Then for any $j\geq j_1$ the last sum is equal to

$$d_A^j\int|\chi_{(A^*)^{-j}E}(\mathbf{t})\widehat{\phi}(\mathbf{t})|^2 d\mathbf{t},$$

because the terms in the sum are the Fourier coefficients of the function $\widehat{\phi}\chi_{(A^*)^{-j}E}$. Therefore by (12) we obtain (11). □

Let $G_k = A^{-k}(\mathbb{Z}^n)$ for any $k\in\mathbb{N}$ and denote $G = \bigcup_{k=1}^{\infty} G_k$. Then the following lemma is true.

Lemma 6. *The set G is dense in $\mathbb{R}^n$.*

Proof. If the assertion of Lemma 6 is not true, then for some $\mathbf{x}\in\mathbb{R}^n$ and $r>0$ we will have that

$$B_r(\mathbf{x})\cap G = \emptyset. \tag{13}$$

Let $j\in\mathbb{N}$ be such that

$$\|A^j\mathbf{y}\| > \sqrt{n} \qquad \text{for all } \mathbf{y} \text{ such that } \|\mathbf{y}\| = r.$$

By the definition of j we have $B_{\sqrt{n}}(A^j\mathbf{x})\subset A^j(B_r(\mathbf{x}))$; hence there exists $\mathbf{k}\in\mathbb{Z}^n$ such that $\mathbf{k}\in A^j(B_r(\mathbf{x}))$. This means $A^{-j}\mathbf{k}\in B_r(\mathbf{x})$, which contradicts (13), because $A^{-j}\mathbf{k}\in G$. □

Proof of Theorem 1. Let us prove first the implication $\mathbf{B_1}\Rightarrow\mathbf{A_1}$. We observe that W is invariant under translations. At first we show that W is invariant under translations by vectors $\mathbf{v}\in G$. We fix some $\mathbf{v}\in G$. Then $\mathbf{v}\in G_l$ for some $l\in\mathbb{N}$.

For any $f\in W$ and $\forall\varepsilon>0, \exists h\in V_{j_0}$ such that $||f-h||_2<\varepsilon$. By (ii$_1$) we have that for every $j\geq j_0$, $h\in V_j$ and therefore $h(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}}^j\phi(A^j\mathbf{x}-\mathbf{k})$. Hence,

$$\tau_{\mathbf{v}}h(\mathbf{x}) = h(\mathbf{x}-\mathbf{v}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}}^j\phi(A^j\mathbf{x}-A^j\mathbf{v}-\mathbf{k}).$$

If $j > \max\{\, l, j_0 \,\}$, then $A^j\mathbf{v} \in \mathbb{Z}^n$. Consequently $\tau_{\mathbf{v}}h \in V_j$ and therefore $\tau_{\mathbf{v}}f \in W$. The set G is dense in $\mathbb{R}^n$. Thus closedness of the subspace W and the continuity of the operator $\tau_{\mathbf{u}}$ in $L^2(\mathbb{R}^n)$ yields the invariance of W under translations.

To show that $W = L^2(\mathbb{R}^n)$ we take any $g \in W^\perp$. Then for every $f \in W$ and for all $\mathbf{x} \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} f(\mathbf{x}+\mathbf{t})\overline{g(\mathbf{t})}d\mathbf{t} = 0$$

and therefore, by Plancherel's identity,

$$\int_{\mathbb{R}^n} \mathrm{e}^{i\mathbf{t}\cdot\mathbf{x}}\widehat{f}(\mathbf{t})\overline{\widehat{g}(\mathbf{t})}d\mathbf{t} = 0.$$

This shows that the Fourier transform of $\widehat{f}\overline{\widehat{g}}$ is identically zero, which immediately yields $\widehat{f}(\mathbf{y})\overline{\widehat{g}}(\mathbf{y}) = 0$ almost everywhere (a.e.) on $\mathbb{R}^n$. If we take $f(\mathbf{x}) = d_A^{\frac{j}{2}}\phi(A^j\mathbf{x}) \in V_j$, then

$$\widehat{f}(\mathbf{y}) = d_A^{-\frac{j}{2}}\widehat{\phi}((A^*)^{-j}\mathbf{y})$$

and $\widehat{\phi}((A^*)^{-j}\mathbf{y})\overline{\widehat{g}(\mathbf{y})} = 0$ (a.e.) or $\widehat{\phi}(\mathbf{t})\overline{\widehat{g}((A^*)^j\mathbf{t})} = 0$ (a.e.).

According to our hypothesis, for any positive integer N and $r > 1$ there exists $k \in \mathbb{N}$ such that

$$|\{\, \mathbf{t} \in (A^*)^{-k}B_r : \widehat{\phi}(\mathbf{t}) = 0 \,\}|_n < \frac{|(A^*)^{-k}B_r|_n}{N}.$$

Then

$$|\{\, \mathbf{t} \in (A^*)^{-k}B_r : \widehat{g}((A^*)^j\mathbf{t}) \neq 0 \,\}|_n < \frac{|(A^*)^{-k}B_r|_n}{N}$$

and therefore taking $j = k$ we obtain

$$|\{\, \mathbf{y} \in B_r : \widehat{g}(\mathbf{y}) \neq 0 \,\}|_n < \frac{|B_r|_n}{N}. \tag{14}$$

Letting $N \to \infty$ we obtain

$$|\{\, \mathbf{y} \in B_r : \widehat{g}(\mathbf{y}) \neq 0 \,\}|_n = 0.$$

Thus $\widehat{g} = 0$ a.e., and therefore $W^\perp = \{\, \mathbf{0} \,\}$.

Let us prove the implication $\mathbf{A_1} \Rightarrow \mathbf{B_1}$. Take any $r > 0$ and denote $E_j = \{\, \mathbf{t} \in (A^*)^{-j}B_r : \widehat{\phi}(\mathbf{t}) = 0 \,\}$ for $j \in \mathbb{N}$. By Lemma A it follows that

$$|\widehat{\phi}(\mathbf{t})| \leq 1 \qquad \text{a.e. on } \mathbb{R}^n; \tag{15}$$

hence,

$$\begin{aligned}|(A^*)^{-j}B_r|_n^{-1}\int_{(A^*)^{-j}B_r}|\widehat{\phi}(\mathbf{t})|^2d\mathbf{t} &= |(A^*)^{-j}B_r|_n^{-1}\int_{(A^*)^{-j}B_r\setminus E_j}|\widehat{\phi}(\mathbf{t})|^2d\mathbf{t}\\ &\leq |(A^*)^{-j}B_r|_n^{-1}(|(A^*)^{-j}B_r|_n - |E_j|_n).\end{aligned}$$

Applying Lemma 5 we obtain that

$$|(A^*)^{-j}B_r|_n^{-1}|E_j|_n \to 0 \quad \text{when } j \to \infty,$$

which finishes the proof.

To prove $\mathbf{A_1} \Rightarrow \mathbf{C_1}$ we have to show that there exists $E \subset \mathbb{R}^n$, $|E|_n > 0$, such that the origin is a point of A^*-density for the set E and

$$\lim_{\substack{\mathbf{y} \to \mathbf{0} \\ \mathbf{y} \in E}} |\widehat{\phi}(\mathbf{y})| = 1, \tag{16}$$

which is equivalent to the following.

For any $\varepsilon > 0$ and any $r > 0$,

$$\lim_{j\to\infty} \frac{|\{\, \mathbf{y} \in (A^*)^{-j} B_r : \big||\widehat{\phi}(\mathbf{y})| - 1\big| < \varepsilon \,\}|_n}{|(A^*)^{-j} B_r|_n} = 1.$$

If the implication $\mathbf{A_1} \Rightarrow \mathbf{C_1}$ is not true then, having in mind (15), we obtain that there exist $0 < \varepsilon_0 < 1$, $r_0 > 0$ and an increasing sequence of natural numbers $\{m_j\}_{j=1}^{\infty}$ such that

$$|\Gamma_j|_n = |\{\, \mathbf{y} \in (A^*)^{-m_j} B_{r_0} : |\widehat{\phi}(\mathbf{y})| < 1 - \varepsilon_0 \,\}|_n \geq \varepsilon_0 |(A^*)^{-m_j} B_{r_0}|_n.$$

By Lemma 1 and (15) we have

$$\begin{aligned}
1 &= \lim_{j\to\infty} |(A^*)^{-m_j} B_{r_0}|_n^{-1} \int_{(A^*)^{-m_j} B_{r_0}} |\widehat{\phi}(\mathbf{t})|^2 d\mathbf{t} \\
&= \lim_{j\to\infty} |(A^*)^{-m_j} B_{r_0}|_n^{-1} \left(\int_{(A^*)^{-m_j} B_{r_0} \setminus \Gamma_j} |\widehat{\phi}(\mathbf{t})|^2 d\mathbf{t} + \int_{\Gamma_j} |\widehat{\phi}(\mathbf{t})|^2 d\mathbf{t} \right) \\
&\leq \lim_{j\to\infty} |(A^*)^{-m_j} B_{r_0}|_n^{-1} \left(|(A^*)^{-m_j} B_{r_0}|_n - |\Gamma_j|_n + (1-\varepsilon_0)|\Gamma_j|_n \right) \\
&\leq \lim_{j\to\infty} |(A^*)^{-m_j} B_{r_0}|_n^{-1} \left(|(A^*)^{-m_j} B_{r_0}|_n - \varepsilon_0^2 |(A^*)^{-m_j} B_{r_0}|_n \right) \leq 1 - \varepsilon_0^2.
\end{aligned}$$

The obtained contradiction finishes the proof. The implication $\mathbf{C_1} \Rightarrow \mathbf{B_1}$ is trivial; hence, the proof of Theorem 1 is complete. □

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