TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 10, Pages 4089–4112
S 0002-9947(99)02185-6
Article electronically published on July 1, 1999

CHARACTERIZATION OF SMOOTHNESS OF MULTIVARIATE REFINABLE FUNCTIONS IN SOBOLEV SPACES

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Abstract. Wavelets are generated from refinable functions by using multiresolution analysis. In this paper we investigate the smoothness properties of multivariate refinable functions in Sobolev spaces. We characterize the optimal smoothness of a multivariate refinable function in terms of the spectral radius of the corresponding transition operator restricted to a suitable finite dimensional invariant subspace. Several examples are provided to illustrate the general theory.

1. Introduction

We are concerned with functional equations of the form

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(M \cdot -\alpha), \tag{1.1}$$

where ϕ is the unknown function defined on the s-dimensional Euclidean space $\mathbb{R}^s$, a is a finitely supported sequence on $\mathbb{Z}^s$, and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. The equation (1.1) is called a **refinement equation**, and the matrix M is called a **dilation matrix**. Correspondingly, the sequence a is called the **refinement mask**. Any function satisfying a refinement equation is called a **refinable function**.

Wavelets are generated from refinable functions. In [18], Jia and Micchelli discussed how to construct multivariate wavelets from refinable functions associated with a general dilation matrix. The approximation and smoothness properties of wavelets are determined by the corresponding refinable functions.

Our goal is to characterize the smoothness of a refinable function strictly in terms of the refinement mask. This information is important for our study of multivariate wavelets.

Before proceeding further, we introduce some notation. For $j = 1, \ldots, s$, let e_j be the jth coordinate unit vector in $\mathbb{R}^s$. The norm in $\mathbb{R}^s$ is defined by

$$|y| := |y_1| + \cdots + |y_s|, \qquad y = (y_1, \cdots, y_s) \in \mathbb{R}^s.$$

Received by the editors June 11, 1996 and, in revised form, April 14, 1997.

1991 *Mathematics Subject Classification.* Primary 42C15, 39B99, 46E35.

Key words and phrases. Refinement equations, refinable functions, wavelets, smoothness, regularity, approximation order, Sobolev spaces, Lipschitz spaces, subdivision operators, transition operators.

Supported in part by NSERC Canada under Grant OGP 121336.

The distance between two points x and y in $\mathbb{R}^s$ is defined by $\operatorname{dist}(x,y) := |x-y|$. Let E be a subset of $\mathbb{R}^s$. The distance from x to E is given by

$$\operatorname{dist}(x,E) := \inf\{\operatorname{dist}(x,y) : y \in E\}.$$

Let $\mathbb{N}_0$ denote the set of nonnegative integers. An element $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is called a multi-index. The length of μ is $|\mu| := \mu_1 + \cdots + \mu_s$, and the factorial of μ is $\mu! := \mu_1! \cdots \mu_s!$. For $j = 1, \ldots, s$, D_j denotes the partial derivative with respect to the jth coordinate. For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, D^μ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$. Moreover, p_μ denotes the monomial given by

$$p_\mu(x) := x_1^{\mu_1} \cdots x_s^{\mu_s}, \qquad x = (x_1, \ldots, x_s) \in \mathbb{R}^s.$$

The total degree of p_μ is $|\mu|$. For a nonnegative integer k, we denote by Π_k the linear span of $\{p_\mu : |\mu| \le k\}$.

We denote by $\ell(\mathbb{Z}^s)$ the linear space of all (complex-valued) sequences on $\mathbb{Z}^s$, and by $\ell_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequences on $\mathbb{Z}^s$. The difference operator ∇_j on $\ell(\mathbb{Z}^s)$ is defined by $\nabla_j a := a - a(\cdot - e_j)$, $a \in \ell(\mathbb{Z}^s)$. For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, ∇^μ is the difference operator $\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}$. We use δ to denote the sequence on $\mathbb{Z}^s$ given by $\delta(0) = 1$ and $\delta(\beta) = 0$ for all $\beta \in \mathbb{Z}^s \setminus \{0\}$. For $b \in \ell(\mathbb{Z}^s)$ and $c \in \ell_0(\mathbb{Z}^s)$, the (discrete) convolution of b with c is defined by

$$b*c(\alpha) := \sum_{\beta \in \mathbb{Z}^s} b(\alpha - \beta) c(\beta), \qquad \alpha \in \mathbb{Z}^s.$$

In particular, $b*\delta = b$ for $b \in \ell(\mathbb{Z}^s)$.

The **symbol** of an element $b \in \ell_0(\mathbb{Z}^s)$ is the Laurent polynomial $\tilde{b}(z)$ given by

$$\tilde{b}(z) := \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) z^\alpha, \qquad z \in (\mathbb{C} \setminus \{0\})^s.$$

The complex conjugate of a complex number z is denoted by $\overline{z}$. For $z = (z_1, \ldots, z_s)$ in $\mathbb{C}^s$ we write $\overline{z}$ for $(\overline{z}_1, \ldots, \overline{z_s})$. Let $\mathbb{T}^s$ denote the s-torus

$$\{(z_1, \ldots, z_s) \in \mathbb{C}^s : |z_1| = \cdots = |z_s| = 1\}.$$

Then $\overline{z} = z^{-1}$ for $z \in \mathbb{T}^s$. Consequently, if $b \in \ell_0(\mathbb{Z}^s)$ is real-valued, then we have $\overline{\tilde{b}(z)} = \tilde{b}(z^{-1})$ for $z \in \mathbb{T}^s$.

For $1 \le p \le \infty$, by $\ell_p(\mathbb{Z}^s)$ we denote the Banach space of all sequences b on $\mathbb{Z}^s$ such that $\|b\|_p < \infty$, where

$$\|b\|_p := \left(\sum_{\alpha \in \mathbb{Z}^s} |b(\alpha)|^p \right)^{1/p} \qquad \text{for } \ 1 \le p < \infty,$$

and $\|b\|_\infty$ is the supremum of b on $\mathbb{Z}^s$.

For $1 \le p \le \infty$, by $L_p(\mathbb{R}^s)$ we denote the Banach space of all (complex-valued) measurable functions f on $\mathbb{R}^s$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p \, dx \right)^{1/p} \qquad \textit{for } 1 \le p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of f on $\mathbb{R}^s$.

The Fourier transform of a function $f \in L_1(\mathbb{R}^s)$ is defined to be

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi} \, dx, \qquad \xi \in \mathbb{R}^s,$$

where $x \cdot \xi$ denotes the inner product of two vectors x and ξ in $\mathbb{R}^s$. The domain of the Fourier transform can be naturally extended to include compactly supported distributions.

If a satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = m := |\det M|, \tag{1.2}$$

then it is known that there exists a unique compactly supported distribution ϕ satisfying the refinement equation (1.1) subject to the condition $\hat{\phi}(0) = 1$. This distribution is said to be **the normalized solution** of the refinement equation (1.1). This fact was essentially proved by Cavaretta, Dahmen, and Micchelli in [2, Chap. 5] for the case in which the dilation matrix is 2 times the $s \times s$ identity matrix I. The same proof applies to the general refinement equation (1.1). Throughout this paper we assume that (1.2) is satisfied.

For $\nu \geq 0$, we denote by $W_2^\nu(\mathbb{R}^s)$ the Sobolev space of all functions $f \in L_2(\mathbb{R}^s)$ such that

$$\int_{\mathbb{R}^s} \big|\hat{f}(\xi)\big|^2 \big(1 + |\xi|^\nu\big)^2 \, d\xi < \infty.$$

The **critical exponent** of a function $f \in L_2(\mathbb{R}^s)$ (see [25]) is defined by

$$\nu(f) := \sup\big\{\nu : f \in W_2^\nu(\mathbb{R}^s)\big\}.$$

Sobolev spaces are related to Lipschitz spaces, which are defined on the basis of the modulus of smoothness. For $y \in \mathbb{R}^s$, the shift operator τ_y is given by

$$\tau_y f := f(\cdot - y),$$

and the difference operator ∇_y is given by

$$\nabla_y f := f - f(\cdot - y),$$

where f is a function defined on $\mathbb{R}^s$. The **modulus of continuity** of a function f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega(f, h)_p := \sup_{|y| \leq h} \big\|\nabla_y f\big\|_p, \qquad h \geq 0.$$

Let k be a positive integer. The kth **modulus of smoothness** of $f \in L_p(\mathbb{R}^s)$ is defined by

$$\omega_k(f, h)_p := \sup_{|y| \leq h} \big\|\nabla_y^k f\big\|_p, \qquad h \geq 0.$$

Thus, $\omega_1(f, \cdot)_p = \omega(f, \cdot)_p$ is the modulus of continuity.

For $1 \leq p \leq \infty$ and $0 < \nu \leq 1$, the Lipschitz space $\mathrm{Lip}(\nu, L_p(\mathbb{R}^s))$ consists of all functions $f \in L_p(\mathbb{R}^s)$ for which

$$\omega(f, h)_p \leq C\, h^\nu \qquad \forall\, h > 0,$$

where C is a positive constant independent of h. For $\nu > 0$ we write $\nu = r + \eta$, where r is an integer and $0 < \eta \leq 1$. The Lipschitz space $\mathrm{Lip}(\nu, L_p(\mathbb{R}^s))$ consists of those functions $f \in L_p(\mathbb{R}^s)$ for which $D^\mu f \in \mathrm{Lip}(\eta, L_p(\mathbb{R}^s))$ for all multi-indices μ with $|\mu| = r$. For $\nu > 0$, let k be an integer greater than ν. The generalized Lipschitz space $\mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s))$ consists of those functions $f \in L_p(\mathbb{R}^s)$ for which

$$\omega_k(f, h)_p \leq C\, h^\nu \qquad \forall\, h > 0,$$

where C is a positive constant independent of h. If $\nu > 0$ is not an integer, then

$$\mathrm{Lip}(\nu, L_p(\mathbb{R}^s)) = \mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s)), \qquad 1 \le p \le \infty.$$

See [6, Chap. 2] for a discussion on Lipschitz spaces.

It is well known that, for $\nu > \varepsilon > 0$, the inclusion relations

$$\mathrm{Lip}(\nu, L_2(\mathbb{R}^s)) \subseteq \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s)) \subseteq \mathrm{Lip}(\nu - \varepsilon, L_2(\mathbb{R}^s))$$

and

$$W_2^\nu(\mathbb{R}^s) \subseteq \mathrm{Lip}(\nu, L_2(\mathbb{R}^s)) \subseteq W_2^{\nu-\varepsilon}(\mathbb{R}^s)$$

hold true. See [23, Chap. V] for these facts. Therefore we have

$$\nu(f) = \sup\{\nu : f \in \mathrm{Lip}(\nu, L_2(\mathbb{R}^s))\} = \sup\{\nu : f \in \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))\}.$$

The concept of stability plays an important role in the study of the smoothness properties of refinable functions. Let ϕ be a compactly supported function in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. We say that the shifts of ϕ are **stable** if there are two positive constants C_1 and C_2 such that

$$C_1\|\lambda\|_p \le \left\| \sum_{\alpha\in\mathbb{Z}^s} \lambda(\alpha)\phi(\cdot - \alpha) \right\|_p \le C_2\|\lambda\|_p \qquad \forall\, \lambda \in \ell_0(\mathbb{Z}^s). \tag{1.3}$$

It was proved by Jia and Micchelli in [17] that a compactly supported function $\phi \in L_p(\mathbb{R}^s)$ satisfies the L_p-stability condition in (1.3) if and only if, for any $\xi \in \mathbb{R}^s$, there exists an element $\beta \in \mathbb{Z}^s$ such that

$$\hat{\phi}(\xi + 2\pi\beta) \ne 0.$$

If a compactly supported distribution ϕ satisfies this condition, then we still say that the shifts of ϕ are stable. In the binary case ($s = 1$ and $M = (2)$), Jia and Wang [19] gave a characterization for stability of a refinable function in terms of the refinement mask. Their results were extended by Zhou [27] to the case where the scaling factor m is an arbitrary integer greater than 1. In the case $s > 1$ and $M = 2I$, Hogan [11] gave a characterization of stability for a class of refinable functions in terms of the mask. When M is a general dilation matrix and the normalized solution ϕ of (1.1) lies in $L_2(\mathbb{R}^s)$, Lawton, Lee, and Shen [20] characterized stability of the shifts of ϕ in terms of eigenvalues and eigenvectors of a certain linear operator associated with the refinement mask.

Let us review the binary case where $s = 1$ and $M = (2)$. Denote by ϕ the normalized solution of the refinement equation with mask a. Under the condition that $\tilde{a}(e^{i\xi}) \ne 0$ for all $\xi \in (-\pi, \pi)$, Eirola in [7] established a formula for the critical exponent of ϕ. His results were improved by Villemoes in [25]. In this case, even the requirement for stability of ϕ can be relaxed (see [25] and [13]). Recently, Cohen and Daubechies [4] studied the regularity of refinable functions for the case where the refinement mask is not necessarily finitely supported.

The results in both [7] and [25] rely on factorization of the symbol of the mask. In the multivariate case, however, the symbol of the refinement mask is often irreducible. For example, let $s = 2$, $M = 2I$, and a the mask given by its symbol

$$\tilde{a}(z) := z_1^2 + z_2 + z_1z_2 + z_1z_2^2.$$

Then $\tilde{a}(z)$ is irreducible (see [16]). But the refinable function ϕ associated with a lies in $L_2(\mathbb{R}^2)$ and has stable shifts. In Section 4 we will show

$$\nu(\phi) = 1 - \log_4 3.$$

Because of the difficulty mentioned above, the existent literature did not provide decisive results for the smoothness analysis of *multivariate* refinable functions. Some special cases were studied by several authors, including Cohen and Daubechies [3], Villemoes [26], Goodman, Micchelli, and Ward [8], Dahlke, Dahmen, and Latour [5], and Riemenschneider and Shen [21].

The purpose of this paper is to provide a conclusive characterization for the smoothness of a multivariate refinable function in terms of the refinement mask and the dilation matrix. This goal will be achieved by employing my previous work [13]–[16] and my joint work with Han [10].

Our methods apply to isotropic dilation matrices. Let M be an $s\times s$ matrix with its entries in $\mathbb{C}$. We say that M is **isotropic** if M is similar to a diagonal matrix $\operatorname{diag}\{\lambda_1,\dots,\lambda_s\}$ with $|\lambda_1|=\cdots=|\lambda_s|$. For example, for $a,b\in\mathbb{R}$, the matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is isotropic.

For a compactly supported distribution ϕ on $\mathbb{R}^s$ and a sequence $b\in\ell(\mathbb{Z}^s)$, the **semi-convolution** of ϕ with b is defined by

$$\phi *' b := \sum_{\alpha\in\mathbb{Z}^s} \phi(\cdot-\alpha)b(\alpha).$$

Let $\mathbb{S}(\phi)$ denote the linear space $\{\phi *' b : b\in\ell(\mathbb{Z}^s)\}$. We call $\mathbb{S}(\phi)$ the **shift-invariant space** generated by ϕ.

Section 3 is devoted to a study of the subdivision and transition operators associated with the refinement equation (1.1). Let a be an element in $\ell_0(\mathbb{Z}^s)$ and let M be a dilation matrix. The **subdivision operator** S_a is the linear operator on $\ell(\mathbb{Z}^s)$ defined by

$$S_a u(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)u(\beta), \qquad \alpha\in\mathbb{Z}^s, \tag{1.4}$$

where $u\in\ell(\mathbb{Z}^s)$. The **transition operator** T_a is the linear operator on $\ell_0(\mathbb{Z}^s)$ defined by

$$T_a v(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(M\alpha - \beta)v(\beta), \qquad \alpha\in\mathbb{Z}^s, \tag{1.5}$$

where $v\in\ell_0(\mathbb{Z}^s)$.

In Section 2 we will establish the following results. Let M be an isotropic dilation matrix with $m=|\det M|$. Let $\nu>0$ and let k be a positive integer. If the normalized solution ϕ of (1.1) lies in $L_2(\mathbb{R}^s)$, and if there exists a constant $C>0$ such that

$$\left\|\nabla_j^k S_a^n \delta\right\|_2 \le C\, m^{(1/2-\nu/s)n} \qquad \forall\, n\in\mathbb{N} \text{ and } j=1,\dots,s, \tag{1.6}$$

then ϕ belongs to $\operatorname{Lip}^*(\nu, L_2(\mathbb{R}^s))$. Conversely, if ϕ lies in $\operatorname{Lip}^*(\nu, L_2(\mathbb{R}^s))$, and if the shifts of ϕ are stable, then (1.6) holds true for $k>\nu$. Note that Han and Jia [10] have already given a characterization for a refinable function to belong to $L_p(\mathbb{R}^s)$, provided it has stable shifts.

In order to apply the preceding results, the integer k should be chosen appropriately. In Section 4, we will demonstrate that k should be chosen to be the largest integer such that $\mathbb{S}(\phi)$ contains Π_{k-1}. This choice of k is related to the approximation order provided by $\mathbb{S}(\phi)$. Moreover, if the shifts of ϕ are stable, then k can be easily determined by checking the order of the so-called sum rules satisfied by the refinement mask (see [15] and [16]).

It turns out that the critical exponent of the normalized solution of (1.1) can be computed by finding the spectral radius of the corresponding transition operator restricted on a suitable invariant subspace. For an integer $k \geq 0$, let

$$V_k := \Big\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha \in \mathbb{Z}^s} p(\alpha) v(\alpha) = 0 \quad \forall\, p \in \Pi_k \Big\}.$$

Choose k to be the largest integer such that $\mathbb{S}(\phi) \supset \Pi_{k-1}$. If the shifts of ϕ are stable, then V_{k-1} is an invariant subspace of T_a. Moreover, let $b := a * a^*$, where a^* is the sequence given by $a^*(\alpha) = \overline{a(-\alpha)}$, $\alpha \in \mathbb{Z}^s$. Then V_{2k-1} is invariant under T_b. Let ρ be the spectral radius of $T_b|_{V_{2k-1}}$. In Section 4, we will establish the following formula for the critical exponent of ϕ:

$$\nu(\phi) = (1 - \log_m \rho) s/2. \tag{1.7}$$

Finally, in Section 5, we will provide several examples to illustrate the general theory. In particular, one example shows that (1.7) may fail to hold if the stability condition is not satisfied.

In a forthcoming paper we will give a comprehensive study of the smoothness of multivariate refinable functions in Besov spaces.

2. Characterization of smoothness

In this section we give a characterization for the smoothness of a refinable function in terms of the refinement mask. Our characterization is based on the following theorem.

Theorem 2.1. *Let $\nu > 0$ and let k be a positive integer. Let M be an isotropic dilation matrix with $m = |\det M|$. If $\phi \in Lip^*(\nu, L_2(\mathbb{R}^s))$ and $k > \nu$, then there exists a constant $C > 0$ such that*

$$\big\|\nabla^k_{M^{-n}e_j}\phi\big\|_2 \leq C\,(m^{-\nu/s})^n \qquad \forall\, n \in \mathbb{N} \text{ and } j = 1, \ldots, s. \tag{2.1}$$

Conversely, if a function $\phi \in L_2(\mathbb{R}^s)$ satisfies the conditions in (2.1), then ϕ belongs to $Lip^*(\nu, L_2(\mathbb{R}^s))$.

The proof of this theorem is elementary but technical. Thus, we postpone its proof to the end of this section.

Given an element $a \in \ell_0(\mathbb{Z}^s)$ and a dilation matrix M, let S_a be the subdivision operator given by (1.4). If ϕ satisfies the refinement equation (1.1), then

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} S_a^n \delta(\alpha)\, \phi(M^n \cdot - \alpha). \tag{2.2}$$

This can be verified by induction on n. When $n = 1$, (2.2) is just the refinement equation (1.1). Suppose $n > 1$ and (2.2) is valid for $n - 1$. Then

$$\begin{aligned}\phi &= \sum_{\beta\in\mathbb{Z}^s} S_a^{n-1}\delta(\beta)\phi(M^{n-1}\cdot -\beta)\\ &= \sum_{\beta\in\mathbb{Z}^s} S_a^{n-1}\delta(\beta)\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(M^n\cdot -M\beta-\alpha)\\ &= \sum_{\alpha\in\mathbb{Z}^s}\left[\sum_{\beta\in\mathbb{Z}^s} a(\alpha-M\beta)S_a^{n-1}\delta(\beta)\right]\phi(M^n\cdot -\alpha)\\ &= \sum_{\alpha\in\mathbb{Z}^s} S_a^n\delta(\alpha)\,\phi(M^n\cdot -\alpha).\end{aligned}$$

This completes the induction procedure.

The following theorem gives a characterization of the smoothness of a refinable function.

Theorem 2.2. *Let M be an isotropic dilation matrix with $m = |\det M|$. Let ϕ be the normalized solution of the refinement equation*

$$\phi = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(M\cdot -\alpha),$$

where $a \in \ell_0(\mathbb{Z}^s)$ with $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m$. Let $\nu > 0$ and let k be a positive integer. If $\phi \in L_2(\mathbb{R}^s)$, and if there exists a constant $C > 0$ such that

$$\|\nabla_j^k S_a^n\delta\|_2 \le C\, m^{(1/2-\nu/s)n} \qquad \forall\, n\in\mathbb{N} \text{ and } j = 1,\ldots,s, \tag{2.3}$$

then $\phi \in Lip^*(\nu, L_2(\mathbb{R}^s))$. Conversely, if $\phi \in Lip^*(\nu, L_2(\mathbb{R}^s))$, and if the shifts of ϕ are stable, then (2.3) holds true for $k > \nu$.

Proof. The proof is based on Theorem 2.1. Suppose the mask a satisfies (2.3). It follows from (2.2) that

$$\nabla_{M^{-n}e_j}^k\phi = \sum_{\alpha\in\mathbb{Z}^s}\nabla_j^k S_a^n\delta(\alpha)\,\phi(M^n\cdot -\alpha). \tag{2.4}$$

Since $\phi \in L_2(\mathbb{R}^s)$ is compactly supported, there exists a positive constant C_1 independent of n and j such that

$$\|\nabla_{M^{-n}e_j}^k\phi\|_2 \le C_1 m^{-n/2}\|\nabla_j^k S_a^n\delta\|_2.$$

This in connection with (2.3) tells us that (2.1) holds true. Thus, by Theorem 2.1, ϕ belongs to $\mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))$.

Conversely, suppose the shifts of ϕ are stable. It follows from (2.4) that

$$m^{-n/2}\|\nabla_j^k S_a^n\delta\|_2 \le C_2\|\nabla_{M^{-n}e_j}^k\phi\|_2, \tag{2.5}$$

where C_2 is a constant independent of n and j. If $\phi \in \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))$, then (2.1) is valid for $k > \nu$, by Theorem 2.1. Therefore, (2.3) follows from (2.1) and (2.5). $\square$

The proof of Theorem 2.1 is based on the following two lemmas.

Lemma 2.3. *Suppose $v_1,\ldots,v_k$ are vectors in $\mathbb{R}^s$, and $t_1,\ldots,t_k$ are positive real numbers such that $t_1\cdots t_k = 1$. Then for every $f \in L_2(\mathbb{R}^s)$,*

$$\|\nabla_{v_1}\cdots\nabla_{v_k}f\|_2^2 \le \frac{1}{k}\sum_{j=1}^k t_j^2\|\nabla_{v_j}^k f\|_2^2.$$

Proof. We observe that, for $v \in \mathbb{R}^s$,

$$(\nabla_v f)\hat{}(\xi) = \left(1 - e^{-iv\cdot\xi}\right)\hat{f}(\xi), \qquad \xi \in \mathbb{R}^s.$$

Thus, by Parseval's identity we get

$$\left\|\nabla_{v_1}\cdots\nabla_{v_k} f\right\|_2^2 = (2\pi)^{-s}\int_{\mathbb{R}^s}\left[\prod_{j=1}^k \left|1 - e^{-iv_j\cdot\xi}\right|^2\right]|\hat{f}(\xi)|^2\, d\xi.$$

It is well known that the geometric mean does not exceed the arithmetic mean; hence $t_1\cdots t_k = 1$ implies that

$$\prod_{j=1}^k \left|1 - e^{-iv_j\cdot\xi}\right|^2 = \left[\prod_{j=1}^k\left(t_j^2\left|1 - e^{-iv_j\cdot\xi}\right|^{2k}\right)\right]^{1/k} \le \frac{1}{k}\sum_{j=1}^k t_j^2\left|1 - e^{-iv_j\cdot\xi}\right|^{2k}.$$

Consequently, we obtain

$$\left\|\nabla_{v_1}\cdots\nabla_{v_k} f\right\|_2^2 \le \frac{1}{k}\sum_{j=1}^k t_j^2\,(2\pi)^{-s}\int_{\mathbb{R}^s}\left|1 - e^{-iv_j\cdot\xi}\right|^{2k}|\hat{f}(\xi)|^2\,d\xi = \frac{1}{k}\sum_{j=1}^k t_j^2\left\|\nabla_{v_j}^k f\right\|_2^2,$$

as desired. □

Lemma 2.4. *Let M be an isotropic matrix with spectral radius $\sigma \ne 0$. For any vector norm $\|\cdot\|$ on $\mathbb{R}^s$, there exist two positive constants C_1 and C_2 such that the inequalities*

$$C_1\sigma^n\|v\| \le \|M^n v\| \le C_2\sigma^n\|v\|$$

hold true for every integer n and every vector $v \in \mathbb{R}^s$.

Proof. Since M is isotropic, M is similar to a diagonal matrix $\operatorname{diag}\{\lambda_1,\dots,\lambda_s\}$ with $|\lambda_1| = \cdots = |\lambda_s| = \sigma$. Hence, we can find a basis $\{v_1,\dots,v_s\}$ for $\mathbb{C}^s$ such that $Mv_j = \lambda_j v_j$. Recall that two norms on a finite dimensional linear space are equivalent. Hence there exist two positive constants C_1 and C_2 such that

$$C_1\sum_{j=1}^s |a_j| \le \|v\| \le C_2\sum_{j=1}^s|a_j| \qquad \text{for } v = \sum_{j=1}^s a_j v_j.$$

But for $v = \sum_{j=1}^s a_j v_j$ we have $M^n v = \sum_{j=1}^s a_j\lambda_j^n v_j$. It follows that

$$\|M^n v\| \le C_2\sum_{j=1}^s |a_j\lambda_j^n| = C_2\sigma^n\sum_{j=1}^s|a_j| \le C_2 C_1^{-1}\sigma^n\|v\|$$

and

$$\|M^n v\| \ge C_1\sum_{j=1}^s |a_j\lambda_j^n| = C_1\sigma^n\sum_{j=1}^s|a_j| \ge C_1 C_2^{-1}\sigma^n\|v\|.$$

This completes the proof of the lemma. □

Proof of Theorem 2.1. Since M is isotropic, its spectral radius is $\sigma := m^{1/s}$. If $k > \nu$ and $\phi \in \operatorname{Lip}^*(\nu, L_2(\mathbb{R}^s))$, then there exists a constant $C_1 > 0$ such that

$$\left\|\nabla_{M^{-n}e_j}^k\phi\right\|_2 \le C_1|M^{-n}e_j|^\nu \qquad \forall\, n\in\mathbb{N} \text{ and } j = 1,\dots,s.$$

By Lemma 2.4, there exists a constant $C_2 > 0$ such that

$$|M^{-n}e_j|^\nu \le C_2(\sigma^{-n})^\nu \qquad \forall\, n\in\mathbb{N} \text{ and } j = 1,\dots,s.$$

Therefore, it follows that

$$\left\|\nabla^k_{M^{-n}e_j}\phi\right\|_2 \le C_1C_2(m^{-\nu/s})^n \qquad \forall\, n\in\mathbb{N} \text{ and } j=1,\dots,s.$$

This verifies (2.1).

Conversely, suppose (2.1) is true for a function $\phi\in L_2(\mathbb{R}^s)$. We wish to prove $\phi\in \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))$. Let y be a nonzero vector in $\mathbb{R}^s$. We choose $\alpha_j\in\mathbb{Z}^s$ $(j=1,2,\dots)$ inductively as follows. Choose $\alpha_1\in\mathbb{Z}^s$ such that $|My-\alpha_1| = \mathrm{dist}(My,\mathbb{Z}^s)$. Suppose $\alpha_1,\dots,\alpha_j$ have been chosen. Let $y_j := M^jy-(M^{j-1}\alpha_1+\cdots+\alpha_j)$ and choose $\alpha_{j+1}\in\mathbb{Z}^s$ such that $|My_j-\alpha_{j+1}| = \mathrm{dist}(My_j,\mathbb{Z}^s)$. By our choice of α_j $(j=1,2,\dots)$ we have $|y_j|\le 1$, and hence $|\alpha_{j+1}|\le |My_j|+1$ for $j=1,2,\dots$. Let $N:=\sup\{|Mu| : |u|\le 1\}$. Then $N<\infty$ and $|\alpha_{j+1}|\le N+1$ for $j=1,2,\dots$. Since $\lim_{n\to\infty} M^{-n}=0$, the vector y has the following representation:

$$y=\sum_{j=1}^{\infty} M^{-j}\alpha_j. \tag{2.6}$$

Let n be the smallest positive integer such that $\alpha_n\ne 0$. Then there exists some r, $1\le r\le n$, such that $|M^ry|\ge 1/2$, for otherwise we would have $\alpha_1=\cdots=\alpha_n=0$ by our choice of α_j, $j=1,2\dots$. By Lemma 2.4, there exists a constant $C_1>0$ such that $|M^jv|\le C_1\sigma^j|v|$ for all $v\in\mathbb{R}^s$ and all $j=1,2,\dots$. Consequently,

$$1/2\le |M^ry|\le C_1\sigma^r|y|\le C_1\sigma^n|y|.$$

In other words,

$$\sigma^{-n}\le 2C_1|y|. \tag{2.7}$$

Write v_j for $M^{-j}\alpha_j$, $j=1,2,\dots$. Since $\alpha_1=\cdots=\alpha_{n-1}=0$, (2.6) implies that

$$\nabla_y\phi=\sum_{j=n}^{\infty} Q_j\phi,$$

where $Q_j:=\tau_{u_j}\nabla_{v_j}$ with $u_n:=0$ and $u_j:=\sum_{\ell=n+1}^{j} v_{\ell-1}$ for $j>n$. It follows that

$$\left\|\nabla^k_y\phi\right\|_2\le \sum_{(j_1,\dots,j_k)\in\mathbb{N}_n^k} \left\|Q_{j_1}\cdots Q_{j_k}\phi\right\|_2,$$

where $\mathbb{N}_n$ denotes the set $\{n,n+1,\dots\}$. Note that

$$\left\|Q_{j_1}\cdots Q_{j_k}\phi\right\|_2 = \left\|\nabla_{v_{j_1}}\cdots\nabla_{v_{j_k}}\phi\right\|_2.$$

For $r=1,\dots,k$, let

$$t_r:=\sigma^{-\nu(-j_r+(j_1+\cdots+j_k)/k)}. \tag{2.8}$$

Clearly, $t_1\cdots t_k=1$. By Lemma 2.3 we have

$$\left\|\nabla_{v_{j_1}}\dots\nabla_{v_{j_k}}\phi\right\|_2^2\le \frac{1}{k}\sum_{r=1}^{k} t_r^2\left\|\nabla^k_{v_{j_r}}\phi\right\|_2^2. \tag{2.9}$$

But $v_j=M^{-j}\alpha_j$ for $\alpha_j\in\mathbb{Z}^s$ with $|\alpha_j|\le N+1$. By Lemma 2.3 it follows from (2.1) that

$$\left\|\nabla^k_{v_j}\phi\right\|_2\le C_2\,\sigma^{-\nu j},$$

where C_2 is a constant independent of j. This together with (2.8) and (2.9) yields

$$\left\|\nabla_{v_{j_1}}\dots\nabla_{v_{j_k}}\phi\right\|_2^2\le C_2^2\sigma^{-2\nu(j_1+\cdots+j_k)/k}.$$

To summarize, we have proved that

$$\left\|\nabla_y^k \phi\right\|_2 \le C_2 \sum_{(j_1,\dots,j_k)\in\mathbb{N}_n^k} \sigma^{-\nu(j_1+\cdots+j_k)/k}. \tag{2.10}$$

The sum on the right-hand side of (2.10) can be computed as follows:

$$\sum_{(j_1,\dots,j_k)\in\mathbb{N}_n^k} \sigma^{-\nu(j_1+\cdots+j_k)/k} = \left(\sum_{j=n}^{\infty} \sigma^{-\nu j/k}\right)^k = \sigma^{-\nu n}\big/(1-\sigma^{-\nu/k})^k.$$

Thus, there exists a positive constant C_3 independent of y such that

$$\left\|\nabla_y^k \phi\right\|_2 \le C_3 \sigma^{-\nu n}.$$

By (2.7) we have $\sigma^{-n} \le 2C_1|y|$; hence there exists a constant $C > 0$ such that

$$\left\|\nabla_y^k \phi\right\|_2 \le C\,|y|^{\nu} \qquad \forall\, y \in \mathbb{R}^s.$$

This shows $\phi \in \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))$, as desired. □

3. Spectral radius

Theorem 2.2 gives a characterization for the smoothness of a refinable function ϕ in terms of the refinement mask a. In order to calculate the critical exponent of ϕ efficiently, we need to compute the limit

$$\lim_{n\to\infty} \left\|\nabla_j^k S_a^n \delta\right\|_2^{1/n}$$

for $j = 1, \dots, s$. This problem has been solved by Han and Jia in [10]. But the results in [10] rely on the joint spectral radius of certain linear operators on a finite dimensional linear space. In the present setting, only the smoothness in the L_2-norm is concerned. So we can give a self-contained exposition of this topic without using the joint spectral radius.

The support of a distribution ϕ on $\mathbb{R}^s$ is denoted by $\operatorname{supp}\phi$. For an element $b \in \ell_0(\mathbb{Z}^s)$, its support is defined by

$$\operatorname{supp} b := \{\alpha \in \mathbb{Z}^s : b(\alpha) \ne 0\}.$$

For a bounded subset Ω of $\mathbb{R}^s$, we denote by $\ell(\Omega)$ the linear subspace of $\ell_0(\mathbb{Z}^s)$ consisting of all sequences supported on $\Omega \cap \mathbb{Z}^s$.

For $\beta \in \mathbb{Z}^s$, we denote by δ_β the sequence on $\mathbb{Z}^s$ given by

$$\delta_\beta(\alpha) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \in \mathbb{Z}^s \setminus \{\beta\}. \end{cases}$$

The shift operator τ^β on $\ell(\mathbb{Z}^s)$ is defined by

$$\tau^\beta u = u(\cdot - \beta), \qquad u \in \ell(\mathbb{Z}^s).$$

An element $v \in \ell_0(\mathbb{Z}^s)$ induces the Laurent polynomial $\tilde{v}(z) = \sum_{\alpha\in\mathbb{Z}^s} v(\alpha) z^\alpha$, which in turn induces the difference operator

$$\tilde{v}(\tau) = \sum_{\alpha\in\mathbb{Z}^s} v(\alpha)\tau^\alpha.$$

For a given element $a \in \ell_0(\mathbb{Z}^s)$ and a dilation matrix M, let T_a be the transition operator given by (1.5). The following lemma gives some information about the spectral properties of the transition operator T_a.

Lemma 3.1. *Suppose that a is an element of $\ell_0(\mathbb{Z}^s)$ and M is a dilation matrix. Let $H := \operatorname{supp} a$ and*

$$\Omega := \sum_{n=1}^{\infty} M^{-n}H := \left\{ \sum_{n=1}^{\infty} M^{-n}h_n : h_n \in H \quad \forall\, n \in \mathbb{N} \right\}. \tag{3.1}$$

Then Ω has the following properties:

(a) $\operatorname{supp}\phi \subseteq \Omega$*,*

(b) $\ell(\Omega)$ is invariant under the transition operator T_a, and

(c) for any $v \in \ell_0(\mathbb{Z}^s)$, there exists some integer r such that $T_a^r v \in \ell(\Omega)$.

Proof. To prove (a) we deduce from (1.1) that $x \in \operatorname{supp}\phi$ implies $Mx - \alpha \in \operatorname{supp}\phi$ for some $\alpha \in \operatorname{supp} a = H$. Hence $x \in M^{-1}\operatorname{supp}\phi + M^{-1}H$. In other words,

$$\operatorname{supp}\phi \subseteq M^{-1}H + M^{-1}\operatorname{supp}\phi.$$

Iterating the above relation n times, we get

$$\operatorname{supp}\phi \subseteq M^{-1}H + \cdots + M^{-n}H + M^{-n}\operatorname{supp}\phi.$$

Since $\lim_{n\to\infty} M^{-n} = 0$, it follows that

$$\operatorname{supp}\phi \subseteq \sum_{n=1}^{\infty} M^{-n}H = \Omega.$$

In order to verify (b) we pick an element $v \in \ell(\Omega)$ and observe that $T_a v(\alpha) \neq 0$ implies that $a(M\alpha - \beta) \neq 0$ for some $\beta \in \operatorname{supp} v$. Hence

$$\operatorname{supp}(T_a v) \subseteq M^{-1}H + M^{-1}\Omega = \Omega.$$

This shows that $\ell(\Omega)$ is invariant under T_a.

Finally, for an element $v \in \ell_0(\mathbb{Z}^s)$, we have

$$\operatorname{supp}(T_a v) \subseteq M^{-1}H + M^{-1}\operatorname{supp} v.$$

Iterating the above relation n times, we obtain

$$\operatorname{supp}(T_a^n v) \subseteq M^{-1}H + \cdots + M^{-n}H + M^{-n}\operatorname{supp} v.$$

Note that Ω is a compact set, and so $d := \operatorname{dist}(\Omega, \mathbb{Z}^s \setminus \Omega)$ is positive. Since $\lim_{n\to\infty} M^{-n} = 0$, there exists a positive integer r such that $\operatorname{dist}(\alpha, \Omega) < d$ for all $\alpha \in \operatorname{supp}(T_a^r v)$. For this r, we have $\alpha \in \Omega$ for all $\alpha \in \operatorname{supp}(T_a^r v)$. Hence $T_a^r v \in \ell(\Omega)$. ☐

Let us draw several useful consequences from Lemma 3.1. If $v \in \ell_0(\mathbb{Z}^s)$ is an eigenvector of T_a corresponding to an eigenvalue σ, then $\sigma^r v = T_a^r v \in \ell(\Omega)$ for sufficiently large r. Hence $\sigma \neq 0$ implies $v \in \ell(\Omega)$, and $v \notin \ell(\Omega)$ implies $\sigma = 0$. This shows that T_a only has finitely many nonzero eigenvalues. Moreover, any eigenvector of T_a corresponding to a nonzero eigenvalue is supported in Ω. For an invariant subspace V of T_a we define the spectral radius of $T_a|_V$ by

$$\rho(T_a|_V) := \rho(T_a|_{\ell(\Omega)\cap V}).$$

In particular, $\rho(T_a) := \rho(T_a|_{\ell(\Omega)})$. Note that the subdivision operator S_a and the transition operator T_a have the same nonzero eigenvalues (see [16]).

If U is a finite subset of $\ell_0(\mathbb{Z}^s)$, then the minimal invariant subspace of T_a generated by U is finite dimensional. To see this, let $E := \bigcup_{u\in U} \operatorname{supp} u$, $G := ME \cup H \cup \{0\}$, and $K := \bigcup_{n=1}^{\infty} M^{-n}G$. Then $\ell(K)$ is a finite dimensional invariant subspace of T_a containing U.

Lemma 3.2. *Let a and v be two elements of $\ell_0(\mathbb{Z}^s)$. Then for $n = 1, 2, \ldots,$*

$$T_a^n v(\alpha) = \tilde{v}(\tau) S_a^n \delta(M^n \alpha) \qquad \forall\, \alpha \in \mathbb{Z}^s. \tag{3.2}$$

Consequently,

$$\|T_a^n v\|_\infty \le \|\tilde{v}(\tau) S_a^n \delta\|_\infty. \tag{3.3}$$

Proof. The proof proceeds by induction on n. For $n = 1$ and $\alpha \in \mathbb{Z}^s$ we have

$$T_a v(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(M\alpha - \beta) v(\beta) = \sum_{\beta \in \mathbb{Z}^s} v(\beta) S_a \delta(M\alpha - \beta) = \tilde{v}(\tau) S_a \delta(M\alpha).$$

This verifies (3.2) for $n = 1$. Suppose $n > 1$ and (3.2) is valid for $n - 1$. Then for $\alpha \in \mathbb{Z}^s$ we have

$$\begin{aligned} T_a^n v(\alpha) &= T_a^{n-1}(T_a v)(\alpha) \\ &= \sum_{\gamma \in \mathbb{Z}^s} (T_a v)(\gamma) S_a^{n-1} \delta(M^{n-1}\alpha - \gamma) \\ &= \sum_{\gamma \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} a(M\gamma - \beta) v(\beta) S_a^{n-1} \delta(M^{n-1}\alpha - \gamma) \\ &= \sum_{\beta \in \mathbb{Z}^s} v(\beta) \sum_{\gamma \in \mathbb{Z}^s} a(M^n\alpha - M\gamma - \beta) S_a^{n-1} \delta(\gamma) \\ &= \sum_{\beta \in \mathbb{Z}^s} v(\beta) S_a^n \delta(M^n \alpha - \beta) \\ &= \tilde{v}(\tau) S_a^n \delta(M^n \alpha). \end{aligned}$$

This completes the induction procedure. □

Theorem 3.3. *Let M be an $s \times s$ dilation matrix. For $a \in \ell_0(\mathbb{Z}^s)$, let $b := a * a^*$, where $a^*(\alpha) = \overline{a(-\alpha)}$ for $\alpha \in \mathbb{Z}^s$. Then for $v \in \ell_0(\mathbb{Z}^s)$,*

$$\lim_{n \to \infty} \|\tilde{v}(\tau) S_a^n \delta\|_2^{1/n} = \sqrt{\rho(T_b|_W)}$$

where W is the minimal T_b-invariant subspace generated by $w := v * v^*$.

Proof. For $n = 1, 2, \ldots,$ write a_n for $S_a^n \delta$ and b_n for $S_b^n \delta$. Note that the symbol of $\tilde{v}(\tau) a_n$ is $\tilde{v}(z) \tilde{a}_n(z)$, and the symbol of $\widetilde{w}(\tau) b_n$ is $\widetilde{w}(z) \tilde{b}_n(z)$. Moreover, for $z \in \mathbb{T}^s$ we have $\widetilde{w}(z) \tilde{b}_n(z) = |\tilde{v}(z) \tilde{a}_n(z)|^2$. By the Parseval identity we obtain

$$\|\tilde{v}(\tau) a_n\|_2^2 = \frac{1}{(2\pi)^s} \int_{[0,2\pi)^s} \left|\tilde{v}(e^{i\xi}) \tilde{a}_n(e^{i\xi})\right|^2 d\xi = \frac{1}{(2\pi)^s} \int_{[0,2\pi)^s} \widetilde{w}(e^{i\xi}) \tilde{b}_n(e^{i\xi})\, d\xi.$$

Since $\widetilde{w}(e^{i\xi}) \tilde{b}_n(e^{i\xi}) \ge 0$ for all $\xi \in \mathbb{R}^s$, it follows that

$$\widetilde{w}(\tau) b_n(0) \le \|\widetilde{w}(\tau) b_n\|_\infty \le \frac{1}{(2\pi)^s} \int_{[0,2\pi)^s} \widetilde{w}(e^{i\xi}) \tilde{b}_n(e^{i\xi})\, d\xi = \widetilde{w}(\tau) b_n(0).$$

This in connection with (3.3) yields

$$\widetilde{w}(\tau) b_n(0) = T_b^n w(0) \le \|T_b^n w\|_\infty \le \|\widetilde{w}(\tau) b_n\|_\infty = \widetilde{w}(\tau) b_n(0).$$

Since W is the minimal T_b-invariant subspace generated by w, we obtain

$$\rho(T_b|_W) = \lim_{n \to \infty} \|T_b^n w\|_\infty^{1/n} = \lim_{n \to \infty} \left[\widetilde{w}(\tau) b_n(0)\right]^{1/n} = \lim_{n \to \infty} \left\|\tilde{v}(\tau) a_n\right\|_2^{2/n},$$

as desired. □

We remark that Goodman, Micchelli, and Ward in [8] established a result similar to Theorem 3.3 for the special case $v = \delta$.

For $j = 1, \ldots, s$, let Δ_j denote the difference operator on $\ell_0(\mathbb{Z}^s)$ given by

$$\Delta_j u := 2u - u(\cdot - e_j) - u(\cdot + e_j), \qquad u \in \ell_0(\mathbb{Z}^s).$$

Theorem 3.4. *Suppose that the normalized solution ϕ of the refinement equation (1.1) lies in $L_2(\mathbb{R}^s)$. Let $b := a*a^*$ and, for a positive integer k, let*

$$\rho := \max\bigl\{\rho(T_b|_{W_j}) : j = 1, \ldots, s\bigr\},$$

where W_j is the minimal T_b-invariant subspace generated by $\Delta_j^k \delta$. Then

$$\nu(\phi) \geq \nu := (1 - \log_m \rho)s/2.$$

Moreover, if $k > \nu$, and if the shifts of ϕ are stable, then $\nu(\phi) = \nu$.

Proof. We observe that $\rho = m^{1-2\nu/s}$ and $(\nabla_j\delta)*(\nabla_j\delta)^* = \Delta_j\delta$. Write ρ_j for $\rho(T_b|_{W_j})$, $j = 1, \ldots, s$. By Theorem 3.3, we have

$$\lim_{n\to\infty} \bigl\|\nabla_j^k S_a^n \delta\bigr\|_2^{1/n} = \rho_j^{1/2} \leq \rho^{1/2} = m^{1/2-\nu/s}.$$

Thus, for any given $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\bigl\|\nabla_j^k S_a^n \delta\bigr\|_2 \leq C\, m^{(1/2-(\nu-\varepsilon)/s)n} \qquad \forall\, n \in \mathbb{N} \text{ and } j = 1, \ldots, s.$$

By Theorem 2.2, ϕ belongs to $\mathrm{Lip}^*(\nu - \varepsilon, L_2(\mathbb{R}^s))$. Since $\varepsilon > 0$ can be arbitrary, we conclude that $\nu(\phi) \geq \nu$.

Now suppose that $k > \nu$ and ϕ has stable shifts. If $\nu(\phi) > \nu$, then there exists some $\varepsilon > 0$ such that $k > \nu + \varepsilon$ and $\phi \in \mathrm{Lip}^*(\nu + \varepsilon, L_2(\mathbb{R}^s))$. By Theorem 2.2, there exists a constant $C > 0$ such that

$$\bigl\|\nabla_j^k S_a^n \delta\bigr\|_2 \leq C\, m^{(1/2-(\nu+\varepsilon)/s)n} \qquad \forall\, n \in \mathbb{N} \text{ and } j = 1, \ldots, s.$$

By Theorem 3.3, it follows that

$$\rho = \max_{j=1,\ldots,s}\{\rho_j\} = \max_{j=1,\ldots,s}\Bigl\{\lim_{n\to\infty} \bigl\|\nabla_j^k S_a^n \delta\bigr\|_2^{2/n}\Bigr\} \leq m^{1-2(\nu+\varepsilon)/s}.$$

On the other hand, $\nu = (1 - \log_m \rho)s/2$ implies $\rho = m^{1-2\nu/s}$. This contradiction shows that $\nu(\phi) = \nu$. $\square$

4. Invariant subspaces

In order to use Theorem 3.4 to calculate the critical exponent of a refinable function ϕ, it is important to choose an appropriate k. Let ρ and ν be given as in Theorem 3.4. If k is chosen so small that $k = \nu$, then ν might not be the optimal smoothness.

This problem is related to the approximation order provided by $\mathbb{S}(\phi)$, the shift-invariant space generated by ϕ. The reader is referred to [15] for a recent survey on approximation by shift-invariant spaces.

Let $S := \mathbb{S}(\phi) \cap L_p(\mathbb{R}^s)$. For $h > 0$, let $S^h := \{g(\cdot/h) : g \in S\}$. For a real number $\kappa \geq 0$, we say that $\mathbb{S}(\phi)$ provides **approximation order** κ if for each sufficiently smooth function f in $L_p(\mathbb{R}^s)$, there exists a constant $C > 0$ such that

$$\inf_{g\in S^h} \|f - g\|_p \leq C\, h^\kappa \qquad \forall\, h > 0.$$

Let $1 \le p \le \infty$, let k be a positive integer, and ϕ a compactly supported function in $L_p(\mathbb{R}^s)$ with $\hat{\phi}(0) \ne 0$. It was proved by Jia [12] that $\mathbb{S}(\phi)$ provides approximation order k if and only if $\mathbb{S}(\phi)$ contains Π_{k-1}.

Smooth refinable functions provide good approximation orders. This fact was observed by Cavaretta, Dahmen, and Micchelli in [2]. Their work was extended by Ron [22] to multiple refinable functions. The following theorem, established by Jia in [16], deals with refinable functions associated with isotropic dilation matrices.

Theorem 4.1. *Suppose M is an $s \times s$ isotropic dilation matrix, and a is an element in $\ell_0(\mathbb{Z}^s)$ satisfying (1.2). Let ϕ be the normalized solution of the refinement equation (1.1).*

If $\phi \in W_2^k(\mathbb{R}^s)$, then $\Pi_k \subset \mathbb{S}(\phi)$ and $\mathbb{S}(\phi)$ provides approximation order $k+1$.

Now it is clear how to choose an appropriate k in Theorem 3.4. We should choose k to be the largest integer such that $\mathbb{S}(\phi)$ contains Π_{k-1}. Indeed, if k is chosen in such a way, then $\nu(\phi) \le k$, by Theorem 4.1. Let ρ and ν be given as in Theorem 3.4. Then $\nu \le \nu(\phi) \le k$; hence $\nu(\phi) = \nu$, provided the shifts of ϕ are stable.

The approximation order provided by a refinable function ϕ can be easily determined by checking the order of the so-called sum rules satisfied by the refinement mask. For an $s \times s$ dilation matrix M, let Γ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$. Let k be a positive integer. An element $a \in \ell_0(\mathbb{Z}^s)$ is said to satisfy the **sum rules** of order k if, for all $p \in \Pi_{k-1}$,

$$\sum_{\beta\in\mathbb{Z}^s} a(M\beta)\, p(M\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)\, p(M\beta+\gamma) \qquad \forall\, \gamma \in \Gamma.$$

The following results were established in [16].

Theorem 4.2. *Let ϕ be the normalized solution of the refinement equation (1.1) with the dilation matrix M and the mask a. If the refinement mask a satisfies the sum rules of order k, then $\mathbb{S}(\phi)$ contains Π_{k-1}. Conversely, if $\mathbb{S}(\phi)$ contains Π_{k-1}, and if the shifts of ϕ are stable, then a satisfies the sum rules of order k.*

Note that the dilation matrix in the above theorem is not necessarily isotropic.

A sequence u on $\mathbb{Z}^s$ is called a **polynomial sequence** if there exists a polynomial p such that $u(\alpha) = p(\alpha)$ for all $\alpha \in \mathbb{Z}^s$. The degree of u is the same as the degree of p. For a nonnegative integer k, let P_k be the linear space of all polynomial sequences of degree at most k, and let

$$V_k := \Big\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha\in\mathbb{Z}^s} p(\alpha)v(\alpha) = 0 \quad \forall\, p \in \Pi_k \Big\}.$$

We observe that V_k is shift-invariant, that is, $v \in V_k$ implies $v(\cdot - \alpha) \in V_k$ for every $\alpha \in \mathbb{Z}^s$.

Theorem 4.3. *For an element $a \in \ell_0(\mathbb{Z}^s)$ and a dilation matrix M, the sequence a satisfies the sum rules of order k if and only if V_{k-1} is invariant under the transition operator T_a. Moreover, V_{k-1} is the linear span of the sequences $\nabla^\mu \delta(\cdot - \alpha)$, $|\mu| = k$, $\alpha \in \mathbb{Z}^s$.*

Proof. The first statement was proved in [16]. Let us prove the second statement. Each element $u \in \ell(\mathbb{Z}^s)$ defines the linear functional on $\ell_0(\mathbb{Z}^s)$ as follows:

$$\langle u, v\rangle := \sum_{\alpha\in\mathbb{Z}^s} u(\alpha)v(\alpha), \qquad v \in \ell_0(\mathbb{Z}^s).$$

It is easily seen that $\ell(\mathbb{Z}^s)$ is the algebraic dual of $\ell_0(\mathbb{Z}^s)$. Let W be the linear span of the sequences $\nabla^\mu \delta(\cdot - \alpha)$, $|\mu| = k$, $\alpha \in \mathbb{Z}^s$. Evidently, V_{k-1} contains W. If $W \neq V_{k-1}$, then we pick an element $v \in V_{k-1} \setminus W$. Since $\ell(\mathbb{Z}^s)$ is the algebraic dual of $\ell_0(\mathbb{Z}^s)$, there exists an element $u \in \ell(\mathbb{Z}^s)$ such that $u \in W^\perp$ and $\langle u, v\rangle \neq 0$. But $u \in W^\perp$ implies that $\nabla^\mu u(\alpha) = 0$ for all $|\mu| = k$ and $\alpha \in \mathbb{Z}^s$. Consequently, $u \in P_{k-1}$, and hence $\langle u, v\rangle = 0$. This contradiction shows $W = V_{k-1}$, as desired. □

We are in a position to establish the following characterization of the smoothness of a refinable function in terms of the refinement mask.

Theorem 4.4. *Let ϕ be the normalized solution of the refinement equation (1.1) with the dilation matrix M and the mask a. Suppose that the dilation matrix M is isotropic. Let $b := a{*}a^*$, where a^* is the sequence given by $a^*(\alpha) = \overline{a(-\alpha)}$, $\alpha \in \mathbb{Z}^s$. If k is the largest integer such that $\mathbb{S}(\phi)$ contains Π_{k-1}, then V_{2k-1} is an invariant subspace of T_b. Moreover, if the shifts of ϕ are stable, then*

$$\nu(\phi) = (1 - \log_m \rho)\, s/2,$$

where ρ is the spectral radius of the linear operator $T_b|_{V_{2k-1}}$.

Proof. Let

$$f := \phi{*}\phi^*,$$

where ϕ^* is the distribution given by $\phi^* := \overline{\phi(-\,\cdot)}$. Then we have $\hat{f}(\xi) = |\hat{\phi}(\xi)|^2$ for $\xi \in \mathbb{R}^s$. Moreover, f satisfies the following refinement equation:

$$f = \sum_{\alpha\in\mathbb{Z}^s} c(\alpha) f(M\cdot - \alpha), \tag{4.1}$$

where $c := b/m = a{*}a^*/m$.

Since the shifts of ϕ are stable, we see that, for any $\xi \in \mathbb{R}^s$, there exists some $\beta \in \mathbb{Z}^s$ such that $\hat{\phi}(\xi + 2\beta\pi) \neq 0$; hence $\hat{f}(\xi + 2\beta\pi) \neq 0$. This shows that the shifts of f are stable. Moreover, $\mathbb{S}(\phi) \supset \Pi_{k-1}$ implies that $D^\mu\hat{\phi}(2\beta\pi) = 0$ for all μ with $|\mu| \le k-1$ and all $\beta \in \mathbb{Z}^s \setminus \{0\}$ (see [16]). By using the Leibniz formula for differentiation, we can easily deduce that $D^\mu \hat{f}(2\beta\pi) = 0$ for all μ with $|\mu| \le 2k-1$ and all $\beta \in \mathbb{Z}^s \setminus \{0\}$. Therefore $\mathbb{S}(f) \supset \Pi_{2k-1}$. Thus, by Theorems 4.2 and 4.3, V_{2k-1} is invariant under T_b.

For $j = 1, \dots, s$, let W_j be the minimal invariant subspace of T_b generated by $\Delta_j^k\delta$. Observe that $\Delta_j^k\delta \in V_{2k-1}$. Since V_{2k-1} is invariant under T_b, we have $W_j \subseteq V_{2k-1}$. Thus, $\rho(T_b|_{W_j}) \le \rho$ for $j = 1, \dots, s$. By Theorem 3.4 we conclude that $\nu(\phi) \ge (1 - \log_m \rho)s/2$.

It remains to prove $\nu(\phi) \le (1 - \log_m \rho)s/2$. For this purpose, we assume that ϕ lies in $W_2^\nu(\mathbb{R}^s)$ for some $\nu > 0$. Then $\nu < k$ by Theorem 4.1. Also,

$$\int_{\mathbb{R}^s} |\hat{f}(\xi)||\xi|^{2\nu}\, d\xi = \int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2 |\xi|^{2\nu}\, d\xi < \infty. \tag{4.2}$$

Let $\mu = (\mu_1, \dots, \mu_s)$ be a multi-index with $|\mu| = 2k$. Let n be a positive integer. For $j = 1, \dots, s$, write v_j for $M^{-n}e_j$. We have

$$\big(\nabla_{v_1}^{\mu_1} \cdots \nabla_{v_s}^{\mu_s} f\big)\hat{}\,(\xi) = \hat{f}(\xi) \prod_{j=1}^{s} \big(1 - e^{-iv_j\cdot\xi}\big)^{\mu_j}, \qquad \xi \in \mathbb{R}^s.$$

Choose $\theta := \nu/k$. Then $0 < \theta < 1$. We observe that

$$\left|1 - e^{it}\right| = |2\sin(t/2)| \le 2|t|^\theta \qquad \forall\, t \in \mathbb{R}.$$

Hence

$$\left|\prod_{j=1}^{s} \left(1 - e^{-iv_j\cdot\xi}\right)^{\mu_j}\right| \le 2^{2k} \prod_{j=1}^{s} |v_j \cdot \xi|^{\theta\mu_j}, \qquad \xi \in \mathbb{R}^s.$$

By Lemma 2.4 we have

$$|v_j \cdot \xi| = |M^{-n} e_j \cdot \xi| \le |M^{-n} e_j||\xi| \le C_1 \sigma^{-n}|\xi|, \qquad \xi \in \mathbb{R}^s,$$

where $C_1 > 0$ is a constant independent of n and $\sigma = m^{1/s}$ is the spectral radius of M. Combining the above estimates together, we see that there exists a constant $C_2 > 0$ such that

(4.3)
$$\left\|\nabla_{v_1}^{\mu_1} \cdots \nabla_{v_s}^{\mu_s} f\right\|_\infty \le \int_{\mathbb{R}^s} \left|(\nabla_{v_1}^{\mu_1} \cdots \nabla_{v_s}^{\mu_s} f)\hat{}(\xi)\right| d\xi \le C_2 \sigma^{-n2\nu} \int_{\mathbb{R}^s} |\hat{f}(\xi)||\xi|^{2\nu}\, d\xi.$$

It follows from (4.1) that

$$f = \sum_{\alpha\in\mathbb{Z}^s} S_c^n \delta(\alpha)\, f(M^n \cdot - \alpha).$$

Applying the difference operator $\nabla_{v_1}^{\mu_1} \cdots \nabla_{v_s}^{\mu_s}$ to both sides of this equation, we obtain

$$\nabla_{v_1}^{\mu_1} \cdots \nabla_{v_s}^{\mu_s} f = \sum_{\alpha\in\mathbb{Z}^s} \nabla^\mu S_c^n \delta(\alpha) f(M^n \cdot - \alpha).$$

Since the shifts of f are stable, there exists a constant $C_3 > 0$ such that

$$\left\|\nabla^\mu S_c^n \delta\right\|_\infty \le C_3 \left\|\nabla_{v_1}^{\mu_1} \cdots \nabla_{v_s}^{\mu_s} f\right\|_\infty \qquad \forall\, n = 1, 2, \ldots .$$

This together with (4.2) and (4.3) tells us that there exists a constant $C > 0$ such that

$$\left\|\nabla^\mu S_c^n \delta\right\|_\infty \le C \sigma^{-n2\nu} \qquad \forall\, n = 1, 2, \ldots .$$

By Lemma 3.2 we have

$$T_c^n(\nabla^\mu \delta_\beta)(\alpha) = \tau^\beta \nabla^\mu S_c^n \delta(M^n \alpha) \qquad \forall\, \alpha, \beta \in \mathbb{Z}^s.$$

It follows that

$$\left\|T_c^n \nabla^\mu \delta_\beta\right\|_\infty \le \left\|\nabla^\mu S_c^n \delta\right\|_\infty \le C\sigma^{-n2\nu} \qquad \forall\, n = 1, 2, \ldots .$$

By Theorem 4.3, V_{2k-1} is spanned by $\nabla^\mu \delta_\beta$, $|\mu| = 2k$ and $\beta \in \mathbb{Z}^s$; hence we conclude that

$$\rho(T_c|_{V_{2k-1}}) = \max_{|\mu|=2k} \left\{ \lim_{n\to\infty} \left\|\nabla^\mu S_c^n \delta\right\|_\infty^{1/n} \right\} \le \sigma^{-2\nu} = m^{-2\nu/s}.$$

But $\rho(T_c|_{V_{2k-1}}) = \rho(T_b|_{V_{2k-1}})/m = \rho/m$. Therefore, $\rho/m \le m^{-2\nu/s}$, which implies that $\nu \le (1 - \log_m \rho)\, s/2$. This shows $\nu(\phi) \le (1 - \log_m \rho)\, s/2$, as desired. $\square$

5. Examples

In this section we give several examples to illustrate the general theory. Our first two examples are concerned with self-similar tilings (see [9]).

Example 5.1. Let $s = 2$ and $M = 2I$, where I is the 2×2 identity matrix. Let ϕ be the normalized solution of the refinement equation $\phi = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha)\phi(2 \cdot -\alpha)$, where a is the sequence on $\mathbb{Z}^2$ given by its symbol

$$\tilde{a}(z) := z_1^2 + z_2 + z_1 z_2 + z_1 z_2^2, \qquad z = (z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2.$$

Then

$$\nu(\phi) = 1 - \log_4 3.$$

Proof. Let $b := a * a^*$. Then the symbol of b is

$$\begin{aligned}\tilde{b}(z) =& 4 + z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_1 z_2 + z_1^{-1} z_2^{-1} + z_1 z_2^{-1} + z_1^{-1} z_2 \\ & + z_1 z_2^{-2} + z_1^{-1} z_2^2 + z_1^2 z_2^{-1} + z_1^{-2} z_2.\end{aligned}$$

Let T_b be the transition operator associated with b. Set

$$\begin{aligned} v_1 &:= -\delta_{-e_1} + 2\delta - \delta_{e_1}, \\ v_2 &:= -\delta_{-e_2} + 2\delta - \delta_{e_2}, \\ v_3 &:= -\delta_{-e_1-e_2} + 2\delta - \delta_{e_1+e_2}, \\ v_4 &:= -\delta_{-e_1+e_2} + 2\delta - \delta_{e_1-e_2}. \end{aligned}$$

Then the T_b-invariant subspace W generated by v_1 and v_2 is the linear span of v_1, v_2, v_3, and v_4. Moreover,

$$T_b \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

The eigenvalues of the above matrix are $0, 0, 1$, and 3. Hence $\rho := \rho(T_b|_W) = 3$. Since $\rho(T_b|_W)/4 < 1$, the subdivision scheme associated with the mask a converges in the L_2-norm and ϕ lies in $L_2(\mathbb{R}^2)$ (see [10]). Hence the shifts of ϕ are orthonormal (see [9]). Therefore, by Theorem 3.4 we obtain

$$\nu(\phi) = (1 - \log_m \rho)s/2 = 1 - \log_4 3.$$

Note that the symbol $\tilde{a}(z)$ is irreducible (see [16]). $\square$

Example 5.2. Let ϕ be the normalized solution of the refinement equation

$$\phi = \phi(M\cdot) + \phi(M \cdot -e_1),$$

where

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then the critical exponent of ϕ is

$$\nu(\phi) = 1 - \log_2 \lambda \approx 0.2382,$$

where λ is the real root of the cubic polynomial $x^3 - x^2 - 2$.

Proof. Note that M is isotropic. It is known that ϕ lies in $L_2(\mathbb{R}^2)$ and has orthonormal shifts (see [9]). In this case, the mask a is given by $a(0,0) = a(1,0) = 1$ and $a(\alpha) = 0$ for $\alpha \in \mathbb{Z}^2 \setminus \{(0,0),(1,0)\}$. Let $b = a * a^*$. Then $b(0,0) = 2$, $b(-1,0) = b(1,0) = 1$ and $b(\alpha) = 0$ otherwise. Set

$$v_1 := -\delta_{-e_1} + 2\delta - \delta_{e_1}, \quad v_2 := -\delta_{-e_2} + 2\delta - \delta_{e_2}, \quad v_3 := -\delta_{-e_1+e_2} + 2\delta - \delta_{e_1-e_2}.$$

Then the T_b-invariant subspace generated by v_1 and v_2 is the linear span of v_1, v_2 and v_3. Moreover,

$$T_b \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The characteristic polynomial of the above matrix is $\lambda^3 - \lambda^2 - 2$. It has one real zero $\lambda_1 \approx 1.6956$ and two complex zeros $\lambda_{2,3} \approx -0.3478 \pm 1.0289i$. We have $\rho(T_b|_W) = \lambda_1$. Therefore

$$\nu(\phi) = 1 - \log_2(\lambda_1) \approx 0.2382.$$

□

Example 5.3. Let ϕ be the normalized solution of the refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha)\phi(M \cdot - \alpha),$$

where

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and the mask $a \in \ell_0(\mathbb{Z}^2)$ is given by

$$a(\alpha) = \begin{cases} 1/2 & \text{for } \alpha \in \{(0,0),(0,1),(1,0),(1,1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the critical exponent of ϕ is

$$\nu(\phi) = 5/2.$$

Proof. Let f be the function given by its Fourier transform

$$\hat{f}(\xi_1, \xi_2) := g(\xi_1)\, g(\xi_2)\, g(\xi_1 + \xi_2)\, g(-\xi_1 + \xi_2), \qquad (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where g is the function on $\mathbb{R}$ given by $\xi \mapsto (1 - e^{-i\xi})/(i\xi)$, $\xi \in \mathbb{R}$. The function f is a box spline, known as the Zwart-Powell element (see [1, p. 181]). It was observed by Villemoes [24] that $\phi = f(\cdot + 2e_2)$. Therefore, $\nu(\phi) = \nu(f) = 5/2$.

Let us find $b := a * a^*$. It is easily seen that

$$\big(b(\alpha_1, \alpha_2)\big)_{-1 \le \alpha_1, \alpha_2 \le 1} = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix},$$

and $b(\alpha) = 0$ for $\alpha \in \mathbb{Z}^2 \setminus [-1,1]^2$. Let V be the minimal invariant subspace of T_b generated by $\Delta_1^3\delta$ and $\Delta_2^3\delta$. By computation we obtain $\rho(T_b|_V) = 1/2$. Theorem 3.4 tells us that

$$\nu(\phi) \ge 1 - \log_2\big(\rho(T_b|_V)\big) = 2,$$

which does not give the optimal smoothness of ϕ, because the shifts of ϕ are *not* stable.

However, the optimal smoothness of ϕ can be recovered by the following consideration. Set

$$u_1 := \nabla_{e_1}\nabla_{e_1+e_2}\nabla_{e_1-e_2}\delta, \quad u_2 := \nabla_{e_1}\nabla_{e_2}\nabla_{e_1-e_2}\delta,$$
$$u_3 := \nabla_{e_2}\nabla_{e_1+e_2}\nabla_{e_1-e_2}\delta, \quad u_4 := \nabla_{e_1}\nabla_{e_2}\nabla_{e_1+e_2}\delta,$$

and $v_j := u_j * u_j^*$, $j = 1,2,3,4$. Let W be the linear span of v_j, $j = 1,2,3,4$. We have

$$T_b \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/2 \\ 1/4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

Thus, W is an invariant subspace of T_b. The characteristic polynomial of the above 4×4 matrix is $\lambda^4 - 1/64$. Hence $\rho(T_b|_W) = \sqrt{2}/4$. From the discussion in Section 2 we can derive that

$$\nu(\phi) \geq 1 - \log_2\big(\rho(T_b|_W)\big) = 5/2,$$

as desired. $\square$

Example 5.4. Let $s = 2$ and $M = 2I$, where I is the 2×2 identity matrix. Let a be the sequence on $\mathbb{Z}^2$ given by its symbol

$$\tilde{a}(z) := (1+z_1)(1+z_2)(1+z_1z_2)\big[1+t+(1-t)z_1+(1-t)z_2+(1+t)z_1z_2\big]/8,$$

where t is a real number. Let ϕ_t be the normalized solution of the refinement equation with the mask a corresponding to the parameter t. Then

$$\nu(\phi_t) = \begin{cases} 4 - \log_4 \sigma_t & \text{for } t \in \mathbb{R} \setminus \{0, -1\}, \\ 5/2 & \text{for } t = 0 \text{ or } t = -1, \end{cases}$$

where $\sigma_t = 10 + 4t + 2t^2 + \sqrt{36 - 48t + 56t^2 + 144t^3 + 68t^4}$.

Proof. For $t = -1, 0$, or 1, ϕ_t is a box spline. For our purpose, we only need box splines of the following type. For nonnegative integers j, k, p, q, let $B_{j,k,p,q}$ be the box spline given by its Fourier transform:

$$\widehat{B}_{j,k,p,q}(\xi_1,\xi_2) := g^j(\xi_1)\, g^k(\xi_2)\, g^p(\xi_1+\xi_2)\, g^q(-\xi_1+\xi_2), \qquad (\xi_1,\xi_2) \in \mathbb{R}^2,$$

where g is the function on $\mathbb{R}$ given by $\xi \mapsto (1 - e^{-i\xi})/(i\xi)$, $\xi \in \mathbb{R}$. We can easily verify that $\phi_0 = B_{2,2,1,0}$, $\phi_1 = B_{1,1,2,0}$, and $\phi_{-1} = B_{1,1,1,1}(\cdot - e_1)$. Note that the shifts of $B_{j,k,p,0}$ are always stable. But the shifts of the box spline $B_{1,1,1,1}$ are not stable. Moreover, $\nu(\phi_0) = \nu(\phi_{-1}) = 5/2$ and $\nu(\phi_1) = 3/2$. See [1] for these facts. We will prove that the shifts of ϕ_t are stable for $t \neq -1$ at the end of this section.

In order to calculate the critical exponent of ϕ_t, we need to find

$$\lim_{n\to\infty} \big\|\nabla_{e_j}^2 S_a^n \delta\big\|_2^{2/n}$$

for $j = 1, 2$. For this purpose, we set

$$\rho_1 := \lim_{n\to\infty} \left\| \nabla_{e_2} \nabla_{e_1+e_2} S_a^n \delta \right\|_2^{2/n},$$

$$\rho_2 := \lim_{n\to\infty} \left\| \nabla_{e_1} \nabla_{e_1+e_2} S_a^n \delta \right\|_2^{2/n},$$

$$\rho_3 := \lim_{n\to\infty} \left\| \nabla_{e_1} \nabla_{e_2} S_a^n \delta \right\|_2^{2/n}.$$

Let $\rho := \max\{\rho_1, \rho_2, \rho_3\}$. It is easily seen that

$$\max_{j=1,2} \left\{ \lim_{n\to\infty} \left\| \nabla_{e_j}^2 S_a^n \delta \right\|_2^{2/n} \right\} = \rho.$$

To facilitate the computation of ρ_j, $j = 1, 2, 3$, we introduce the sequence c given by its symbol

$$\tilde{c}(z) = (1+t) + (1-t)z_1 + (1-t)z_2 + (1+t)z_1 z_2.$$

Furthermore, let a_j $(j = 1, 2, 3)$ be the sequences given by

$$\tilde{a}_1(z) = (1+z_1)\tilde{c}(z)/8, \quad \tilde{a}_2(z) = (1+z_2)\tilde{c}(z)/8, \quad \text{and } \tilde{a}_3(z) = (1+z_1 z_2)\tilde{c}(z)/8.$$

We claim that, for $j = 1, 2, 3$,

$$\rho_j = \lim_{n\to\infty} \left\| S_{a_j}^n \delta \right\|_2^{2/n}. \tag{5.1}$$

Indeed, by [13, Theorem 3.3], there exist two positive constants C_1 and C_2 such that

$$C_1 \left\| S_{a_1}^n \delta \right\|_2 \le \left\| \nabla_{e_2} \nabla_{e_1+e_2} S_a^n \delta \right\|_2 \le C_2 \left\| S_{a_1}^n \delta \right\|_2$$

for all $n = 1, 2, \ldots$. This verifies (5.1) for $j = 1$. The cases $j = 2$ and $j = 3$ can be proved in the same way.

Now we are in a position to compute ρ_j, $j = 1, 2, 3$. For this purpose, let $b_j := a_j * a_j^*$, $j = 1, 2, 3$. Then the nonzero entries of b_1 are given by the matrix

$$\frac{1}{64} \begin{bmatrix} 1-2t+t^2 & 4-4t & 6-2t^2 & 4+4t & 1+2t+t^2 \\ 2-2t^2 & 8 & 12+4t^2 & 8 & 2-2t^2 \\ 1+2t+t^2 & 4+4t & 6-2t^2 & 4-4t & 1-2t+t^2 \end{bmatrix},$$

where $12 + 4t^2$ is the entry at the origin. Set

$$v_1 := \delta_{-e_1} + \delta_{e_1}, \quad v_2 := \delta_{-e_2} + \delta_{e_2}, \quad v_3 := \delta_{-e_1-e_2} + \delta_{e_1+e_2}.$$

Then

$$T_{b_1} \begin{bmatrix} \delta \\ v_1 \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 12+4t^2 & 2-2t^2 \\ 16 & 8 \end{bmatrix} \begin{bmatrix} \delta \\ v_1 \end{bmatrix}.$$

Hence the minimal T_{b_1}-invariant subspace generated by δ is the linear span of δ and v_1. By Theorem 3.3 we obtain

$$\rho_1 = \rho(T_{b_1}|_W) = \max\{16, 4+4t^2\}/64.$$

By symmetry we also have

$$\rho_2 = \max\{16, 4+4t^2\}/64.$$

The nonzero entries of b_3 is given by the matrix

$$\frac{1}{64}\begin{bmatrix} 0 & 0 & 1-2t+t^2 & 2-2t^2 & 1+2t+t^2 \\ 0 & 2-4t+2t^2 & 6-6t^2 & 6+4t+6t^2 & 2-2t^2 \\ 1-2t+t^2 & 6-6t^2 & 10+4t+10t^2 & 6-6t^2 & 1-2t+t^2 \\ 2-2t^2 & 6+4t+6t^2 & 6-6t^2 & 2-4t+2t^2 & 0 \\ 1+2t+t^2 & 2-2t^2 & 1-2t+t^2 & 0 & 0 \end{bmatrix},$$

where $10+4t+10t^2$ is the entry at the origin. By computation we obtain

$$T_{b_3}\begin{bmatrix} \delta \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \frac{1}{64}\begin{bmatrix} 10+4t+10t^2 & 1-2t+t^2 & 1-2t+t^2 & 1+2t+t^2 \\ 12-12t^2 & 6-6t^2 & 2-2t^2 & 2-2t^2 \\ 12-12t^2 & 2-2t^2 & 6-6t^2 & 2-2t^2 \\ 12+8t+12t^2 & 2-4t+2t^2 & 2-4t+2t^2 & 6+4t+6t^2 \end{bmatrix}\begin{bmatrix} \delta \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The eigenvalues of the above matrix are $4(1+t^2)/64$, $4(1-t^2)/64$, and

$$\left[10+4t+2t^2 \pm \sqrt{36-48t+56t^2+144t^3+68t^4}\,\right]/64.$$

It can be easily verified that

$$\rho_3 = \sigma_t/64,$$

where

$$\sigma_t := 10+4t+2t^2+\sqrt{36-48t+56t^2+144t^3+68t^4}.$$

Since $\rho_3 \geq \rho_1 = \rho_2$, we obtain

$$\rho = \sigma_t/64.$$

By using the results in [10] we can prove that ϕ_t lies in $L_2(\mathbb{R}^2)$ if and only if $\sigma_t < 256$. Note that $\sigma_t \geq 16$ with equality if and only if $t=0$ or -1. Thus, for $t \neq 0$, $\rho > 1/4$ and $1-\log_4\rho < 2$. Therefore, by Theorem 3.4, we conclude that the critical exponent of ϕ_t is

$$\nu(\phi_t) = 4-\log_4\sigma_t \quad \text{for } \ t\in\mathbb{R}\setminus\{0,-1\}.$$

When $t=-1$, we have $\sigma_{-1}=16$ and $4-\log_4\sigma_{-1}=2$. But $\nu(\phi_{-1})=5/2$. In this case, Theorem 3.4 is not applicable, because the shifts of ϕ_{-1} are not stable. When $t=1$, we have $\sigma_1=32$ and $4-\log_4\sigma_1=3/2$. This agrees with the fact that the critical exponent of the box spline $B_{1,1,2,0}$ is $3/2$.

It remains to prove that the shifts of ϕ_t are stable for $t\neq -1$. For simplicity we write ϕ for ϕ_t. We observe that

$$\hat{\phi}(\xi) = H(\xi/2)\hat{\phi}(\xi/2), \qquad \xi=(\xi_1,\xi_2)\in\mathbb{R}^2, \tag{5.2}$$

where

$$H(\xi) = \left[1+t+(1-t)e^{-i\xi_1}+(1-t)e^{-i\xi_2}+(1+t)e^{-i(\xi_1+\xi_2)}\right]G(\xi)$$

with

$$G(\xi) := (1+e^{-i\xi_1})(1+e^{-i\xi_2})(1+e^{-i(\xi_1+\xi_2)})/32, \qquad \xi=(\xi_1,\xi_2)\in\mathbb{R}^2.$$

Let

$$N(\phi) := \left\{\xi\in\mathbb{R}^2 : \hat{\phi}(\xi+2\beta\pi)=0 \quad \forall\,\beta\in\mathbb{Z}^2\right\}.$$

The shifts of ϕ are stable if and only if $N(\phi)$ is the empty set. We claim that $(0,\xi_2) \notin N(\phi)$ for any $\xi_2 \in \mathbb{R}$. Indeed,

$$H(0,\xi_2) = (1+e^{-i\xi_2})^3/8, \qquad \xi_2 \in \mathbb{R}.$$

It follows that

$$\hat{\phi}(0,\xi_2) = \Big[\big(1-e^{-i\xi_2}\big)/(i\xi_2)\Big]^3, \qquad \xi_2 \in \mathbb{R}.$$

Hence $(0,\xi_2) \notin N(\phi)$. The same argument tells us that $(\xi_1, 0) \notin N(\phi)$ for any $\xi_1 \in \mathbb{R}$. Furthermore, we have

$$H(\xi,-\xi) = (1+e^{-i\xi})(1+e^{i\xi})\big[2+2t+(1-t)e^{-i\xi}+(1-t)e^{i\xi}\big]/16, \qquad \xi \in \mathbb{R}.$$

Consider the function $\xi \mapsto \hat{\phi}(\xi,-\xi)$, $\xi \in \mathbb{R}$. We have

$$\hat{\phi}(\xi,-\xi) = \prod_{j=1}^{\infty} H(\xi/2^j, -\xi/2^j), \qquad \xi \in \mathbb{R}.$$

By using [19, Theorem 1] we can easily prove that this function does not have 2π-periodic zeros, provided $t \neq -1$. Hence $(\xi,-\xi) \notin N(\phi)$ for any $\xi \in \mathbb{R}$. Thus, $\xi_1 + \xi_2 = 2\pi$ implies $(\xi_1,\xi_2) \notin N(\phi)$.

Let $K(\phi) := N(\phi) \cap [0,2\pi)^2$. We have proved that $(\xi_1,\xi_2) \in K(\phi)$ implies $\xi_1 \neq 0$, $\xi_2 \neq 0$, and $\xi_1+\xi_2 \neq 2\pi$. We claim that

$$(5.3) \quad (\xi_1,\xi_2) \in K(\phi) \quad \Longrightarrow \quad (\xi_1/2, \xi_2/2) \in K(\phi) \quad \text{or} \quad (\xi_1/2, \xi_2/2+\pi) \in K(\phi).$$

Indeed, if $(\xi_1/2,\xi_2/2) \notin K(\phi)$ and $(\xi_1/2, \xi_2/2+\pi) \notin K(\phi)$, then there exist integers $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$\hat{\phi}(\xi_1/2 + 2\alpha_1\pi, \xi_2/2 + 2\alpha_2\pi) \neq 0 \quad \text{and} \quad \hat{\phi}(\xi_1/2 + 2\beta_1\pi, \xi_2/2 + 2\beta_2\pi + \pi) \neq 0.$$

But $\hat{\phi}(\xi_1 + 4\alpha_1\pi, \xi_2 + 4\alpha_2\pi) = 0$ and $\hat{\phi}(\xi_1 + 4\beta_1\pi, \xi_2 + 4\beta_2\pi + 2\pi) = 0$. Taking (5.2) into account, we deduce that $H(\xi_1/2, \xi_2/2) = 0$ and $H(\xi_1/2, \xi_2/2+\pi) = 0$. Hence

$$1+t+(1-t)e^{-i\xi_1/2} + (1-t)e^{-i\xi_2/2} + (1+t)e^{-i(\xi_1+\xi_2)/2} = 0$$

and

$$1+t+(1-t)e^{-i\xi_1/2} - (1-t)e^{-i\xi_2/2} - (1+t)e^{-i(\xi_1+\xi_2)/2} = 0.$$

It follows that $(1+t)+(1-t)e^{-i\xi_1/2} = 0$, which is impossible for $\xi_1 \in [0,2\pi)$ and $t \in \mathbb{R}$. Thus, (5.3) has been verified.

Suppose $(\xi_1,\xi_2) \in K(\phi)$. By (5.3) we can find a sequence $\xi^{(n)} = (\xi_1^{(n)}, \xi_2^{(n)}) \in K(\phi)$, $n = 1,2,\ldots$, such that $\xi_1^{(1)} = \xi_1$, $\xi_2^{(1)} = \xi_2$, and $\xi_1^{(n+1)} = \xi_1^{(n)}/2$, $\xi_2^{(n+1)} = \xi_2^{(n)}/2$ or $\xi_2^{(n)}/2+\pi$. Thus, $\lim_{n\to\infty} \xi_1^{(n)} = 0$. Also, $(\xi_2^{(n)})_{n=1,2,\ldots}$ has a subsequence which converges to some $\omega \in [0,2\pi]$. It follows that $(0,\omega) \in N(\phi)$, which is a contradiction.

We conclude that the shifts of ϕ_t are stable if and only if $t \neq -1$. □

Acknowledgement

I am grateful to Dr. Villemoes for his comments on Example 5.3.

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