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CHARACTERIZATION OF SOME APPORTIONMENT METHODS

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Abstract. In this work known main characteristics of 11 apportionment methods are systemized, including the Hondt, Hamilton, Sainte-Laguë and Huntington-Hill methods, and some new ones are determined by computer simulation. To such characteristics, refer the disproportionality of solutions and the percentage of Quota rule violation, of the Alabama, Population and New state paradoxes occur and of favoring of beneficiaries. For a large range of initial data, where determined the preferences order of the explored apportionment methods by each of these characteristics. No one of methods is preferable by all of the six characteristics-criteria. Of the immune to the three paradoxes, d'Hondt, Huntington-Hill, Sainte-Laguë and adapted Sainte-Laguë methods, by disproportionality of solutions the best is Sainte-Laguë method, followed by the adapted Sainte-Laguë, then by the Huntington-Hill and, finally, by the d'Hondt one. The same order for these four methods is by compliance with the Quota rule. The percentage of Sainte-Laguë method's Quota rule violation is little influenced by the total number of seats value, but it strongly decreases (over 400 times) with the increase of the number of states – from approx. 0.045% for 4 states, up to 0.0001-0.0004% for 30 states. Based on multi-aspectual comparative analyses, it is shown that from explored methods there is reasonable to use, in specific areas, only three or four: the Hamilton, Sainte-Laguë and Adapted Sainte-Laguë methods and may be the Quota linear divisor one.

Keywords: *apportionment paradoxes, comparative analyses, computer simulation, disproportionality of solutions, favoring of* beneficiaries*, qualitative characteristics, quantitative characteristics.*

1. Introduction

In the examined multi-optional systems, the decision is made on two or more options, consisting of parts of a homogeneous resource measured in integers (a set of identical objects); the resource in question is apportioned to beneficiaries-options (states, regions, parties, institutions, subdivisions, etc.), aiming to ensure the extreme value of a given criterion. The decision may be individual or collective. Collective decisions are taken by voting. It is considered that both, individual multi-optional decisions and the collective multi-optional ones are based on solving a deterministic apportionment optimization problem, all the necessary initial data being known.

Examples of apportionments:

1) apportionment of M mandates in the elective body to n parties by the number of votes cast V_i for each party i, $i = \overline{1,n}$;
2) apportionment of M mandates in the European Parliament to the n EU Member States by population V_i in each country i, $i = \overline{1,n}$;
3) apportionment of M seats in the US Congress House of Representatives to the n = 50 states by population V_i in each state i, $i = \overline{1,n}$;
4) distribution of M computers to the n public lyceums of Moldova by the number of scholars V_i in each lyceum i, $i = \overline{1,n}$;
5) distribution of M sector policemen to the n sectors of Chisinau municipality by the number of inhabitants V_i in each sector i, $i = \overline{1,n}$.

One of the most known practices regarding the use of multi-optional decisions systems is related to elections. Thus further on the systems in question will be investigated, without diminishing the universality of the approach, through elections by party-list proportional representation (LPR).

To solve the optimization problem an apportionment method (algorithm) is used. There is not yet a universally accepted apportionment (APP) method to be used in similar situations. Therefore, for this purpose several such methods are used, including: Jefferson [1, 2], Hamilton (Hare) [2, 3], Webster {1, 2], d'Hondt [4], Sainte-Laguë [5] and Huntington-Hill [6]. For example, the Hamilton (Hare) method is used in elections in Germany, Russia, Ukraine, Mexico and Iceland, the d'Hondt method – in Belgium, Japan, Israel, Peru, Portugal, Romania, Spain, Hungary and Thailand, the Sainte-Laguë method – in Poland, Denmark, Latvia and New Zealand [7- 9].

Studies [1, 2, 10] show that in different situations APP methods behave differently: methods that in some situations allow for better solutions, in other situations yield to other methods. The obtained solution may vary considerably from one method to the other. This can lead to unexpected effects.

The knowledge of characteristics of APP methods, especially of quantitative ones, would facilitate the comparative analysis of methods and a successful selection of a particular method for a specific mission. To such characteristics refer [1, 2]: disproportionality of solutions, compliance with the Quota rule, favoring of parties, monotonicity and so on.

In addition to the known ones, some new aspects of APP methods are examined in this paper, including the estimation, by computer simulation, of disproportionality of solutions, of the percentage of Quota rule violation, of Alabama, Population and New state paradoxes occurrence and of favoring of parties. Initially, the apportionment optimization problem is defined, then the examined APP methods are described, next the characteristics of APP methods are obtained, outlined and systemized, and, finally, the comparative multi-aspectual analysis of methods is made.

2. Preliminary considerations

The departure point of APP methods is to minimize the disproportion of allocation of mandates (seats) to parties (states). To estimate this disproportion various indices were proposed, including the Gallagher [2], d'Hondt [4], Sainte-Laguë [5], Rae [12], Loosemore-Handby (Duncan and Duncan) [13, 18], Rose [14], Grofman [15], Lijphart [16], Monroe [17], Square deviation, Relative standard deviation and Average relative deviation [10] ones. Depending on the index used as criterion in the apportionment problem, the solution may vary, sometimes considerably [1, 2, 10]. As shown in [10], minimizing the disproportionality, within the meaning of each of the above-nominated 11 indices (excluding the Monroe one), is ensured, as appropriate, by one of the three methods: Hamilton (Hare) [2, 3], Sainte-Laguë (Webster) [2, 5] or d'Hondt (Jefferson) [2, 4]. But other methods are known, too, including that of Huntington-Hill [6], Largest remainders with Droop quota [11], Largest remainders with Hagenbach-Bischoff quota [2], Largest remainders with Imperiali quota [2], etc.

Thus, selecting the relevant index of disproportionality is one of the important steps in implementing an adequate multi-optional decision system. A successful selection requires as more complete as possible comparative analysis of the known indices. Some of such issues are addressed in many papers, including [2, 10, 14-17, 19-21]. In [19], the disproportionality is split into two types: (1) forced or unavoidable, due to the very nature of the apportionment problem, and (2) the non-forced one. Based on this approach, an index just measuring avoidable disproportionality was proposed. A characterization of the Duncan and Duncan [18] (also called Loosemore-Hanby [13]) and Lijphart [16] indices, from the point of view of Homogeneity, Radial Linearity, Substitution and Invariance, is proposed in [21].

In LPR apportionments, M. Gallagher [2] highlights two broad categories of measures of disproportionality: (1) measures based on the absolute difference between the party's seats and votes; (2) measures focused on the ratio between a party's seats and its votes. In both these categories, parties are primary in assessing the disproportionality. In reality, however, primary are the voters; voters should be represented equally in the elective body or, if it is not possible, with the smallest possible disproportion. Therefore, at the base of the index of disproportionality the value of each vote should stand – vote that reflects unequivocally the rights r of each voter in the election. Namely, starting from the value r of a vote (r = total number of seats/total number of votes) and basing on a comparative multi-aspectual analysis, in [10], the opportunity of using the Average relative deviation index (ARD) for this purpose is argued. Namely, the ARD index is used in this paper.

If so, let [10]: M – number of mandates (seats) in the elective body; n – number of parties that have reached or exceeded the representation threshold; V_i, v_i – number and, respective, percentage of valid votes cast for party i; $V = V_1 + V_2 + \ldots + V_n$ – total valid votes cast for the n parties; $r = M/V$ – equal rights of each elector (decider) in the overall decision; x_i, m_i – number and, respectively, percentage of mandates to be allocated to party i; I – index of disproportionality (ARD). Then the apportionment problem can be formulated as follows [10]. Knowing quantities (positive natural numbers) M, n and V_i, $i=\overline{1,n}$, the values of sizes x_i ($i=\overline{1,n}$) are required to be determined – natural numbers that would ensure the minimization of the Average relative deviation index

$$I=\sum_{i=1}^{n}\left|v_i-m_i\right|=100\sum_{i=1}^{n}\left|\frac{V_i}{V}-\frac{x_i}{M}\right|\to \min \tag{1}$$

in compliance with the restriction

$$\sum_{i=1}^{n}x_i=M. \tag{2}$$

Problem (1)-(2) is one of mathematical programming in integers. From (1) it can be seen that I = 0, i.e. the distribution x_i ($i = \overline{1,n}$) is proportional, only if equalities $m_i = v_i$, $i = \overline{1,n}$ occur. However, the probability that these equalities occur in particular elections, at integer sizes x_i ($i = \overline{1,n}$) and $V >> M \geqslant n$, is very small. Thus, in real LPR elections certain disproportionality of apportionment occurs. In such cases it is important that the disproportionality in question to be assessed.

Sometimes the problem (1)-(2) also includes additional requirements such as [1, 2]: compliance with the Quota rule, not favoring of parties, monotonicity, and so on.

The Quota rule entails compliance with relations $a_i \leqslant x_i \leqslant a_i + 1$, $i = \overline{1,n}$, i.e., taking into account that x_i, $i = \overline{1,n}$ are integers, one has $x_i = a_i$ or $x_i = a_i + 1$, $i = \overline{1,n}$, where $a_i = \lfloor V_i/Q \rfloor$ is the Lower quota of mandates, $a_i + 1$ is the Upper quota of mandates for party i and $Q = V/M$ is the Standard divisor (Hare quota or simple quota) for the scrutiny.

In case of successive decisions, for example in the case of periodic elections, sometimes the monotonicity of APP methods is important. It is expressed, first of all, by the immunity to Alabama, Population and New state paradoxes (the "three paradoxes") [1, 2], but may be also [1, 10] by modifications in decision-makers' preferences, by merging of parties, by partitioning of parties and by changing of the number of parties. Of these aspects of monotonicity, further on just those of the three paradoxes will be examined.

The essence of the three paradoxes [1, 2]:

a) Alabama paradox – an increase in the total number of seats (mandates) causes the loss of seats to some states (parties);
b) Population paradox – a state (party) with a higher rate of increase in the number of inhabitants (votes cast) loses at least one seat to the benefit of states (parties) with a smaller increase in the rate in question;
c) New state paradox – adding of a new state (party), with a proportional increase in the total number of seats, leads to the redistribution of seats among other states (parties).

The Balinski and Young impossibility theorem [1] asserts, in general, that there is no an APP method, which, at three or more parties (states), would have the following three properties: follows the Quota rule; is immune to Alabama paradox; is immune to Population paradox.

For example, the divisor methods such as Jefferson, d'Hondt, Webster, Sainte-Laguë and Huntington-Hill are immune to the three paradoxes, but may not follow the Quota rule [1, 2]. The Hamilton method follows the Quota rule, but is not immune to the three paradoxes [1, 2]. So, we have to make a compromise, depending on the situation, with regards to which APP method to apply.

Known theoretical results sometimes do not give an unambiguous answer to the preferences of using a particular APP method in a specific situation. In such cases the comparative analysis of methods by computer simulation may be appropriate. For this purpose, the computer application SIMAP has been elaborated and respective calculations have been made. The initial data used in calculations are: M = 6, 11, 21, 51, 101, 201, 501; n = 2, 3, 4, 5, 7, 10, 15, 20, 30, 50; $n \leqslant M - 1$; $V = 10^8$; uniform distribution of values V_i, $i = \overline{1,n}$; sample size 10^6. The use of small values of M is useful, for example, when determining the M members of a parliamentary committee basing on the number of deputies (V_i) of each of the n parties in the Parliament.

3. The essence of the investigated apportionment methods

There are distinguished two well-known categories of apportionment methods with proportional representation [2]: divisor methods (Jefferson, d'Hondt, Sainte-Laguë, modified Sainte-Laguë, Webster and Huntington-Hill) and remainders methods (Hamilton, Hare; Droop, Hagenbach-Bischoff and Imperiali). Divisor methods are based on the "highest average" principle and the remainders methods – on the "largest remainder" one. A special case is the Huntington-Hill method [6]. This method is based on the geometric mean at the border of the number of mandates for each party (state), but it still uses a divisor as divisor methods do. To divisor methods also refer the General linear divisor (GLD), adapted Sainte-Laguë and Variable linear divisor (VLD) ones [10].

One can also talk about a third category of methods – the mixed methods, which combines aspects from both previous categories. Of the explored in this paper, to mixed methods refer [10]: Quota linear divisor (QLD), Lower quota linear divisor (LQLD), Quota dependent linear divisor (QDLD), Quota variable linear divisor (QVLD) and Lower quota variable linear divisor (LQVLD).

At the same time, in [2, 22], for example, it is mentioned that, regarding the results of the distribution of seats, the Jefferson and d'Hondt methods and the Webster and Sainte-Laguë methods are equivalent. But, as shown in [10], the direct use of Jefferson and Webster methods do not always guarantee the solution. Also, the disproportionality of solutions obtained using the Hamilton method is lower than those of Droop, Hagenbach-Bischoff and Imperiali methods [2, 10]. Finally, the Hamilton and Hare methods coincide [2], and the General linear divisor method defines a class of divisor methods [10].

So, the following APP methods are investigated in this paper: Hamilton (H), Sainte-Laguë (SL), d'Hondt (d'H), Huntington-Hill (HH), adapted Sainte-Laguë (ASL) and, mainly for research purposes, the QLD, LQLD, QDLD, VLD, QVLD and the LQVLD ones.

The Huntington-Hill method provides the calculation of standard divisor $Q = V/M$; then to each party i is initially assigned the lower quota of seats $x_i = a_i = \lfloor V_i/Q \rfloor$, if $V_i/Q < \sqrt{a_i(a_i+1)}$, or – the upper one, i.e. $x_i = a_i + 1$, if $V_i/Q \geqslant \sqrt{a_i(a_i+1)}$, $i = \overline{1,n}$. Thus, the value of ratio V_i/Q, rounded up to integers basing on the respective geometric mean, is assigned to x_i. If the sum of the above-specified quotas of the n parties is equal to M, i.e. $x_1 + x_2 + \ldots + x_n = M$, then the distribution of seats has ended, being proportional.

Otherwise, iteratively, using several attempts, it is found such a new divisor q^* and, respectively, $x_i = u_i = \lfloor V_i/q^* \rfloor$ if $V_i/q^* < \sqrt{u_i(u_i+1)}$, or $x_i = u_i + 1$ if $V_i/q^* \geqslant \sqrt{u_i(u_i+1)}$, $i = \overline{1,n}$, that the equality $x_1 + x_2 + \ldots + x_n = M$ to be achieved. The solution obtained will be a disproportionate one.

The majority of divisor methods are particular cases of **General linear divisor method** (GLD) [10], which is based on the APP rule

$$i \succ k, \text{ if } \frac{V_i}{cu_i + 1} > \frac{V_k}{cu_k + 1}, \tag{3}$$

in which c is a constant and u_j is the number of mandates already allocated to party j.

The apportionment algorithm consists in calculating, for each party $i = \overline{1,n}$, of ratio $V_i/(cu_i + 1)$ consecutively at $u_i = 0, 1, 2, \ldots$. Thus, n series of decreasing numbers are formed, by one for each of the n parties. Of these n series, the largest M numbers are selected, of which x_i numbers belong to series i, $i = \overline{1,n}$. To party i is assigned x_i mandates, $i = \overline{1,n}$.

Particular cases of the GLD method differ only by the value of constant c. So:

- for the **d'Hondt method**, $c = 1$;
- for the **Sainte-Laguë method**, $c = 2$;
- for the **VLD method**, $c = n/\Delta M$ at $\Delta M = M - (a_1 + a_2 + ... + a_n) > 0$.

Thereafter, the terms "GLD method" and "all linear divisor methods" will be used synonymously.

The adapted Sainte-Laguë method [10] differs from the Sainte-Laguë one only by allocating, initially, by $g \geqslant 1$ mandates to small parties, so that finally no party will have less than g mandates. At $g = 1$, the Huntington-Hill method corresponds to such a requirement.

According to the **Hamilton method**, at first step, by a lower quota of mandates $a_i = \lfloor V_i/Q \rfloor$ is allocated to each party i, $i = \overline{1,n}$ and the number $\Delta M = M - (a_1 + a_2 + ... + a_n)$ of still undistributed mandates is determined. If $\Delta M = 0$, then the distribution has ended and is proportional. Otherwise, at the second step, by one mandate of the ΔM is allocated additionally to each of the first ΔM parties with the larger value of remainder $\Delta V_i = V_i - a_i Q$. The distribution has ended, being disproportionate.

Taking into account that the Hamilton method ensures the less disproportionate apportionment [2, 10], the Hamilton method's first step is used also as first step of the **QLD, LQLD, QDLD, QVLD** and **LQVLD** methods. These methods differ only at the second, final, step:

- QLD, QVLD and QDLD methods – by one mandate of the ΔM is allocated additionally to each of the first ΔM parties with the larger value of ratio, respectively, $V_i/(2a_i + 1)$, $V_i/(na_i + \Delta M)$ and $V_i/[ca_i + 1)$, where $c = \max\{2; n - 1\}$, $i = \overline{1,n}$;
- LQLD and LQVLD methods – the ΔM mandates are allocated additionally to parties with the larger value of ratio, respectively, $V_i/[2u_i + 1)$ and $V_i/[nu_i + \Delta M)$, where $u_i \geqslant a_i$, $i = \overline{1,n}$.

The second step for these methods is defined basing on the following reasons. According to [10], the Sainte-Laguë method (linear divisor method with $c = 2$) ensures, in average, the less disproportionate apportionment among all linear divisor methods (at c = constant); the value $c = 2$ is used in the QLD and LQLD methods, too. At the same time [10], the linear divisor method with $c = n - 1$ (QDLD) does not exceed the Upper quota. Also, the linear divisor method with $c = n/\Delta M$ (VLD) less of all favors parties, when allocating mandates to them, and therefore it is the least disproportionate among the linear divisor methods [10].

Let's examine the characteristics of the nominated APP methods.

4. Compliance with the Quota rule

Of the methods explored in this paper, the Hamilton, QLD, QDLD and QVLD ones met the requirement of Quota rule, by definition. Also, in the case of two-party voting, the DLG method at $c \geqslant 1$, including d'Hondt and Sainte-Laguë ones and the Huntington-Hill, VLD and LQLD methods follow the Quota rule [2, 10]. The Sainte-Laguë and LQLD methods follow the Quota rule also at $n = 3$ [1, 10].

The D'Hondt method is compliant only with the Lower quota rule [10]. The LQLD and LQVLD methods are also compliant with this quota rule, by definition.

Statement 1 [10]. At $c \geqslant n - 1$, all linear divisor methods are compliant with the Upper quota rule.

Statement 2 [10]. At $c \geqslant 1$, all linear divisor methods can't violate simultaneously the Lower quota and the Upper quota rules in a poll.

The last result is used when improving (less calculations) the d'Hondt, Jefferson, Webster, Saint-Laguë and Adams methods [10].

Figure 1 [10] outlines the various possibilities of DLG method complying with the Lower quota and Upper quota rules, depending on the value of constant c and the larger or smaller the parties are. The numbers of consequences are from [10].

The case $c = 1$ (d'Hondt method) is a special one – it is the only case of complying with the Lower quota rule, regardless of the party's size. An opposite is the case $c = n - 1$ (generally, the class of $c \geqslant n - 1$ cases), which complies with the Upper quota rule. Also, smaller parties do not exceed the Upper quota, regardless of the value of c, but they can violate the Lower quota rule at $c < 1$.

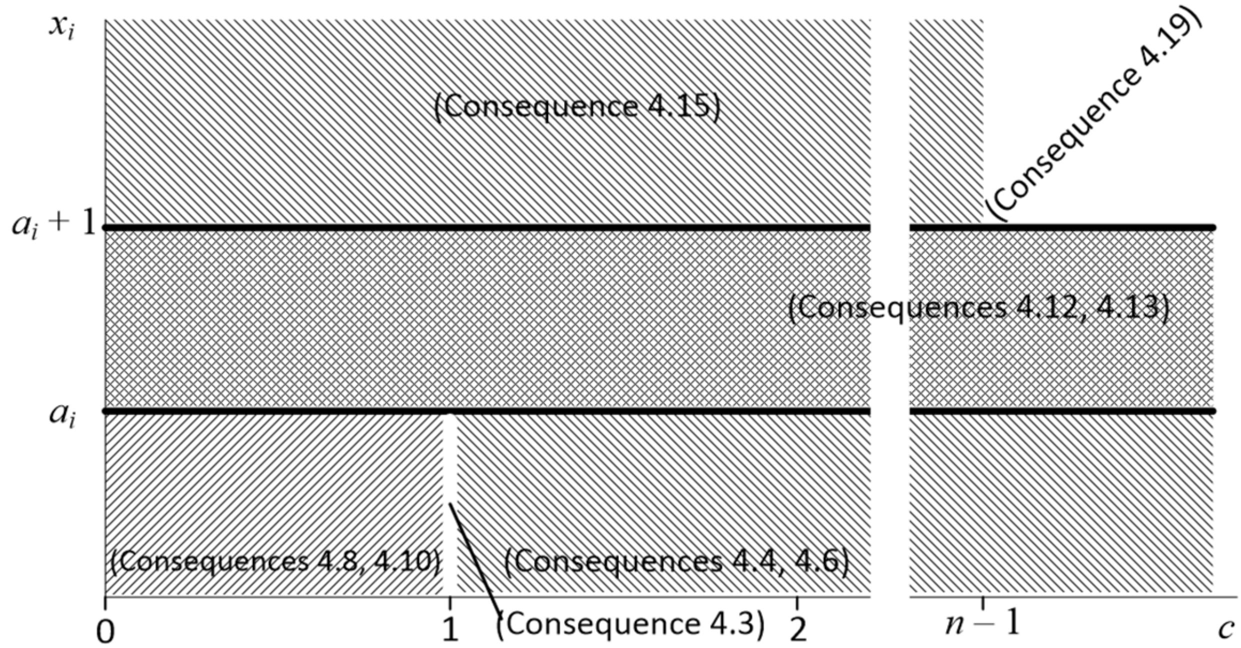

Figure 1. Definition domain of parameters $x_i, i = \overline{1,n}$.

At the same time, larger parties can violate both the Lower quota (at $c > 1$) and the Upper quota (at $0 < c < n - 1$) rules.

It is estimated that for $M = 435$ and $n = 50$:

1) the probability of Quota rule violation by Sainte-Laguë (Webster) method is, according to [1], of approx. 0.00061, and according to [23] - of approx. 0.0016;
2) the probability of Quota rule violation by Huntington-Hill method is of approx. 0.000286 [1].

Some results of computer simulation of polls using SIMAP (see section 2), regarding the violation of Quota rule, are shown in Table 1.

With regard to data of Tables 1-5, the calculations made do not guarantee the accuracy of four-digits after comma, the representation being, however, useful in some comparative research.

The P_Q percentage of Quota rule violation depends on the method used and on the values of sizes M (less) and n. Of the four divisor methods, SL, ASL, HH and d'H, the value of P_Q is the lowest for the Sainte-Laguë method, and the largest – for the d'Hondt one.

Table 1

Percentage P_Q of Quota rule violation by divisor APP methods, %

Sainte-Laguë (Webster) method					D'Hondt (Jefferson) method				
n\M	51	101	201	501	n\M	51	101	201	501
3	0	0	0	0	3	3.9843	4.1367	4.2105	4.2355
4	0.0472	0.0468	0.0461	0.0452	4	6.3046	6.5047	6.5789	6.6259
5	0.0790	0.0817	0.0827	0.0821	5	7.7247	8.0367	8.1421	8.2393
7	0.0698	0.0736	0.0701	0.0698	7	9.5227	9.9902	10.2082	10.3114
10	0.0355	0.0354	0.0310	0.0309	10	10.8147	11.4797	11.9164	12.1932
15	0.0094	0.0080	0.0074	0.0077	15	11.3990	12.7932	13.5830	14.0348
20	0.0018	0.0014	0.0016	0.0018	20	12.2000	13.5427	14.5451	15.3240
30	0.0004	0.0001	0.0001	0.0002	30	9.8856	13.4113	15.4380	16.9379
Adapted Sainte-Laguë method					**Huntington-Hill method**				
n\M	51	101	201	501	n\M	51	101	201	501
3	0.2656	0.1288	0.0673	0.0232	3	0.2928	0.1377	0.0713	0.0237
4	0.4088	0.2086	0.1171	0.0721	4	0.4875	0.2377	0.1276	0.0769
5	0.4944	0.2516	0.1649	0.1171	5	0.6126	0.2985	0.1815	0.1209
7	0.5782	0.2541	0.1447	0.0977	7	0.7831	0.3204	0.1654	0.1019
10	0.6941	0.2182	0.0967	0.0518	10	1.0293	0.3141	0.1251	0.0598
15	1.0459	0.1781	0.0511	0.0164	15	1.6536	0.3045	0.0831	0.0208
20	2.0325	0.1873	0.0292	0.0062	20	3.2937	0.3691	0.0585	0.0103
30	7.0067	0.3059	0.0166	0.0012	30	10.347	0.6589	0.0449	0.0028

The P_Q percentage of the Sainte-Laguë method is to a small extent influenced by the value of M, but it decreases strongly (over 400 times) with the increase of n – from approx. 0.045% at $n = 4$, up to 0.0001-0.0004% at $n = 30$.

Of a smaller dynamics is the P_Q value of the d'Hondt method; this is changing, at initial data of Table 1 ($51 \leqslant M \leqslant 501$, $3 \leqslant n \leqslant 50$), only approx. 4 times (from 4.0% to 16.9%).

Regarding the Huntington-Hill method, the P_Q value decreases with the increase of M (from about 0.3% at $M = 51$ to 0.02% at $M = 501$ for $n = 3$ and from about 10.3% at $M = 51$ to 0.003% at $M = 501$ for $n = 30$).

The P_Q value of the advanced Sainte-Laguë method is lower than for the Huntington-Hill one.

Thus, by percentage P_Q of compliance with the Quota rule, the preferences of APP divisor methods are: SL ≻ ASL ≻ HH ≻ d'H.

In general, the following preferences of APP methods were identified regarding the compliance with:

- the Lower Quota rule: {H, QVLD, LQVLD, QLD, LQLD, QDLD, d'H} ≻ SL ≻ VLD ≻ ASL ≻ HH. Exceptions exist regarding the VLD method: at $n \leqslant 10$ and $M \geqslant 101$, this in some cases is positioned between the ASL and HH methods, and in others - even after the HH method;
- the Upper Quota rule: {H, QVLD, QLD, QDLD} ≻ HH ≻ ASL ≻ SL ≻ LQLD ≻ VLD ≻ LQVLD ≻ d'H;
- the Quota rule: {H, QVLD, QLD, QDLD} ≻ LQLD ≻ SL ≻ LQVLD ≻ VLD ≻ ASL ≻ HH ≻ d'H. Exceptions exist regarding:

a) the d'Hondt method: at small values of $M - n$, this in some cases is positioned between the ASL and HH methods, and in others – even between the VLD and ASL methods;

b) the {LQVLD $\succ$ VLD} methods: at $M \geqslant 101$ and $n \leqslant 10$, these in some cases are positioned between the ASL and HH methods, and in others – even between the HH and d'H methods;

c) the VLD method: at $M \geqslant 51$ and $n \leqslant 20$, this in some cases is positioned between the ASL and HH methods, and in others – even between the HH and d'H methods.

5. Party favoring

The distribution of mandates between parties in LPR electoral systems is usually disproportionate, based on the passage of a part of influence power from some parties to others.

Favoring of parties leads to the increase of disproportionality. This is examined in details in [10].

Definition 1. It is considered that, as a result of apportionment, a party i is favored if it obtains an excess of influence power ($x_i > D_i$, where $D_i = MV_i/V$ is the influence power of party i, delegated to it by the V_i voters' votes), is disfavored, if it has a deficit of influence power ($x_i < D_i$), and is neutral (neither favored nor disfavored) if it obtains a power of influence equal to that delegated to it by voters' votes ($x_i = D_i$).

So, for a party i:

1) favored, occurs $x_i > a_i$;

2) disfavored, occurs $x_i \leqslant a_i$ at $V_i > a_iQ$;

3) neutral, occurs $x_i = a_i$ at $V_i = a_iQ$.

Definition 2. It is considered that the APP method favors large (small) parties, if as a result of apportionment, the summary influence power in excess ($x_i - D_i > 0$), obtained by parties with more (fewer) votes, is greater than that, obtained by parties with fewer (more) votes.

Definition 3. It is considered that the APP method is neutral (as a whole, favors neither large nor small parties), if the summary influence power in excess ($x_i - D_i > 0$), obtained by parties with fewer votes, is equal to that obtained by parties with more votes.

Statement 3. The Hamilton method may favor particular parties both large and small, but generally, on infinity of polls, it is neutral in favoring of parties.

Statement 4. When applying the GLD method (Figure 2):

1) if $c < n/(n - 1)$ then, as a rule, larger parties may be favored;

2) if $c > n$ then, as a rule, smaller parties may be favored;

3) if $n/(n - 1) \leqslant c \leqslant n$ then both the larger parties and the smaller ones may be favored, inclusively:

 3.1) at $n/(n - 1) \leqslant c < n/\Delta M$, more larger parties than smaller ones may be favored;

 3.2) at $n/\Delta M < c \leqslant n$, more smaller parties than larger ones may be favored;

 3.3) at $c = n/\Delta M$, both larger parties and smaller ones may be favored, but rarely.

Statement 5. For the GLD method, the optimal value of constant c (in sense of (1) and of not favoring of parties) is equal to 2. In average, the GLD method at $c < 2$ favors larger parties, and at $c > 2$ – the smaller ones. So:

- the d'Hondt method can favor particular parties, both large and small, but overall it favors large parties;

- the Sainte-Laguë method can favor particular parties, both large and small, but overall it is neutral. It may favor particular parties, predominantly, large – at $\Delta M \in [1; n/2)$, small – at $\Delta M \in (n/2; n - 1]$, and usually does not favor any party at $\Delta M = n/2$.

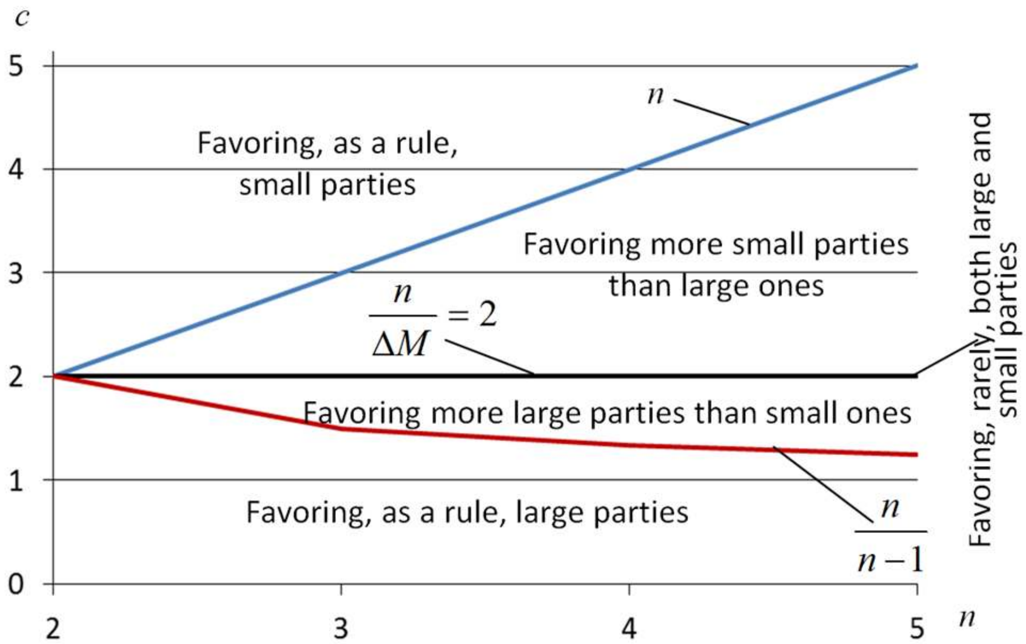

Figure 2. Favoring of parties by GLD method.

Statement 6. From the multitude of GLD method alternatives that differ only by the value of constant *c*, in average, on an infinite number of polls, the Sainte-Laguë method is the least predisposed to favoring particular parties, ensuring the least disproportionality *I* of apportionments.

Statement 7. From the multitude of GLD method alternatives that differ only by the value of constant *c*, even dependent on sizes *n* and Δ*M*, the least predisposed, probably, to favoring of parties apart is the VLD method, ensuring the lowest disproportionality of apportionments.

Statement 8. The Hamilton method is characterized, in average, by a less favoring of particular parties than the adapted Sainte-Laguë one.

Statement 9. The Huntington-Hill method may favor particular parties, both large and small, but generally favors small parties, the tendency in question decreasing with the increase of the number of votes cast by parties and tending towards neutrality.

Statement 10. The Huntington-Hill method is characterized, in average, by a higher favoring of parties than the adapted Sainte-Laguë one.

So, by not favoring of parties, we have preferences H ≻ SL ≻ ASL ≻ HH ≻ d'H.

6. Immunity to paradoxes

The so-called "paradoxes" have influenced the replacement of some APP methods with others in practice. It is well known the immunity to Alabama, Population, and New State paradoxes of such APP methods as Jefferson, Webster, d'Hondt, Sainte-Laguë, modified Sainte-Laguë and Huntington-Hill ones [1, 2]. The adapted Sainte-Laguë and GLD methods are immune to these paradoxes, too [10]. There are also a number of particular cases of elections for which other APP methods also are immune to the three paradoxes [10].

It is well-known the non-immunity to the three paradoxes of APP methods with remain-ders: Hamilton, Droop quota, Hagenbach-Bischoff quota and Imperiali quota [1, 2]. The QLD, LQLD, QDLD, VLD, QVLD and LQVLD methods are not immune to these paradoxes, too [10].

Statement 11 [10]. In case of two-party elections, the QLD, LQLD, QDLD and QVLD methods are immune to Alabama, Population and New state paradoxes, and the Hamilton and VLD methods – to Alabama and Population paradoxes.

Statement 12 [10]. In case of three-party elections, the QLD, LQLD and QDLD methods are immune to Alabama and Population paradoxes.

It is estimated [1] that for M = 435 and n = 50, the Hamilton method exhibits a monotonicity paradox about once in every 18 apportionments. Under certain assumptions as the number of seats tends to infinity, the expected number of states that will suffer from the Alabama paradox when applying the Hamilton method is asymptotically bounded above by 1/e and, on average, is approximately 0.123 [24].

Some results of computer simulation using SIMAP (see section 2) with refer to the three paradoxes are described below.

Alabama paradox. Calculations were performed for the increase δM of the total number of mandates M by 1 to M mandates ($1 \leqslant \delta M \leqslant M$). It was found that the percentage P_A of Alabama paradox is increasing to n and is decreasing (with rare deviations at M and n large, and δM small) to the number δM of additional mandates, becoming equal to 0 for at most $\delta M = M$.

Exception, regarding the dependence of P_A on M, are the QLD and LQLD methods – the value of P_A(QLD) and P_A(LQLD) slightly depends on M. At n = 2, occurs P_A = 0 for all examined methods – the Hamilton method and the six mixed methods; obviously there exist, however, cases for which the Alabama paradox occurs.

The character of percentage P_A dependence to M and n can be seen in Figure 3 and Table 2.

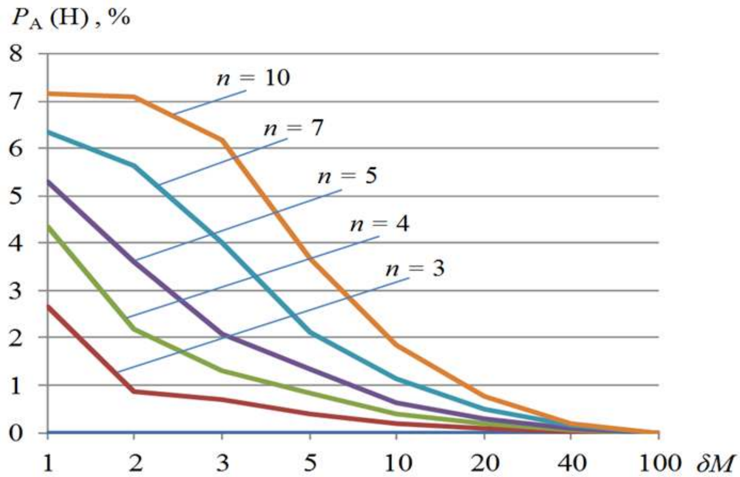

Figure 3. Parameter P_A dependence to M and n for Hamilton method.

The P_A percentage for QVLD, LQVLD, VLD and DQLD methods is slightly higher, and for QLD and LQLD methods it is much lower than for the Hamilton method. Moreover, if the maximum value of P_A for Hamilton method is 9.5827% (at M = 501, n = 50 and δM = 2), then the maximum value of P_A for QLD method is 0.0129% (at M = 51, n = 7 and δM = 1), i.e. more than 742 times smaller, this being 0% in most of other cases of 10^6 alternatives each. At 6 ⩽

$M \leqslant 501$, $3 \leqslant n \leqslant 50$, $n < M$ and $1 \leqslant \delta M \leqslant M$, the relationships occur: $0 \leqslant P_A(\mathrm{H}) \leqslant 9.58$ and $0 \leqslant P_A(\mathrm{QLD}) \leqslant 0.013$.

Table 2

Percentage P_A of Alabama paradox for the Hamilton and QLD methods

M	n	δM	P_A(H)	P_A(QLD)	M	n	δM	P_A(H)	P_A(QLD)
101	3	1	2.6646	0	501	3	1	2.6827	0
		2	0.8522	0			2	0.8504	0
		99	0	0			99	0	0
	4	1	4.3549	0		4	1	4.2981	0
		2	2.2120	0			2	2.1988	0
		100	0	0			100	0	0
	5	1	5.3302	0.0035		5	1	5.3046	0.0040
		2	3.6361	0.0003			2	3.6049	0.0003
		100	0	0			100	0	0

By immunity of APP methods to Alabama paradox, the following preferences occur:

{SL, ASL, HH, d'H} ≻ LQLD ≻ QLD ≻ QDLD ≻ H ≻ QVLD ≻ LQVLD ≻ VLD.

New state paradox. Calculations were performed for the $(n + 1)$ state's population V_{n+1} – random size of uniform distribution in interval $[Q; Q\lfloor M/n \rfloor]$, where $Q = (V_1 + V_2 + V_3 + \ldots + V_n)/M$.

The results of calculations are largely similar to those for Alabama paradox. The percentage P_S of New state paradox occurrence slightly depends on M, but it is increasing with respect to n (with some deviations for QLD and LQLD methods).

The value of P_S for QVLD, LQVLD, VLD and QDLD methods is slightly larger, and for QLD and LQLD methods it is over 1000 times smaller than for the Hamilton method. Moreover, in many cases of 10^6 alternatives each, the P_S value for LQLD and QLD methods is 0%; obviously there exist, however, cases for which the New state paradox occurs. At $6 \leqslant M \leqslant 501$, $2 \leqslant n \leqslant 50$ and $n < M$, the following relations occur: $0 \leqslant P_S(\mathrm{H}) \leqslant 8.46$ and $0 \leqslant P_S(\mathrm{QLD}) \leqslant 0.0041$. The character of percentage P_S dependence on M and n can be seen in Figure 4.

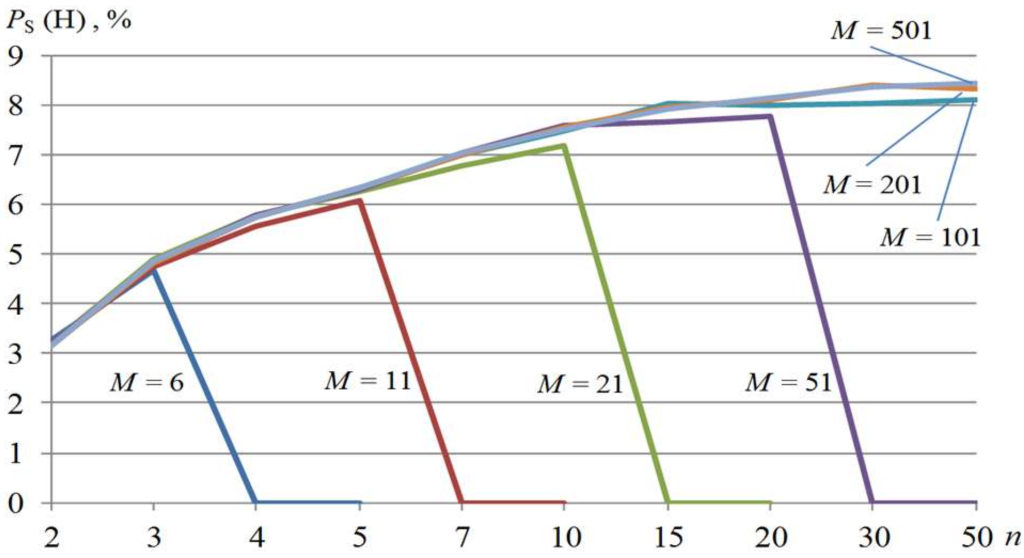

Figure 4. Parameter P_S dependence to M and n for Hamilton method.

Some results of calculations are shown in Table 3.

By immunity of APP methods to New state paradox, the following preferences occur:

{SL, ASL, HH, d'H} ≻ LQLD ≻ QLD ≻ QDLD ≻ H ≻ QVLD ≻ LQVLD ≻ VLD.

Population paradox. The total number of votes in the second poll V_+ is determined as $V_+ = (1 + b)V$. Here V is the total number of votes in the first poll, and $(1 + b)$ is the rate of V increase to the second poll. Values V_{i+}, $i = \overline{1,n}$ in the second poll are random sizes determined as $V_{i+} = p_i V_i$, $i = \overline{1,n}$, with corrections needed to make $V_+ = V_{1+} + V_{2+} + \ldots + V_n$; p_i is a stochastic size of uniform distribution in range $[b(1 - d); b(1 + d)]$ and d is a constant.

Table 3

Percentage P_S of New State paradox for Hamilton (H) and QLD methods

M	n	P_S(H)	P_S(QLD)	M	n	P_S(H)	P_S(QLD)
101	2	3.1705	0	501	2	3.1694	0
	3	4.8226	0		3	4.8615	0
	4	5.7543	0.0020		4	5.7436	0.0022
	5	6.3341	0.0038		5	6.3324	0.0040
	7	7.0230	0.0035		7	7.0311	0.0028
	10	7.4950	0.0011		10	7.5344	0.00063
	20	8.0157	0.00007		20	8.1542	0
	30	8.0408	0.00003		30	8.3556	0
	50	8.1086	0		50	8.4606	0

The character of percentage P_P dependence to n and d can be seen on Figure 5 and Table 4.

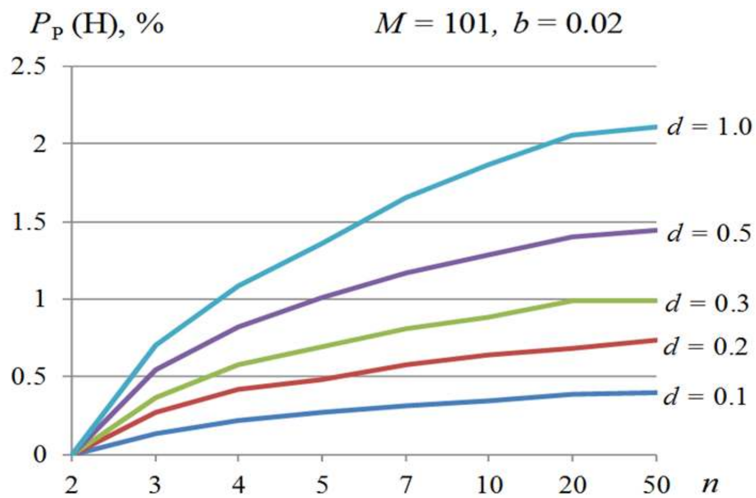

Figure 5. Parameter P_p dependence to M and n for Hamilton method.

The percentage P_P of Population paradox is slightly lower compared to those of Alabama (P_A) and New State (P_S) paradoxes; at n = 2 this is even equal to 0 for all seven explored methods. Also, for QVLD, LQVLD, VLD and QDLD methods it is almost the same; it is, however, slightly larger than that for the Hamilton method (P_P(H)). At the same time, the P_P value for QLD (P_P(QLD)) and LQLD (P_P(LQLD)) methods is at least 200 times (usually over 1000 times) lower than that for the Hamilton method. At $6 \leqslant M \leqslant 501$, $3 \leqslant n \leqslant 50$, $0.02 \leqslant b \leqslant 0.1$, $0.1 \leqslant d \leqslant 1$ and $n < M$, the relations $0.018 \leqslant P_P(\mathrm{H}) \leqslant 4.66$ and $0 \leqslant P_P(\mathrm{QLD}) \leqslant 0.0061$ occur.

Table 4

Percentage P_P of Population paradox for Hamilton and QLD methods

M	n	b	d	P_P(H)	P_P(QLD)	M	n	b	d	P_P(H)	P_P(QLD)
101	3	0.02	0.1	0.1372	0	501	3	0.02	0.1	0.5393	0
			0.3	0.3703	0				0.3	0.6064	0
			1	0.7057	0				1	0.2589	0
		0.1	0.1	0.4977	0			0.1	0.1	0.4634	0
			0.3	0.6587	0				0.3	0.2018	0
			1	0.2629	0				1	0.0585	0
	5	0.02	0.1	0.2656	0.0031		5	0.02	0.1	0.9848	0.0041
			0.3	0.6951	0.0043				0.3	1.3498	0.0028
			1	1.3589	0.0039				1	0.692	0.0008
		0.1	0.1	0.9433	0.0041			0.1	0.1	1.1284	0.0018
			0.3	1.3602	0.0028				0.3	0.5487	0.0003
			1	0.7264	0.0006				1	0.2010	0

By immunity of APP methods to Population paradox, the following preferences occur:
{SL, ASL, HH, d'H} ≻ LQLD ≻ QLD ≻ QDLD ≻ H ≻ QVLD ≻ LQVLD ≻ VLD.

7. Disproportionality of solutions

If not taking into account other constraints, the optimal solution of problem (1)-(2) is ensured by the Hamilton method [2, 10]. The mathematical expectancy $\bar{I}^*$ of disproportionality of such solutions is determined, approximately, as

$$\bar{I}^* = \frac{25}{M}\begin{cases} n, \text{ at } n = 2 \\ (n+\frac{1}{2})(1-\frac{1}{n^2}), \text{ at } n > 2, \end{cases} \tag{4}$$

the error not exceeding 0.5% mandates, and, in most cases, met in practice - 0.05% mandates [10]. Function $\bar{I}^*(M,n)$ is ascending to n and descending to M. Some results of calculations according to (4) are shown in Table 5.

Table 5.

Average disproportionality of Hamilton method's apportionments, % mandates

M	n									
	2	3	4	5	7	10	15	20	30	50
51	1.02	1.53	2.07	2.59	3.60	5.10	7.56	10.02	14.93	24.75
101	0.51	0.77	1.04	1.31	1.82	2.57	3.82	5.06	7.54	12.50
201	0.26	0.39	0.52	0.66	0.91	1.29	1.92	2.54	3.79	6.28
501	0.10	0.16	0.21	0.26	0.37	0.52	0.77	1.02	1.52	2.52

As proven in [10], in average, the Sainte-Laguë method is the least disproportionate of all linear divisor methods, i.e. the value c = 2 is optimal for the APP rule (3). So, the Sainte-Laguë method is less disproportionate than the d'Hondt one. Also, in average, the Sainte-Laguë method is less disproportionate than the Huntington-Hill and the adapted Sainte-Laguë ones [10]. In their turn, the adapted Sainte-Laguë method is less disproportionate than the Huntington-Hill one [10].

The average disproportion $\bar{I}$, when applying the compared methods, is determined by simulation using the SIMAP application for the initial data specified in section 2.

Some results of calculations are shown in Figure 6.

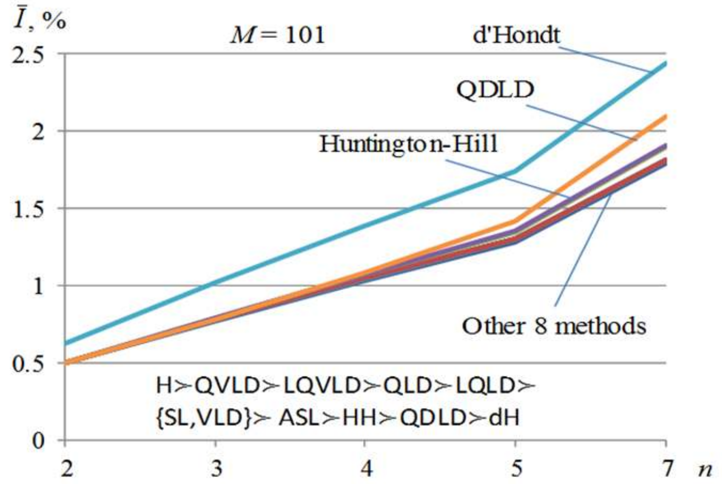

Figure 6. Dependence of disproportion $\bar{I}$ to *n*, at *M=101*.

From Figure 6 one can see that the highest average disproportions has the d'Hondt method, followed by the QDLD one and, after, by the Huntington-Hill method. The other eight methods are close to each other, but, at *M* = 101 and 2 ⩽ *n* ⩽ 15, the order by the increase of disproportion is: H ≻ QVLD ≻ LQVLD ≻ QLD ≻ LQLD ≻ SL ≻ VLD ≻ ASL ≻ HH ≻ QDLD ≻ d'H. The value of $\bar{I}$ is ascending to *n*. When using the d'Hondt method, for example, the value of $\bar{I}$ is:

a) 15.97% at *M* = 101 and *n* = 50;
b) 8.30% at *M* = 201 and *n* = 50;
c) 3.45% at *M* = 501 and *n* = 50.

More informatively, in graphical form, the situation is reflected by the difference between the disproportions of APP methods, Figures 7, 8.

From Figures 7 and 8 one can see to what extent the Sainte-Laguë method ($\bar{I}(\mathrm{SL})$) is better than the d'Hondt one ($\bar{I}(\mathrm{d'H})$), and the adapted Sainte-Laguë method ($\bar{I}(\mathrm{ASL})$) is better than the Huntington-Hill (*I*(HH)) one.

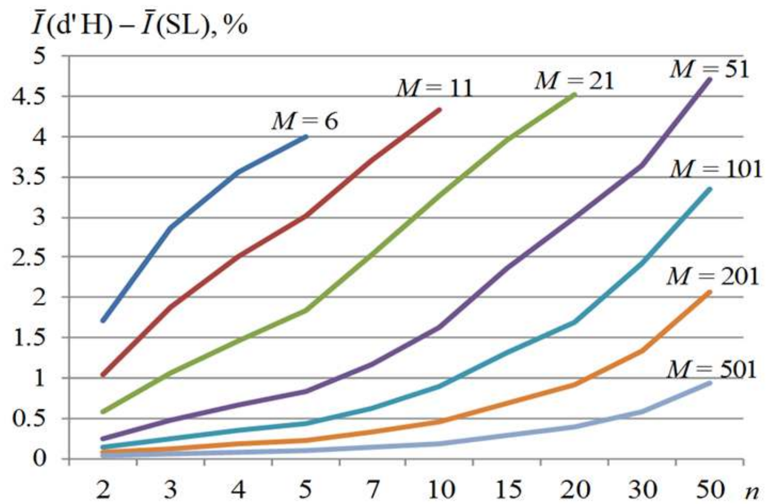

Figure 7. Dependence of $\bar{I}(\mathrm{d'H}) - \bar{I}(\mathrm{SL})$ to *M,* and *n*.

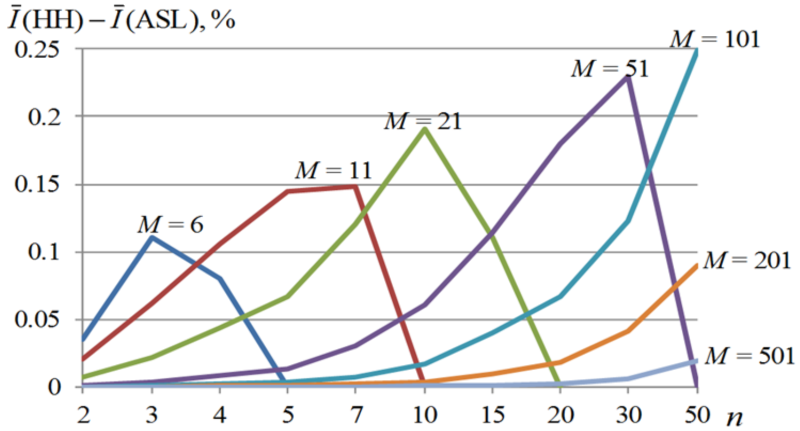

Figure 8. Dependence of $\bar{I}(\mathrm{HH})$ – $\bar{I}(\mathrm{ASL})$ to *M,* and *n*.

The value of difference $\bar{I}(\mathrm{d'H})$ – $\bar{I}(\mathrm{SL})$ is descending to *M* and is ascending to *n* (Figure 7). With reference to the difference $\bar{I}(\mathrm{HH})$ – $\bar{I}(\mathrm{ASL})$ (Figure 8), it is nonnegative and is also descending to *M*, but to *n* it is ascending at large values of ratio $(M - n)/M$, and is descending at small values of $(M - n)/M$, becoming 0 at $(M - n)/M = 1/M$.

Although, by disproportionality $\bar{I}$, in average, Sainte-Laguë method is better than the d'Hondt one, and the adapted Sainte-Laguë method is better than the Huntington-Hill one, no matter of parameters *M* and *n* values ($\bar{I}(\mathrm{d'H}) - \bar{I}(\mathrm{SL}) > 0$, $\bar{I}(\mathrm{HH}) - \bar{I}(\mathrm{ASL}) \geqslant 0$), there may be particular cases, when d'Hondt method, as well as Huntington-Hill method, provides a lower value of parameter *I* than the Sainte-Laguë and adapted Sainte-Laguë ones, respectively. To characterize such situations, parameters $R_{\mathrm{SL\text{-}d'H}}$ and $R_{\mathrm{ASL\text{-}HH}}$ are used. Parameter $R_{\mathrm{SL\text{-}d'H}}$ is the ratio of the percentage of apportionments, for which $\bar{I}(\mathrm{d'H}) > \bar{I}(\mathrm{SL})$, to the percentage of apportionments, for which $\bar{I}(\mathrm{d'H}) < \bar{I}(\mathrm{SL})$. Similarly, $R_{\mathrm{ASL\text{-}HH}}$ is the ratio of the percentage of apportionments, for which $\bar{I}(\mathrm{HH}) > \bar{I}(\mathrm{ASL})$, to the percentage of apportionments, for which $\bar{I}(\mathrm{HH}) < \bar{I}(\mathrm{ASL})$.

Some results of parameters $R_{\mathrm{SL\text{-}dH}}$, $R_{\mathrm{SL\text{-}HH}}$, $P_{\mathrm{SL=d'H}}$ and $P_{\mathrm{ASL=HH}}$ calculations are shown in Table 6 and Figures 9 and 10. Here $P_{\mathrm{SL=d'H}}$ is the percentage of apportionments for which $\bar{I}(\mathrm{d'H}) = \bar{I}(\mathrm{SL})$, and $P_{\mathrm{ASL=HH}}$ is the percentage of apportionments for which $\bar{I}(\mathrm{HH}) = \bar{I}(\mathrm{ASL})$.

Table 6

Parameters $P_{\mathrm{SL=d'H}}$ and $P_{\mathrm{ASL=HH}}$ dependence to *M* and *n*

	$P_{\mathrm{SL=d'H}}$						$P_{\mathrm{ASL=HH}}$					
*M**n*	2	3	4	5	7	50	2	3	4	5	7	50
101	78.9	63.0	50.5	40.7	26.4	0.0156	99.4	98.4	97.2	95.8	92.7	41.9
201	78.8	62.8	50.2	40.1	25.8	0.0033	99.6	99.0	98.3	97.3	95.1	42.1
501	78.8	62.7	50.0	40.0	25.5	0.0014	99.8	99.5	99.1	98.6	97.4	48.4

Data of Table 6 show that, in cases of $101 \leqslant M \leqslant 501$ and $2 \leqslant n \leqslant 5$, for over 40% of apportionments, the d'Hondt method gives the same allocation of seats as the Sainte-Laguë one does ($\bar{I}(\mathrm{d'H}) = \bar{I}(\mathrm{SL})$). In case of the adapted Sainte-Laguë and Huntington-Hill methods, the value of such an index is over 95%. Both characteristics, $P_{\mathrm{SL=d'H}}$ and $P_{\mathrm{ASL=HH}}$, are descending to n, but $P_{ASL=HH}$ is slightly descending to M, too, while $P_{SL=d'H}$ is slightly ascending to M.

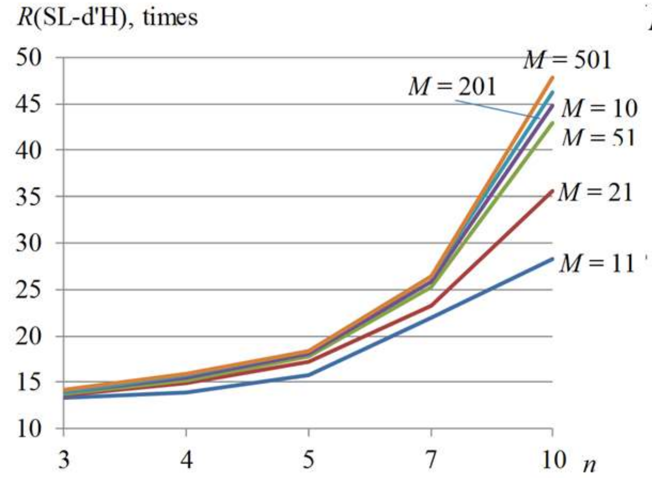

Figure 9. Dependence of ratio R(SL-d'H) to M and n.

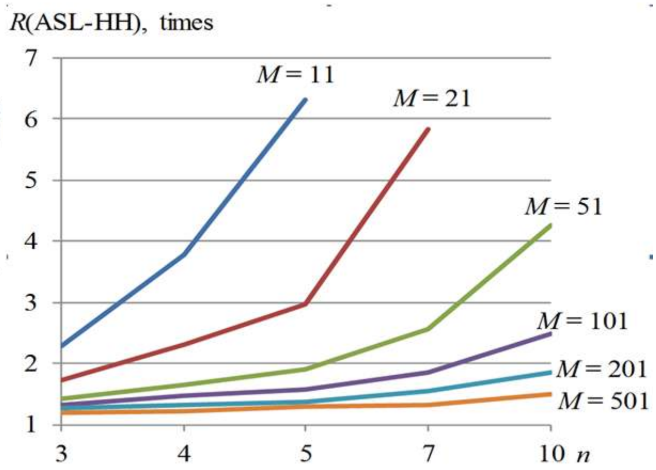

Figure 10. Dependence of ratio R(ASL-HH) to M and n.

Figure 9 shows that ratio $R_{SL-d'H}$ is ascending to n and to M, while ratio R_{ASL-HH} (Figure 10) is ascending to n, but is descending to M. For the same initial data, ratio $R_{SL-d'H}$ is considerably larger (13 – 20 times and more) than the R_{ASL-HH} one (1.2-1.5 times and more). Also, at n = 2 occurs $\bar{I}(\mathrm{d'H}) = \bar{I}(\mathrm{SL})$ and there are no cases for which $\bar{I}(\mathrm{HH}) < \bar{I}(\mathrm{ASL})$.

Overall, of the monotonous linear divisor methods the most efficient (less disproportionate) is the Sainte-Laguë method; also, the adapted Sainte-Laguë method is less disproportionate than the Huntington-Hill one.

The quantitative comparison of the Sainte-Laguë method (the less disproportionate APP method which complies to the tree paradoxes) with the Hamilton one (the less disproportionate APP method) is made in Figure 11 and Table 7.

Table 7

Percentage $P_{SL=H}$ of cases in which $\bar{I}(\mathrm{SL}) = \bar{I}(\mathrm{H})$, %

*M**n*	2	3	4	5	7	10	15	20	30	50
101	100	93.4	90.5	88.0	84.1	79.9	75.0	71.0	65.5	55.7
201	100	93.4	90.5	88.0	84.2	79.9	74.8	71.1	65.5	58.2
501	100	93.4	90.5	88.0	84.1	80.0	74.9	71.1	65.6	58.4

It is known that there are no cases in which the Sainte-Laguë method ensures a less disproportionate solution than the Hamilton method does.

At the same time, the percentage of cases in which the Sainte-Laguë method ensures the same solution as Hamilton method does $(P(\bar{I}(\mathrm{SL}) = \bar{I}(\mathrm{H})))$, practically does not depend on M, but is descending to n, being 100% at n = 2, and more than 55% at n = 50 (Table 7).

Figure 11 shows that the average value of the difference I(SL) − I(H) is ascending to n but is descending to M, beginning with 0% at n = 2, and for n = 50 reaching no more than 0.038 % at M = 101, no more than 0.017% at M = 201 and no more than 0.0065% at M = 501. Of the five mixt methods, the LQLD and QLD ones have considerably better features with regard to the three paradoxes (the probability of their occurrence is at least 200 times (usually over 1000 times) lower than for the Hamilton method.

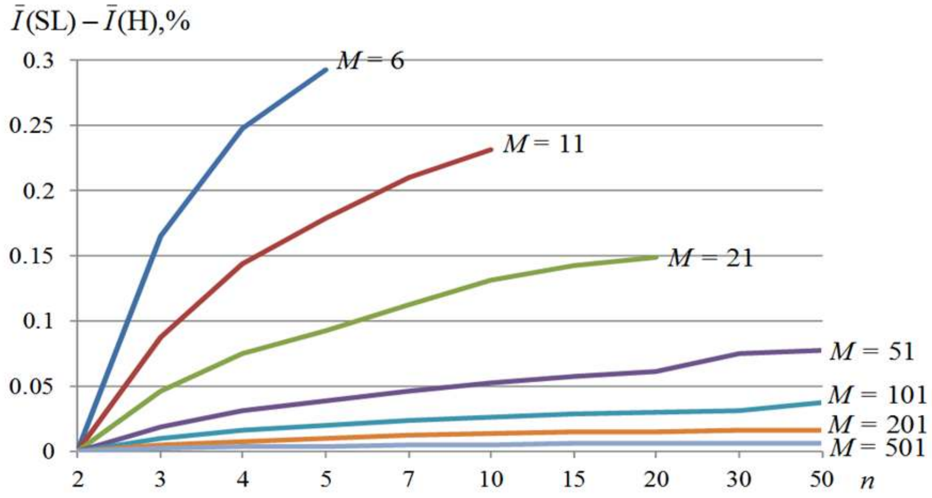

Figure 11. Dependence of $\bar{I}(\text{SL}) - \bar{I}(\text{H})$ to M and n.

Also, they are the second and, respectively, the third, after QVLD and LQVLD, less disproportionate mixt method at a very little difference. At the same time, QLD method follows the Quota rule, while the LQLD one doesn't. Therefore, the QLD method is the only method that is compared quantitatively, by disproportionality, with the Sainte-Laguë and Hamilton methods.

To be mentioned that, by computer simulation using SIMAP and the initial data specified in section 2, there have not beed identified cases when $\bar{I}(\text{SL}) < \bar{I}(\text{QLD})$; although such cases exist [10], they are rear. At the same time, at n = 2 and n = 3 (no matter the value of parameter M), and at n = 50 for 51 ⩽ M ⩽ 501 the equality $\bar{I}(\text{SL}) = \bar{I}(\text{QLD})$ occurs. The percentage of cases for which $\bar{I}(\text{SL}) = \bar{I}(\text{QLD})$ is equal to 100% − P_Q(SL) (see Table 1). From Figure 12 it results that $\bar{I}(\text{QLD}) - \bar{I}(\text{H})$ ⩽ $\bar{I}(\text{SL}) - \bar{I}(\text{H})$, but taking into account the little value of difference $\bar{I}(\text{SL}) - \bar{I}(\text{QLD})$, the dependence of difference $\bar{I}(\text{QLD}) - \bar{I}(\text{H})$ to M and n is close to that of difference $\bar{I}(\text{SL}) - \bar{I}(\text{H})$ to M and n (Figure 11).

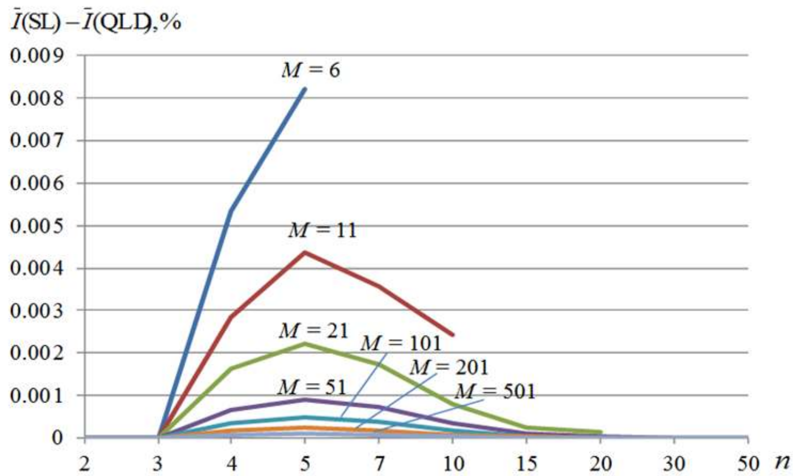

Figure 12. Dependence of $\bar{I}(\text{SL}) - \bar{I}(\text{QLD})$ to M and n.

By disproportionality of solutions, the following preferences of APP methods occur: H ≻ QVLD ≻ LQVLD ≻ QLD ≻ LQLD ≻ {SL, VLD} ≻ ASL ≻ HH ≻ d'H ≻ QDLD; at n < 10÷15 (also, when M = 6 and n = 5), we have $\bar{I}(\mathrm{SL}) < \bar{I}(\mathrm{VLD})$, and at $n \geqslant 10 \div 15$, we have $\bar{I}(\mathrm{SL}) > \bar{I}(\mathrm{VLD})$.

8. Multi-aspectual comparisons of apportionment methods

Qualitative and quantitative comparisons contribute to the selection of a suitable APP method in specific situations. Some characteristics of APP methods taken from sections 4-7 are systematized in Table 8. It should be noted that the disproportionality of representation of each party in the decision is less than one mandate if the Quota rule is followed, and may be more than one mandate for some parties if it is not followed.

In addition to Table 8, by not favoring of parties, the following preferences occur: H ≻ SL ≻ ASL ≻ HH ≻ d'H.

Table 8

Qualitative characteristics of some apportionment methods

Method	Following the Quota rule/ rating			Immunity to paradoxes/ rating			Not favoring parties		Dispropor-tionality rating
	lower	upper	total	Alabama	Population	New state	small	large	
Hamilton	yes/ 1-7	yes/ 1-4	yes/ 1-4	n = 2/ 7	n = 2/ 7	no/ 7	yes	yes	1
D'Hondt	yes/ 1-7	n = 2/ 11	n = 2/ 11	yes/1-4	yes/1-4	yes/1-4	yes	no	10
Sainte-Laguë	$n = \overline{2,3}$/ 8	$n = \overline{2,3}$/ 7	$n = \overline{2,3}$/ 6	yes/1-4	yes/1-4	yes/1-4	yes	yes	6-7
Huntington-Hill	n = 2/ 11	n = 2/ 5	n = 2/ 10	yes/1-4	yes/1-4	yes/1-4	no	yes	9
ASL	n = 2/ 10	$n = \overline{2,3}$ /6	n = 2/ 9	yes/1-4	yes/1-4	yes/1-4	yes*	yes	8
VLD	n = 2/ 9	n = 2/ 9	n = 2/ 8	n = 2/ 10	n = 2/ 10	no/ 9	yes	yes	6-7
QVLD	yes/ 1-7	yes/ 1-4	yes/ 1-4	n = 2/ 8	n = 2/ 8	no/ 8	yes	yes	2
LQVLD	yes/ 1-7	n = 2/ 10	n = 2/ 7	n = 2/ 9	n = 2/ 9	no/ 10	yes	no	3
QLD	yes/ 1-7	yes/ 1-4	yes/ 1-4	$n = \overline{2,3}$/ 5	$n = \overline{2,3}$/ 5	n = 2/ 5	yes	yes	4
LQLD	yes/ 1-7	$n = \overline{2,3}$/ 8	$n = \overline{2,3}$/ 6	$n = \overline{2,3}$/ 6	$n = \overline{2,3}$/ 6	n = 2/ 6	yes	no	5
QDLD	yes/ 1-7	yes/ 1-4	yes/ 1-4	$n = \overline{2,3}$/ 7	$n = \overline{2,3}$/ 7	n = 2/ 7	$n = \overline{2,3}$	yes	11

* Except parties for which a_i = 0.

To mention that criteria of following the Lower quota and the Upper quota rules are auxiliary, the main, resultant, being the criterion of following the Quota rule (total). The last will be used in comparing the 11 APP methods.

According to Table 8, none of the compared methods completely prevails over the others. But if to consider the other constraints too, such a prevalence exists in many cases. As such constraints used in practice are the mandatory of immunity to the three paradoxes and/or the allocation of minimum one mandate to each party (state).

The Hamilton method yields to the other methods only in that it is not immune to the three paradoxes. So, if the immunity to the three paradoxes is not mandatory, the Hamilton method is the best one (the adapted Hamilton method, if the allocation of minimum one mandate to each party is required).

Of the four methods that are immune to the three paradoxes, D'Hondt, Sainte-Laguë, Huntington-Hill and adapted Sainte-Laguë ones, the Sainte-Laguë method prevails upon the other three methods. So, if the immunity to the three paradoxes is mandatory but the allocation of a minimum number g > 0 of mandates to each party is not mandatory, the Sainte-Laguë method is the best.

Of the two methods that ensure the allocation of minimum one mandate to each party, the Huntington-Hill and the adapted Sainte-Laguë ones, the last method prevails upon the first by all criteria in Table 8. So, if the immunity to the three paradoxes and the allocation of minimum one (or more) mandate to each party are required, the adapted Sainte-Laguë method is the best.

With reference to the VLD and five mixt methods, their disproportionality, except the QDLD one, differs slightly. However, a position apart is beheld by the QLD method. If not take the disproportionality into account, the QLD method prevails over all the other five methods by all other criteria of Table 8. Moreover, the non-immunity to the three paradoxes of the QLD method is considerably more rear (more than 200÷1000 times) than that of the Hamilton method (0% ⩽ P_P(QLD) ⩽ 0.0061%, 0% ⩽ P_S(QLD) ⩽ 0.0041%, 0 ⩽ P_A(QLD) ⩽ 0.013%). So, the QLD method is nearly immune to the three paradoxes, is less disproportionate than the Sainte-Laguë one and is compliant with the Quota rule.

For details on comparative analysis, in addition to the qualitative characteristics of Table 8, the quantitative characteristics, including that of Tables 1-7 and Figures 1-12, may be useful.

9. Conclusions

A few new main qualitative and quantitative characteristics of 11 apportionment methods have been determined and some known ones have been systemized. It refers to such characteristics of APP methods as: disproportionality of solutions, solutions' compliance with the Quota rule, favoring of parties and the immunity to Alabama, Population and New state paradoxes (Sections 4-8). Formula (4) ensures the estimation of mathematical expectancy of Hamilton method apportionments' disproportionality, the error not exceeding, in most cases met in practice, 0.05% mandates.

By computer simulation (for initial data specified in Section 2 with concretizations and complementation in some cases described below), it was found that:

1) for initial data {M = 101, 201, 501; n = 2, 3, 4, 5, 7, 10, 15, 20}, the mathematical expectancy of solutions' disproportionality varies in the range (% mandates):
   a) 0.100 ÷ 5.03 − for the Hamilton method;
   b) 0.100 ÷ 5.06 - for the Sainte-Laguë and Quota linear divisor methods;
   c) 0.100 ÷ 5.81 - for the adapted Sainte-Laguë method;
   d) 0.100 ÷ 5.88 - for the Huntington-Hill method;
   e) 0.126 ÷ 6.75 - for the d'Hondt method.
2) the percentage of Quota rule violation varies in the range of:
   a) 0% ÷ 0.083% - for the Sainte-Laguë method;
   b) 0.0001% ÷ 5.61% - for the adapted Sainte-Laguë method, at $2 \leqslant n \leqslant M/2$;
   c) 0.0002% ÷ 7.18% - for the Huntington-Hill method, at $2 \leqslant n \leqslant M/2$;
   d) 2.69% ÷ 18.77% - for the d'Hondt method.
3) the percentage of paradoxes occurrence for the Hamilton method varies in the range of:
   a) 2.56% ÷ 8.36% - for the Alabama paradox at δM = 1 and $2 < n < M - 1$;
   b) 3.17% ÷ 8.46% - for the New state paradox at $2 \leqslant n \leqslant M/2$ and V_{n+1} a random value in the range $[Q; Q\lfloor M/n \rfloor]$;
   c) 0.01% ÷ 4.65% - for the Population paradox at {b = 0.02, 0.1; d = 0.1, 0.3, 1} and $2 < n < M - 1$.

To be mentioned that the percentage of Quota rule violation when applying the Sainte-Laguë method is to a small extent influenced by the total number of seats, but it decreases strongly (over 400 times) with the increase of the number of states − from approx. 0.045% at 4 states, up to 0.0001-0.0004% at 30 states.

The performed comparative analysis shows the following preferences of APP methods:

- by the disproportionality of solutions (with few exceptions between the SL and VLD methods),

    H ≻ QVLD ≻ LQVLD ≻ QLD ≻ LQLD ≻ {SL, VLD} ≻ ASL ≻ HH ≻ d'H ≻ QDLD;
- by the percentage of compliance with the Quota rule (with exceptions described in Section 4),

    {H, QVLD, QLD, QDLD} ≻ LQLD ≻ SL ≻ LQVLD ≻ VLD ≻ ASL ≻ HH ≻ d'H;
- by not favoring options (parties, etc.), H ≻ SL ≻ ASL ≻ HH ≻ d'H;
- by the percentage of Alabama paradox,

    {SL, ASL, HH, d'H} ≻ LQLD ≻ QLD ≻ QDLD ≻ H ≻ QVLD ≻ LQVLD ≻ VLD;
- by the percentage of New state paradox,

    {SL, ASL, HH, d'H} ≻ LQLD ≻ QLD ≻ QDLD ≻ H ≻ QVLD ≻ LQVLD ≻ VLD;
- by the percentage of Population paradox,

    {SL, ASL, HH, d'H} ≻ LQLD ≻ QLD ≻ QDLD ≻ H ≻ QVLD ≻ LQVLD ≻ VLD.

Based on these characteristics, it has been found that none of the 11 explored methods is overall preferable by each of the six characteristics-criteria. Of the 11 investigated methods, only three or four are reasonable to use in specific areas:

1) the Hamilton method (adapted Hamilton, if the allocation of minimum $g \geqslant 1$ seats to each state is required), if the immunity to the three paradoxes is not mandatory;
2) the Sainte-Laguë method and, may be, the Quota linear divisor one, if the immunity to the three paradoxes is required but the allocation of minimum one (or more) mandate to each state is not mandatory;
3) the adapted Sainte-Laguë method, if the immunity to the three paradoxes and the allocation of minimum one (or more) mandate to each party are required.

Also, the obtained quantitative characteristics allow estimating the extent to which each of the 11 explored apportionment methods corresponds to the need of a specific beneficiary. At the same time, they may be useful for the quantitative comparative analysis of methods, especially of their difference regarding the disproportionality of apportionments.

References

1. Balinski M.L., Young, H.P. *Fair Representation: Meeting the Ideal of One Man, One Vote*. 2nd ed. Washington, DC: Brookings Institution Press, 2001.
2. Gallagher M. Proportionality, Disproportionality and Electoral Systems. In: *Electoral Studies*, 1991, 10(1), pp. 33-51.
3. Hare T. *The Election of Representatives, Parliamentary and Municipal*. London, 1859.
4. d'Hondt V. *La Répresentation Proportionnelles des Partis par un Electeur*. Ghent, 1878.
5. Lijphart A., Gibberd, R.W. Thresholds and payoffs in list systems of proportional representation. In: *European Journal of Political Research*, 1977, 5(3), pp. 219−44.
6. Huntington, E. A new method of apportionment of representatives. In: *Quart. Publ. Amer. Stat. Assoc*, 1921, 17, pp. 859-1970.

7. Sorescu A., Pirvulescu, C. and al. *Electoral systems*. Bucharest: Pro Democratia, 2006 (Romanian).
8. Norris P. *Electoral Engineering: Voting Rules and Political Behavior*. Cambridge: Cambridge University Press, 2004.
9. European Election Database, 2013 (http://www.nsd.uib.no/european_election_database/country/, accessed 25.06.2019).
10. Bolun, I. *Optimization of multi-optional decisions*. Chisinau: ASEM Publishing House, 2016 (Romanian).
11. Droop, H.R. On Methods of Electing Representatives. In: *Journal of the Statistical Society of London*, 1881, 44(2), pp. 141-202 (Reprinted in *Voting matters*, 2007, 24, pp. 7-46).
12. Rae D.W. *The Political Consequences of Electoral Lows*. New Heaven: Yale University Press, 1967.
13. Loosermore, J., Handby, V. The Theoretical Limits of Maximum Distortion: some Analytic Expressions for Electoral Systems. In: *British Journal of Political Sciences*, 1971, 1(4), pp. 467-477.
14. Rose R., Munro, N., Mackie, T. *Elections in Central and Eastern Europe Since 1990*. Strathclyde: Center for the Study of Public Policy, 1998.
15. Grofman B., Lijphart, A. *Electoral Lows and their Political Consequences*. New York: Agathon Press, 1986.
16. Lijphart A. *Electoral Systems and Party Systems*. Oxford: Oxford University Press, 1994.
17. Karpov A. Measurement of disproportionality in proportional representation. In: *Mathematical and Computer Modelling*, 2008, 48, pp. 1421-1438.
18. Duncan O.D., Duncan, B. A methodological analysis of segregation indexes. In: *American Sociological Review*, 1955, 20(2), pp. 210-217.
19. Martinez-Panero, M., Arredondo, V., Pena, T., Ramirez, V. A New Quota Approach to Electoral Disproportionality. In: *Economies*, 2019, 17(7), pp. 1-17.
20. Benoit K. Which Electoral Formula Is the Most Proportional? A New Look with New Evidence. In: *Political Analysis*, 2000, 8(4), pp. 381-388.
21. Bouyssou D., Marchant, T., Pirlot, M. A characterization of two disproportionality and malapportionment indices: the Duncan and Duncan index and the Lijphart index. In: *Annals of Operations Research*, 2019.
22. Kohler U., Zeh, J. Apportionment methods. In: *The Stata Journal*, 2012, 12(3), pp. 375–392.
23. Smith W.D. Apportionment and rounding schemes, 2007 (http://rangevoting.org/Apportion.html, accessed 16.08.2019).
24. Janson S., Linusson, S. The probability of the Alabama paradox. In: Journal of Applied Probability, 2012, 49(3), pp. 773-794.