The Annals of Probability
1978, Vol. 6, No. 2, 279-293

CHARACTERIZATION OF SUBCLASSES OF CLASS L PROBABILITY DISTRIBUTIONS

BY A. KUMAR AND B. M. SCHREIBER[1]

Northeastern Illinois University and Wayne State University

The subclasses of class L probability distributions recently studied by K. Urbanik are characterized by requiring that certain functions be convex and have derivatives of some fixed order. The extreme points of certain compact convex sets of probability measures are determined, and this information is then used to obtain a representation of the characteristic functions of the probability distributions in those classes, in the same manner as Urbanik has proceeded for the class L.

0. Introduction. The set of self-decomposable, or class L, probability measures plays a fundamental role in the description of the limit laws of sequences of random variables. It is well known that a probability measure on the real line is self-decomposable if and only if certain functions obtained from its Lévy measure are convex and that all stable probability measures are self-decomposable (see, e.g., [3], Section 23). In recent years the self-decomposable measures have been the object of further study by Urbanik and by the authors, some of this work appearing in [1], [2] and [5].

In [6] Urbanik has subclassified the self-decomposable distributions inductively so as to obtain a decreasing sequence $\{L_m\}$ of classes, each of which is closed under shifts, changes of scale, convolution and passages to weak limits, such that their intersection L_∞ is the smallest class closed under these operations containing the stable distributions. He obtained a characterization of the measures in each of these classes in terms of their components.

In this paper, we begin in Section 2 by characterizing each of these classes L_m $(1 \leqq m \leqq \infty)$ in terms of the functions referred to above. These characterizations are then used in Section 3 to find the extreme points of the compact convex sets of Lévy probability measures corresponding to distributions in each of the classes L_m. For the entire class of self-decomposables, this program was carried out by Urbanik in [5]. Finally, in Section 4, we point out how this information leads to the representation of the characteristic functions of the measures in each of the classes L_m, obtained in [6] by other methods.

In a sequel to this paper we shall use the results of Section 3 and the methods

Received February 22, 1976; revised August 3, 1976.

[1] Research of this author was partially supported by the National Science Foundation under Grant No. 75-07115. This work was undertaken while he was a Visiting Professor at the Institute of Mathematics of the Hebrew University of Jerusalem and a Science Research Council Senior Visiting Fellow at the Mathematical Institute of the University of Edinburgh.

AMS 1970 *subject classifications.* Primary 60B15, 60E05, 60F05, 60G50; Secondary 28A50.

Key words and phrases. Characteristic function, Lévy-Khincthine representation, extreme points, infinitely divisible distribution, convex function, completely monotonic function.

of [2] to represent the corresponding subclasses of the class L on higher-dimensional spaces and certain Banach spaces.

1. Preliminaries and notation. Let X_{nj}, $j = 1, 2, \cdots, n$, $n = 1, 2, 3, \cdots$ be a triangular array of uniformly asymptotically negligible random variables such that for each n the X_{nj}, $j = 1, 2, \cdots, n$ are independent. Following [6], we say that a triangular array as above is generated by a sequence $\{X_n\}$ if for each n and j, X_{nj} is distributed as X_j, and we call two triangular arrays equivalent if they have the same limiting distribution. It follows that a triangular array $\{X_{nj}\}$ is equivalent to an array generated by a sequence $\{X_n\}$ if and only if for suitably chosen a_n and b_n the sequences

$$\frac{1}{n} \sum_{j=1}^{n} X_{nj} - a_n \qquad \text{and} \qquad \frac{1}{n} \sum_{k=1}^{n} X_k - b_n$$

have the same limiting distribution.

Now define S_m $(m = 0, 1, 2, \cdots)$ inductively as follows. Let S_0 be the class of sequences $\{X_n\}$ of independent random variables generating convergent triangular arrays. Then S_0 is the class of sequence $\{X_n\}$ of independent random variables such that for suitably chosen constants a_n, $(1/n) \sum_{k=1}^{n} X_k - a_n$ has a limit distribution. Define S_m $(m = 1, 2, 3, \cdots)$ to be the class of all sequences $\{X_n\}$ such that $\{X_n\} \in S_0$ and for every positive real number c the triangular array $X_{nj} = X_{[cn]+j}$ is equivalent to an array generated by a sequence from S_{m-1}. Let $L_\infty = \bigcap_{m=0}^{\infty} L_m$, where L_m is the set of all possible limit distributions of normed sums $(1/n) \sum_{k=0}^{n} X_k - a_n$ where $\{X_n\} \in S_m$ and a_n are real constants. We define L_{-1} to be the set of all probability measures on the real line R. It is clear that the S_m form a decreasing sequence of sets and hence the L_m form a decreasing sequence.

For a probability measure μ, $T_a\mu$, $a > 0$, and $\tilde{\mu}$ are defined to be the probability measures given by $T_a\mu(A) = \mu(a^{-1}A)$, where $a^{-1}A = \{a^{-1}x : x \in A\}$, and $\tilde{\mu}(A) = \mu(-A)$. The following theorem was proved by Urbanik in [6], page 227.

THEOREM 1.1. *A probability measure μ belongs to L_m $(m = 0, 1, 2, \cdots, \infty)$ if and only if for each $c \in (0, 1)$ there exists a probability measure $\mu_c \in L_{m-1}$ such that $\mu = T_c\mu * \mu_c$.*

Here $*$ denotes the convolution between two probability measures. The measures μ_c are called the components of μ.

Let $\{X_n\} \in S_0$. Consider the normed sums

$$Y_n = \frac{1}{b_n} \sum_{k=1}^{n} X_k - a_n \qquad n = 1, 2, \cdots,$$

where a_n are real, $b_n > 0$ and the random variables X_k/b_n $(k = 1, 2, \cdots, n)$ are uniformly asymptotically negligible. It is well known that the class of all limiting distributions of such sequences Y_n coincides with the class L_0 of all self-decomposable distributions. The problem of describing the probability measures in L_0 was solved by Lévy, who obtained an explicit representation of the characteristic functions of those measures. Namely, the function ϕ is the characteristic function of a distribution from L_0 if and only if ϕ is the characteristic function of an infinitely divisible probability measure and the functions

$$Q_\mu(x) = \int_{(e^x,\infty)} \frac{1+y^2}{y^2}\, d\mu(y)\,, \qquad Q_{\tilde\mu}(x) = \int_{(e^x,\infty)} \frac{1+y^2}{y^2}\, d\mu(y)$$

are convex functions on R, where μ is the finite Borel measure on R determined uniquely by the Lévy–Khintchine representation. For details see [3], pages 323–326.

The aim of this paper is to characterize the classes L_m $(m = 1, 2, 3, \cdots, \infty)$ by requiring $Q_\mu(x)$ and $Q_{\tilde\mu}(x)$ to be not only convex but also have derivatives of order m at every point. We then use this characterization to find the extreme points of a certain compact convex set in a somewhat similar fashion as Urbanik has done for L_0. After finding the extreme points we characterize the L_m by their Lévy–Khintchine representations.

Let us make some further definitions. If ϕ is the characteristic function of an infinitely divisible probability distribution, then the Lévy–Khintchine representation of ϕ is given by

$$\phi(t) = \exp\left\{i\gamma t + \int_{-\infty}^{\infty} \left(e^{ity} - 1 - \frac{ity}{1+y^2}\right) \frac{1+y^2}{y^2}\, d\mu(y)\right\}\,,$$

where μ is a finite Borel measure and γ is a real constant. The function ϕ uniquely determines μ and γ. We shall call μ the Lévy measure of ϕ. Let $\bar R$ be the compactified line $[-\infty, \infty]$, and if μ is a measure on $\bar R$ we define $\tilde\mu$ as before. Let the symbol μ stand throughout the remainder of this section for a finite Borel measure on $\bar R$. Consider the one-parameter group $\{S_c\}_{c>0}$ of translation operators on functions on R given by $S_c f(x) = f(x - \log c)$. We denote the identity S_1 by I. Now we define sets M_m, $M_m{}^0$, K_m and $K_m{}^0$, and the functions $J_m(u)$ and $Q_\mu(x)$ as follows:

$$Q_\mu(x) = \int_{(e^x,\infty]} \frac{1+y^2}{y^2}\, d\mu(y)\,, \qquad x \in R\,.$$

$$M_0 = \{\mu : Q_\mu(x) \text{ and } Q_{\tilde\mu}(x) \text{ are convex functions}\}.$$

$$M_m = \{\mu : \textstyle\prod_{k=1}^m (I - S_{c_k})Q_\mu(x) \text{ and } \prod_{k=1}^m (I - S_{c_k})Q_{\tilde\mu}(x) \text{ are convex functions for every } c_1, \cdots, c_m \in (0,1)\}\,, \qquad m = 1, 2, 3, \cdots\,.$$

$$M_m{}^0 = \{\mu : \mu \in M_m \text{ and } \mu \text{ is concentrated on } R\}\,, \qquad m = 0, 1, 2, \cdots\,.$$

$$K_m = \{\mu : \mu \in M_m \text{ and } \mu \text{ is a probability measure}\}\,, \qquad m = 0, 1, 2, \cdots\,.$$

$$K_m{}^0 = K_m \cap M_m{}^0\,, \qquad m = 0, 1, 2, \cdots\,.$$

$$J_m(u) = \int_0^u \left(\log \frac{u}{y}\right)^m \frac{y}{1+y^2}\, dy\,, \qquad u \in R,\ m = 0, 1, 2, \cdots\,.$$

Note that $J_m(-u) = J_m(u)$, and one can easily check that if $u > 0$, then $J_m(u) \leqq \int_0^u (\log u/y)^m y\, dy = m/2 \int_{-\infty}^{\log u} (\log u - t)^{m-1} e^{2t}\, dt$ for $m = 1, 2, \cdots$. Thus since $J_0(u) = \frac{1}{2}\log(1 + u^2)$, we conclude that $J_m(u)$ is finite for every u.

2. Characterization of L_m. In this section we obtain some convexity conditions on $Q_\mu(x)$ and $Q_{\tilde{\mu}}(x)$ which characterize the classes L_m.

PROPOSITION 2.1. *Let λ be an infinitely divisible probability measure on R. Then λ is in L_0 if and only if for every $c \in (0, 1)$, $\nu_c = \nu - T_c\nu$ is a nonnegative Borel measure on R, where $\nu(E) = \int_E ((1 + y^2)/y^2)\, d\mu(y)$ and μ is the Lévy measure of λ.*

PROOF. We know by Theorem 1.1 that $\lambda \in L_0$ if and only if $\lambda = T_c\lambda * \lambda_c$ for every $c \in (0, 1)$. Since λ and λ_c are infinitely divisible by [3], page 323, we have

$$\begin{aligned}\hat{\lambda}(t) = \exp\Big[ict\Big(\gamma + (1 - c^2)\int_R \frac{y^3\, d\nu(y)}{(1 + c^2y^2)(1 + y^2)}\Big) \\ + \int_R \Big(e^{iyt} - 1 - \frac{iyt}{1 + y^2}\Big)\, dT_c(y)\Big] \\ \times \exp\Big[i\gamma_c t + \int_R \Big(e^{iyt} - 1 - \frac{iyt}{1 + y^2}\Big)\, d\nu_c(y)\Big].\end{aligned}$$

Thus $\lambda \in L_0$ if and only if $\nu = T_c\nu + \nu_c$. Hence that for every $c \in (0, 1)$ $\nu - T_c\nu$ is a nonnegative Borel measure on R is a necessary and sufficient condition for λ to belong to L_0.

THEOREM 2.1. *Let λ be an infinitely divisible probability measure on R. Then $\lambda \in L_m$ $(m = 0, 1, 2, \cdots)$ if and only if its Lévy measure μ belongs to $M_m{}^0$.*

PROOF. We proceed by induction on m. The proof in case $m = 0$ is well known, but we include it for completeness. Let $m = 0$ and $\lambda \in L_0$. Then by Proposition 2.1,

$$\nu(E) - T_c\nu(E) = \int_E \frac{1 + y^2}{y^2}\, d\mu(y) - \int_{c^{-1}E} \frac{1 + y^2}{y^2}\, d\mu(y) \geqq 0$$

for every $c \in (0, 1)$ and E a Borel set in R. Take $E = (e^{x-h}, e^x]$, $c = e^{-h}$ and $h > 0$. Then we obtain

$$Q_\mu(x - h) - 2Q_\mu(x) + Q_\mu(x + h) \geqq 0\,.$$

Hence $Q_\mu(x)$ is convex. By taking $E = (-e^x, -e^{x-h}]$ we obtain

$$Q_{\tilde{\mu}}(x - h) - 2Q_{\tilde{\mu}}(x) + Q_{\tilde{\mu}}(x + h) \geqq 0\,.$$

Hence Q_μ and $Q_{\tilde{\mu}}$ are convex which proves that $\mu \in M_0{}^0$.

Now assume the induction hypothesis, namely that $\lambda \in L_{m-1}$ implies $\mu \in M_{m-1}^0$. If $\lambda \in L_m$, by Theorem 1.1 $\lambda_c \in L_{m-1}$ for every $c \in (0, 1)$, so $\mu_c \in M_{m-1}^0$. But for any μ we have

$$Q_\mu(x) = \nu((e^x, \infty)) \qquad \text{and} \qquad Q_{\tilde{\mu}}(x) = \nu((-\infty, -e^x))\,.$$

Therefore

$$(1)\qquad \begin{aligned} Q_{\mu_c}(x) &= \nu_c((e^x, \infty)) \\ &= \nu((e^x, \infty)) - \nu((c^{-1}e^x, \infty)) \\ &= Q_\mu(x) - Q_\mu(x - \log c) \\ &= (I - S_c)Q_\mu(x)\,, \end{aligned}$$

and similarly

$$Q_{\tilde\mu_c}(x) = (I - S_c)Q_{\tilde\mu}(x)\,.$$

It follows that $\mu \in M_m{}^0$.

To prove the sufficiency assume again that $m = 0$ and $Q_\mu(x)$ and $Q_{\tilde\mu}(x)$ are convex. Then there exists a nonnegative, nonincreasing function $g(y)$ such that

$$Q_\mu(x) = \textstyle\int_x^\infty g(y)\,dy\,.$$

Consequently, for every $h > 0$ and $0 < a < b$ we have

$$\int_a^{b+} \frac{1 + y^2}{y^2}\,d\mu(y) \geqq \int_{e^h a}^{e^h b+} \frac{1 + y^2}{y^2}\,d\mu(y)\,.$$

Thus $\nu(a, b] \geqq \nu(e^h a, e^h b]$. Similarly we can show $\nu(a, b] \geqq \nu(e^h a, e^h b]$ if $a < b < 0$, this time using the convexity of $Q_{\tilde\mu}(x)$. By Proposition 2.1, $\lambda \in L_0$. Assume that $\mu \in M_m{}^0$ and that $\mu \in M_{m-1}^0$ implies that $\lambda \in L_{m-1}$. It follows immediately from (1) that $\mu_c \in M_{m-1}^0$, so $\lambda_c \in L_{m-1}$, once we show that $\lambda \in L_0$ so μ_c exists. To do this, observe that $Q_\mu(x) \to 0$ as $x \to \infty$. Hence

$$Q_\mu(x) = \lim_{c\to 0+} (I - S_c)^m Q_\mu(x)$$

is convex, and the same holds for $Q_\mu(x)$. Now by Theorem 1.1, $\lambda \in L_m$. This completes the proof of the theorem.

In the next two theorems we characterize the classes $M_m{}^0$ by means of derivatives of order m of the function $Q_\mu(x)$. To do this we require the following lemma whose proof is left for the reader.

LEMMA 2.1. *Let f be a convex or concave function such that the right-hand derivative, D^+f, and left-hand derivative, D^-f, of f are continuous. Then f is differentiable at every point.*

THEOREM 2.2. *Let $\mu \in M_m{}^0$, for some $0 \leqq m < \infty$. Then the* mth-*order derivatives of Q_μ and $Q_{\tilde\mu}$, denoted respectively by $Q_\mu{}^{(m)}(x)$ and $Q_{\tilde\mu}{}^{(m)}(x)$, exist at every point. Furthermore, if m is even (odd) then $Q_\mu{}^{(m)}(x)$ and $Q_{\tilde\mu}{}^{(m)}(x)$ are convex (concave) functions.*

PROOF. When $m = 0$ we understand $Q_\mu{}^{(0)}(x)$ and $Q_{\tilde\mu}{}^{(0)}(x)$ to mean the functions themselves; in this case there is nothing to prove. Assume the conclusion of the theorem for $m - 1$, and let $\mu \in M_m{}^0$. Then, since the sets $M_m{}^0$ are decreasing with m, by Theorems 1.1 and 2.1 μ and μ_c are in M_{m-1}^0, so by (1) the functions $Q_\mu{}^{(m-1)}(x)$ and

$$Q_{\mu_c}^{(m-1)}(x) = Q_\mu{}^{(m-1)}(x) - Q_\mu{}^{(m-1)}(x - \log c)$$

exist and are concave (convex) if m is even (odd). Consequently,

$$D^+Q_\mu^{(m-1)}(x) = \lim_{c\to 1^-} Q_{\mu_c}^{(m-1)}(x)/\log c$$

and

$$D^-Q_\mu^{(m-1)}(x) = \lim_{c\to 1^-} Q_{\mu_c}^{(m-1)}(x+\log c)/\log c\ .$$

Since the pointwise limit of convex (concave) functions is convex (concave), Lemma 2.1 implies $Q_\mu^{(m)}(x)$ exists and is convex (concave) if m is even (odd). The same argument applies to $Q_{\tilde\mu}(x)$.

THEOREM 2.3. *The necessary and sufficient condition for a measure μ to belong to M_m^0, $m = 0, 1, 2, \cdots$ is that there exist nonnegative and nonincreasing functions $g(y)$ and $h(y)$ on R such that*

$$Q_\mu(x) = \int_0^\infty y^m g(y+x)\,dy \qquad \text{and} \qquad Q_{\tilde\mu}(x) = \int_0^\infty y^m h(y+x)\,dy\ .$$

PROOF. Let $\mu \in M_m^0$. Then $\mu \in M_k^0$ for all $k \leqq m$ since the M_m^0 form a decreasing sequence of sets. By Theorem 2.2, $Q_\mu^{(k)}(x)$ exists at every point and is convex or concave depending on whether k is even or odd. Hence

$$\begin{aligned} Q_\mu(x) &= \int_x^\infty -Q_\mu^{(1)}(y_1)\,dy_1 = \int_x^\infty \int_{y_1}^\infty (-1)^2 Q_\mu^{(2)}(y_2)\,dy_2\,dy_1 \\ &= \cdots = \int_x^\infty \int_{y_1}^\infty \cdots \int_{y_{m-1}}^\infty (-1)^m Q_\mu^{(m)}(y_m)\,dy_m\,dy_{m-1}\cdots dy_1\ . \end{aligned}$$

Since $Q_\mu^{(m)}(y)$ is convex or concave, there exists a function $q(y)$ such that

$$Q_\mu^{(m)}(y) = \int_y^\infty q(z)\,dz\ ;$$

in fact $q(z) = -Q_\mu^{(m+1)}(z)$ except on a countable set. Therefore we can write

$$(2) \qquad Q_\mu(x) = \int_x^\infty \int_{y_1}^\infty \cdots \int_{y_{m-1}}^\infty \int_{y_m}^\infty (-1)^{m+1} D^+Q_\mu^{(m)}(z)\,dz\,dy_m\cdots dy_1\ .$$

Define $g(y) = (m!)^{-1}(-1)^{m+1}D^+Q_\mu^{(m)}(y)$. One can easily check that $h(y)$ is nonnegative and nonincreasing. By interchange of variables in (2) we obtain

$$Q_\mu(x) = \int_x^\infty (y-x)^m g(y)\,dy = \int_0^\infty y^m g(y+x)\,dy\ .$$

Similarly we can obtain the desired expression for $Q_{\tilde\mu}(x)$.

Conversely, suppose the conditions given in the theorem are satisfied for $m = 0$ and some measure μ. Then since $g(y)$ is nonincreasing, for $x_1 < x_2$ we have

$$\begin{aligned} Q_\mu((x_1+x_2)/2) &= \int_{(x_1+x_2)/2}^\infty g(y)\,dy \\ &= \int_{(x_1+x_2)/2}^{x_2} g(y) + \int_{x_2}^\infty g(y)\,dy \\ &\leqq \tfrac{1}{2}\int_{x_1}^{(x_1+x_2)/2} g(y)\,dy + \tfrac{1}{2}\int_{(x_1+x_2)/2}^{x_2} g(y)\,dy + \int_{x_2}^\infty g(y)\,dy \\ &= \tfrac{1}{2}\int_{x_1}^\infty g(y)\,dy + \tfrac{1}{2}\int_{x_2}^\infty g(y)\,dy \\ &= \tfrac{1}{2}Q_\mu(x_1) + \tfrac{1}{2}Q_\mu(x_2)\ . \end{aligned}$$

Thus $Q_\mu(x)$ is convex. Similarly $Q_{\tilde\mu}(x)$ is convex, so $\mu \in M_0^0$.

Now assume that a measure μ is given satisfying the conditions of the theorem for some m and that a measure satisfying those conditions for $m - 1$ must lie in M^0_{m-1}. Then

$$Q_\mu(x) = \int_0^\infty y^m g(y+x)\,dy = m \int_0^\infty \int_0^y t^{m-1} g(y+x)\,dt\,dy$$
$$= m \int_0^\infty \int_t^\infty t^{m-1} g(y+x)\,dy\,dt = m \int_0^\infty \int_{t+x}^\infty t^{m-1} g(y)\,dy\,dt\,.$$

Set $\tilde{g}(y) = m \int_y^\infty g(t)\,dt$. Then $\tilde{g}(y)$ is nonnegative and nonincreasing and

$$Q_\mu(x) = \int_0^\infty y^{m-1}\tilde{g}(y+x)\,dy\,.$$

Proceeding similarly for $Q_{\tilde{\mu}}(x)$, we conclude that $u \in M^0_{m-1}$, so u_c exists for each $c \in (0, 1)$. Moreover,

$$Q_{\mu_c}(x) = \int_0^\infty y^{m-1}(I - S_c)\tilde{g}(y+x)\,dy$$

and

$$(I - S_c)\tilde{g}(y) = m \int_y^{y-\log c} g(t)\,dt$$

is nonincreasing and nonnegative since g was assumed to have these properties. The same observation applies to $Q_{\tilde{\mu}_c}(x)$, so we see that $\mu_c \in M^0_{m-1}$. Thus $\mu \in M_m{}^0$, and the theorem is proved.

We now characterize the set $M_\infty{}^0 = \bigcap_{m=0}^\infty M_m{}^0$ by means of the functions Q_μ and $Q_{\tilde{\mu}}$.

THEOREM 2.4. *A measure μ is in $M_\infty{}^0$ if and only if $Q_\mu(x)$ and $Q_{\tilde{\mu}}(x)$ are completely monotonic functions on R.*

PROOF. Let $\mu \in M_\infty{}^0$. Then $\mu \in M_m{}^0$ for every $m = 0, 1, 2, \cdots$. Hence by Theorem 2.2, $Q_\mu{}^{(m)}(x)$ and $Q_{\tilde{\mu}}{}^{(m)}(x)$ exist at every point and $(-1)^m Q_\mu{}^{(m)}(x) \geqq 0$ and $(-1)^m Q_{\tilde{\mu}}{}^{(m)}(x) \geqq 0$. Thus the functions Q_μ and $Q_{\tilde{\mu}}$ are completely monotonic. Conversely, suppose Q_μ and $Q_{\tilde{\mu}}$ are completely monotonic. Then for each m one can prove as in Theorem 2.3 that

$$Q_\mu(x) = \int_0^\infty y^m g_m(y+x)\,dy \qquad \text{and} \qquad Q_{\tilde{\mu}}(x) = \int_0^\infty y^m h_m(y+x)\,dy$$

for some nonnegative, nonincreasing functions $g_m(y)$ and $h_m(y)$. Thus by Theorem 2.3, $\mu \in M_m{}^0$. Hence $\mu \in M_\infty{}^0$.

3. Extreme points of K_m. This section is devoted to showing that the sets K_m are compact and convex and to finding the extreme points of the sets K_m.

LEMMA 3.1. *The sets K_m, $m = 0, 1, 2, \cdots$ are compact and convex.*

PROOF. The convexity of K_m follows easily from the definition. Since the class of all probability measures on $[-\infty, \infty]$ is weakly compact it is sufficient to prove that the sets K_m are closed. So let $\{\mu_n\}$ be a sequence in K_m converging weakly to μ. Then for all x which are continuity points of $Q_\mu(x)$,

$$\lim_{n\to\infty} Q_{\mu_n}(x) = Q_\mu(x)\,.$$

If $m = 0$, let $F_n(x) = Q_{\mu_n}(x)$ and $F(x) = Q_\mu(x)$. If $m \geqq 1$ and $c_1, \cdots, c_m \in (0, 1)$ are given, let

$$F_n(x) = \prod_{k=1}^m (I - S_{c_k}) Q_{\mu_n}(x)$$

and let $F(x)$ be defined analogously with μ in place of μ_n. Then since $Q_\mu(x)$ is decreasing we see that

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for all but countably many real numbers x. By Theorem 2.1 each of the functions $F_n(x)$ is assumed to be convex. Therefore, by [4], page 47, the above convergence is uniform on bounded intervals, and it follows that $F(x)$ is convex also. A similar argument applies to $\tilde{\mu}_n$ and $\tilde{\mu}$. Since $c_1, \cdots, c_m$ were arbitrarily chosen, we conclude that $\mu \in K_m$.

LEMMA 3.2. *The set* $K_\infty = \bigcap_{m=0}^{\infty} K_m$ *is compact and convex.*

PROOF. The intersection of compact convex sets is compact and convex.

LEMMA 3.3. *The extreme points of* K_m, $m = 0, 1, 2, \cdots, \infty$, *are measures concentrated on one of the following sets*: $\{-\infty\}$, $\{0\}$, $\{+\infty\}$, $(0, \infty)$, $(-\infty, 0)$.

PROOF. The result is obvious as soon as we realize that if $\mu \in K_m$ then so is its normalized restriction to each of the given sets on which it is nonzero. This follows immediately from Theorems 2.3 and 2.4.

Since μ is an extreme point of K_m if and only if $\tilde{\mu}$ is, it is clear from Lemma 3.3 that to find the extreme points of K_m it suffices to consider those μ which are totally concentrated on $(0, \infty)$. So consider such a $\mu \in K_m{}^0$, $m = 0, 1, 2, \cdots$. Then by Theorem 2.3, if $m \geqq 1$ we can write

$$Q_\mu(x) = \frac{1}{m} \int_0^\infty y^m g(y + x)\, dy = \int_x^\infty \int_0^\infty y^{m-1} g(y + t)\, dy\, dt\,.$$

Hence

$$\mu((a, b)) = \int_{\log a}^{\log b} \frac{e^{2t}}{1 + e^{2t}} \int_0^\infty y^{m-1} g(y + t)\, dy\, dt\,, \quad 0 \leqq a < b \leqq \infty\,. \tag{3}$$

Conversely, every nonnegative, nonincreasing function $g(y)$ satisfying

$$\int_{-\infty}^\infty \int_0^\infty y^{m-1} \frac{e^{2t}}{1 + e^{2t}} g(y + t)\, dy\, dt = \int_{-\infty}^\infty \int_0^\infty \frac{g(y)t^{m-1}}{1 + e^{2(t-y)}}\, dt\, dy < \infty \tag{4}$$

defines a measure λ concentrated on $(0, \infty)$ given by (3) with λ in place of μ. Furthermore this λ belongs to $M_m{}^0$.

Denote by $m_{-\infty}$, m_0 and m_∞ the probability measures concentrated at $-\infty$, 0 and ∞, respectively. If $u \in R$, denote by m_u the probability measures given for each $m = 0, 1, 2, \cdots$ by

$$m_u(E) = m_u{}^{(m)}(E) = \frac{1}{J_m(u)} \int_E C_u(t) \left(\log \frac{u}{t}\right)^m \frac{t}{1 + t^2}\, dt\,,$$

where $C_u(t)$ denotes for $u > 0$ the indicator function of the interval $(0, u)$ and for $u < 0$ the indicator function of the interval $(u, 0)$. $J_m(u)$ is as defined in Section 1. Notice that $m_{-u} = \tilde{m}_u$.

Now we shall find the extreme points of K_m. Denote by $e(K_m)$ the set of extreme points of K_m.

LEMMA 3.4. $e(K_m) \subset \{m_u : u \in [-\infty, \infty]\}$, $m = 0, 1, 2, \cdots$.

PROOF. Let $\mu \in e(K_m)$. It is sufficient to assume μ is concentrated on $(0, \infty)$. Furthermore, if $m = 0$ the theorem follows from [5], page 212. So let $m \geqq 1$. Now since $\mu \in K_m{}^0$ (3) holds and we must have

$$\int_{-\infty}^{\infty} \int_0^{\infty} y^{m-1} \frac{e^{2t}}{1+e^{2t}} g(y+t)\, dy\, dt = 1 .$$

Let

$$I_m(y) = \int_0^{\infty} \frac{t^{m-1}}{1+e^{2(t-y)}}\, dt .$$

Then (4) yields

$$\int_{-\infty}^{\infty} h(y) I_m(y)\, dy = 1 . \tag{5}$$

Since it is immediate from (4) that $\int_a^\infty I_m(y)\, dy = \infty$ for every $a \in R$, we conclude that $h(y)$ cannot be a nonzero constant function on (a, ∞) for any a. Suppose there exists a real number ν such that $g(y)$ is not constant on both of the intervals $(-\infty, \nu)$ and (ν, ∞). Then by (5),

$$c = \int_{-\infty}^{\nu} g(\nu) I_m(y)\, dy + \int_\nu^\infty g(y) I_m(y)\, dy \in (0, 1) .$$

Define the functions h_1 and h_2 by

$$\begin{aligned} h_1(y) &= c^{-1} g(\nu) && \text{if } y < \nu \\ &= c^{-1} g(y) && \text{if } y \geqq \nu \\ h_2(y) &= (1-c)^{-1}(g(y) - g(\nu)) && \text{if } y < \nu \\ &= 0 && \text{if } y \geqq \nu . \end{aligned}$$

The functions h_1 and h_2 are nonnegative and nonincreasing, and they satisfy (5). Hence they define two probability measures μ_1 and μ_2 as in (3). It is easy to see that $\mu = c\mu_1 + (1-c)\mu_2$ and $\mu_1 \neq \mu_2$. But this contradicts the assumption that μ is an extreme point. Hence for every a, $g(y)$ is constant on either $(-\infty, a)$ or (a, ∞). Hence $\sup\{a : g(y) \text{ is constant on } (-\infty, a)\} = \inf\{a : g(y) \text{ is constant on } (a, \infty)\}$. Now let $\log u$, $u > 0$, be the point of decrease of $g(y)$, so that $g(y)$ would be a constant K on $(-\infty, \log u)$ and zero on $(\log u, \infty)$. From (4) and (5) we now have

$$\begin{aligned} 1 &= K \int_{-\infty}^{\log u} \int_0^\infty \frac{t^{m-1}}{1+e^{2(t-y)}}\, dt\, dy \\ &= K \int_{-\infty}^{\log u} \int_0^{\log u - t} \frac{e^{2t}}{1+e^{2t}} y^{m-1}\, dy\, dt = \frac{K}{m} J_m(u) , \end{aligned}$$

i.e., $K = m/J_m(u)$. Consequently from (3) we have for $0 \leqq a < b \leqq u$,

$$\begin{aligned} \mu((a, b)) &= \frac{m}{J_m(u)} \int_{\log a}^{\log b} \int_0^{\log u - t} y^{m-1} \frac{e^{2t}}{1+e^{2t}}\, dy\, dt \\ &= m_u((a, b)) , \end{aligned}$$

meaning that $\mu = m_u$. Thus the lemma has been proved.

THEOREM 3.1. $e(K_m) = \{m_u : u \in [-\infty, \infty]\}$, $m = 0, 1, 2, \cdots$.

PROOF. We must show $m_u \in e(K_m)$, $-\infty \leqq u \leqq \infty$. Once again it is sufficient to consider m_u for $0 < u < \infty$. Suppose m_u is not an extreme point. Then it follows from the proof of Lemma 3.4 that there exist functions h_1 and h_2 which are nonnegative and nonincreasing and satisfy (5) such that $h_1 \neq h_2$ and for some $c \in (0, 1)$,

$$g(y) = m/J_m(u)I_{(-\infty, \log u)}(y) = ch_1(y) + (1 - c)h_2(y) .$$

(Here I denotes indicator function.) Thus

$$ch_1(y) = g(y) - (1 - c)h_2(y) .$$

It follows that if $h_2(y)$ was not a constant function on $(-\infty, \log u)$ and on $(\log u, \infty)$ then $h_1(y)$ would be increasing. Hence $h_2(y)$ is a constant function on each of these intervals and so is $h_1(y)$. Thus $h_1 = h_2 = g$. This contradiction implies m_u must be an extreme point.

Now we proceed to find $e(K_\infty)$. By Theorem 2.4 and Bernstein's theorem [7, page 155], $\mu \in M_\infty{}^0$ if and only if we can write

$$Q_\mu(x) = \int_0^\infty e^{-xy}\, d\sigma(y) , \qquad Q_{\tilde{\mu}}(x) = \int_0^\infty e^{-xy}\, d\tau(y) , \tag{6}$$

where σ and τ are finite Borel measures on $[0, \infty)$. Consequently,

$$Q_\mu(x) = \int_x^\infty \int_0^\infty ye^{-yt}\, d\sigma(y)\, dt .$$

Thus for $0 < b \leqq \infty$,

$$\begin{aligned} \mu((0, b)) &= \int_{-\infty}^{\log b} \frac{e^{2t}}{1 + e^{2t}} \int_0^\infty ye^{-yt}\, d\sigma(y)\, dt \\ &= \int_0^\infty \int_{-\infty}^{\log b} \frac{e^{2t}}{1 + e^{2t}} ye^{-yt}\, dt\, d\sigma(y) \\ &= \int_0^\infty \int_0^b \frac{yt^{1-y}}{1 + t^2}\, dt\, d\sigma(y) . \end{aligned}$$

Similarly, for $-\infty \leqq a < 0$ we obtain

$$\mu((a, 0)) = \int_0^\infty \int_0^{-a} \frac{yt^{1-y}}{1 + t^2}\, dt\, d\tau(y) .$$

Moreover, since

$$\int_0^\infty \frac{t^{1-y}}{1 + t^2}\, dt$$

is finite if and only if $y \in (0, 2)$, we conclude that in (6) σ and τ must be concentrated on $(0, 2)$. Thus

$$\mu((0, b)) = \int_0^2 \int_0^b \frac{yt^{1-y}}{1 + t^2}\, dt\, d\sigma(y) . \tag{7}$$

Denote by p_y the probability measure given by

$$p_y(E) = \frac{2}{\pi} \sin \frac{\pi}{2} |y| \int_E \frac{|t|^{1-|y|}}{1+t^2} C(t)\, dt ,$$

where $y \in (-2, 0) \cup (0, 2)$ and $C(t)$ is the indicator function of the interval $(0, \infty)$ if y is positive and $(-\infty, 0)$ if y is negative. We denote by p_∞, $p_{-\infty}$, and p_2 the probability measures concentrated at $\{\infty\}$, $\{-\infty\}$, and $\{0\}$, respectively.

THEOREM 3.2. $e(K_\infty) = \{p_y : y \in \{-\infty\} \cup (-2, 0) \cup (0, 2] \cup \{\infty\}\}$.

PROOF. Once again it is sufficient to consider those $\mu \in K_\infty$ which are concentrated on $(0, \infty)$. It is easy to see from (7) that such a μ is an extreme point of K_∞ if and only if σ is concentrated at one point $y \in (0, 2)$ such that

$$y\sigma(\{y\}) \int_0^\infty \frac{t^{1-y}}{1+t^2}\, dt = 1 . \tag{8}$$

Since the integral appearing in (8) is equal to $\pi/2 \sin(\pi y/2)$, the theorem is proved by considering (7) for such measures σ.

4. The Urbanik representation of measures in L_m. In this section we shall use the results of Section 3 to obtain a formula for the characteristic function of a probability measure in L_m. Our representation is the same as that obtained by Urbanik in [5] for $m = 0$ and in [6] for all m.

LEMMA 4.1. *The mapping $u \to m_u$ is a homeomorphism of $[-\infty, \infty]$ onto $e(K_m)$ for each $m = 0, 1, 2, \cdots$.*

PROOF. Since the functions $J_m(u)$ are clearly continuous, it is easy to see we have continuity at each $u \in R$. To show continuity at $\pm\infty$, suppose first that f is a continuous function on R with compact support disjoint from 0. Then there exists $C > 0$ such that $|t|/(1 + t^2) \leqq C/|t|$ whenever $f(t) \neq 0$. It follows that for such a function f and for $u > 0$,

$$\int_0^u f(t) \left(\log \frac{u}{t}\right)^m \frac{t}{1+t^2}\, dt = O((\log u)^m) \qquad \text{as} \quad u \to \infty . \tag{9}$$

On the other hand, for $u > 1$ we have

$$J_m(u) \geqq \tfrac{1}{2} \int_1^u \left(\log \frac{u}{t}\right)^m \frac{dt}{t} = \frac{1}{2(m+1)} (\log u)^{m+1} .$$

Thus

$$\lim_{u\to\infty} \int_0^\infty f\, dm_u = 0 . \tag{10}$$

Now, the collection of all such functions f, considered as functions in $C(\bar{R})$, is uniformly dense in

$$C_0 = \{f \in C(\bar{R}) : f(0) = f(\pm\infty) = 0\} .$$

Hence (10) holds for all $f \in C_0$, and we may proceed similarly as $u \to -\infty$. Let

$$\begin{aligned} \phi_1(x) &= 1 && x \leqq 0 \\ &= 1 - x && 0 < x < 1 \\ &= 0 && x \geqq 1 \end{aligned}$$

and $\phi_2(x) = \phi_1(-x)$. Then any $f \in C(\bar{R})$ can be written uniquely in the form

$$f = g + a + b\phi_1 + c\phi_2 ,$$

where $g \in C_0$. If $u > 0$, then $\int_0^\infty \phi_2 \, dm_u = 1$, and it is easy to see that the estimate (9) holds for $f = \phi_1$, so we have (10) for $f = \phi_1$. Hence $\int_0^\infty f \, dm_u \to f(\infty)$ as $u \to \infty$ for all $f \in C(\bar{R})$, and a similar argument applies as $u \to -\infty$. Thus $u \to m_u$ is continuous, so a homeomorphism.

LEMMA 4.2. *The mapping $y \to p_y$ is a homeomorphism of $(-2, 0) \cup (0, 2]$ into $e(K_\infty)$.*

PROOF. Set $q_z = p_y$, where $y = (\operatorname{sgn} z)(2 - |z|)$, $0 < |z| < 2$, $q_0 = p_2$, and $q_{\pm 2} = p_{\pm\infty}$. We shall show that $z \to q_z$ is a continuous map (hence a homeomorphism) of $[-2, 2]$ onto $e(K_\infty)$, from which our lemma follows immediately. Continuity at all points except 0, ± 2 is easily verified.

Let f be a continuous function on R with compact support disjoint from 0. The dominated convergence theorem then implies that

$$\Psi_f(z) = \int_R f \, dq_z$$

is continuous on $[-2, 2]$. Recalling the notation established in the proof of Lemma 4.1, we conclude that Ψ_f is continuous for all $f \in C_0$. For all $0 < z < 2$ we have $\Psi_{\phi_2}(z) = 1$, and one can readily check that

$$\lim_{z \to 0} \Psi_{\phi_1}(z) = 0 \qquad \text{and} \qquad \lim_{z \to 2} \Psi_{\phi_1}(z) = 1 .$$

Hence Ψ_f is continuous on $[0, 2]$ for all $f \in C(\bar{R})$. Proceed similarly on $[-2, 0]$.

LEMMA 4.3. *A measure μ is in $M_m{}^0$, $m = 0, 1, 2, \cdots, \infty$, if and only if there exists a finite Borel measure ω_m on R such that for all bounded measurable functions f on R,*

$$\int_{-\infty}^{\infty} f(x) \, d\mu(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x) \, dm_u{}^{(m)}(x) \right) d\omega_m(u)$$

for $m = 0, 1, 2, \cdots$, and for $m = \infty$,

$$\int_{-\infty}^{\infty} f(x) \, d\mu(x) = \int_{-2}^{2} \left(\int_{-\infty}^{\infty} f(x) \, dp_y(x) \right) d\omega_\infty(y) .$$

PROOF. By Lemmas 4.1 and 4.2 $e(K_m)$ is homeomorphic to $[-\infty, \infty]$, $m = 0, 1, 2, \cdots$, and there is a one-to-one bimeasurable mapping of $e(K_\infty)$ onto $(-2, 0) \cup (0, 2] \cup \{\pm\infty\}$. Therefore, by the Krein–Milman–Choquet theorem, a measure μ is in K_m, $m = 0, 1, 2, \cdots$ if and only if there exists a probability measure ω_m on $\bar{R}$ such that

$$\int_{\bar{R}} f(x) \, d\mu(x) = \int_{\bar{R}} \left(\int_{\bar{R}} f(x) \, dm_u{}^{(m)}(x) \right) d\omega_m(u) \tag{11}$$

for all continuous functions on $\bar{R}$. Moreover, the measure ω_m will assign zero mass to $\{\pm\infty\}$ if and only if μ does so. Furthermore the formula (11) with $\bar{R}$ replaced by R extends to all bounded measurable functions f on R. One can proceed similarly for $m = \infty$.

THEOREM 4.1. *A function ϕ is the characteristic function of a probability measure from L_m, $m = 0, 1, 2, \cdots$ if and only if*

$$\phi(t) = \exp\left[irt + it \int_{-\infty}^{\infty} \left(\int_0^u e^{itx}\left(\log \frac{u}{x}\right)^{m+1} dx - (m+1)! \tan^{-1}u\right) \times \frac{d\omega(u)}{(\log(1+|u|^{2/m+1}))^{m+1}}\right],$$

where r is real and ω is a finite Borel measure on R. The function ϕ uniquely determines r and ω.

PROOF. We know from Theorem 2.1 that $\lambda \in L_m$, $m = 0, 1, 2, \cdots$, if and only if λ is infinitely divisible and its Lévy measure μ belongs to $M_m{}^0$. Thus by Theorem 3.1 and Lemma 4.3 this happens if and only if there exist $r \in R$ and a finite Borel measure ω on R such that (with an obvious convention at 0)

$$\begin{aligned}
\hat{\lambda}(t) &= \exp\left[irt + \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\, d\mu(x)\right] \\
&= \exp\left[irt + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\, dm_u{}^{(m)}(x)\, d\omega(u)\right] \\
&= \exp\left[irt - \tfrac{1}{2}\omega(\{0\})t^2 + \int_{R\setminus\{0\}}\int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\right. \\
&\qquad \left.\times\, C_u(x)(\log u/x)^m \frac{x}{1+x^2}\, dx\, \frac{d\omega(u)}{J_m(u)}\right] \qquad (12)\\
&= \exp\left[irt + \int_{-\infty}^{\infty}\int_0^{|u|}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)(\log u/x)^m \frac{dx}{x}\frac{d\omega(u)}{J_m(u)}\right] \\
&= \exp\left[irt + it\left(\frac{1}{1+m}\int_0^u e^{itx}\left(\log\frac{u}{x}\right)^{m+1} dx\right.\right. \\
&\qquad \left.\left. - m\int_0^u \tan^{-1}x\left(\log\frac{u}{x}\right)^{m-1}\frac{dx}{x}\right)\frac{d\omega(u)}{J_m(u)}\right].
\end{aligned}$$

For two functions f and g on $(-\infty, \infty)$ let us say f and g are equivalent and write $f \sim g$ if there exists $C > 0$ such that $f(x) \leqq Cg(x)$ and $g(x) \leqq Cf(x)$ for all x. Let

$$\begin{aligned}
\phi(t) &= |t| & -1 \leqq t \leqq 1 \\
&= \frac{1}{|t|} & |t| > 1
\end{aligned}$$

and

$$\begin{aligned}
\Psi_m(u) &= u^2 & -1 \leqq u < 1 \\
&= 1 + (\log|u|)^{m+1} & |u| \geqq 1\,.
\end{aligned}$$

Then $|t|/(1+t^2) \sim \phi(t)$, from which it follows by integration by parts that $J_m(u) \sim \Psi_m(u)$, which in turn is equivalent to $\log^{m+1}(1+u^{2/m+1})$. Let σ be the measure given by

$$d\sigma(u) = \frac{\log^{m+1}(1+u^{2/m+1})}{J_m(u)}\, d\omega'(u)\,.$$

Then σ is a finite measure if ω is, and conversely, and we may replace $J_m^{-1}(u)\, d\omega(u)$ in (12) by $\log^{-m-1}(1+u^{2/m+1})\, d\sigma(u)$. Finally, further integration by parts shows that the new integrand in (12) is equivalent to the integrand appearing in the statement of the theorem (cf. [6], page 233).

The uniqueness of ω and r follow from the uniqueness assertions in the Krein–Milman–Choquet theorem and the Lévy–Khintchine representation, respectively.

THEOREM 4.2. *A function ϕ is the characteristic function of a probability measure from L_∞ if and only if*

$$\phi(t) = \exp\left[irt + \int_{-2}^{2} y \sin\frac{\pi}{2} y \int_0^\infty \left(\int_0^{tx} \frac{e^{iv}-1}{v}\, dv - it\tan^{-1} x\right) \frac{dx}{x^{1+|y|}}\, d\omega(y)\right], \tag{13}$$

where r is real and ω is a finite Borel measure on $(-2, 0) \cup (0, 2]$. The function ϕ uniquely determines r and ω.

PROOF. Proceed by analogy with (12), using Theorem 3.2, and note that it can be shown ([3], page 330; [6], page 236) that the ω-integrand in (13) converges to $-\pi t^2/4$ as $y \to 2$.

Theorem 4.2 can be refined ([6], pages 236–237) to read as follows.

THEOREM 4.3. *A function ϕ is the characteristic function of a probability measure from L_∞ if and only if*

$$\phi(t) = \exp\left[irt - \int_{-2}^{2}\left(|t|^{|y|}\left(\cos\frac{\pi}{2}y - \frac{it}{|t|}\sin\frac{\pi}{2}y\right) + ity\right)\frac{d\omega(y)}{1-|y|}\right],$$

where r and ω are as in Theorem 4.2 and the integrand is defined as its limiting values $(\pi/2)|t| \pm it\log|t| \mp it$ when $y = \pm 1$.

REFERENCES

[1] KUMAR, A. (1973). A note on the convergence of stable probability measures and class L probability measures on Banach spaces. *Ann. Probability* **4** 716–718.

[2] KUMAR, A. and SCHREIBER, B. M. (1975). Self-decomposable probability measures on Banach spaces. *Studia Math.* **53** 55–71.

[3] LOÈVE, M. (1963). *Probability Theory*, 3rd ed. Van Nostrand, Princeton.

[4] NOSARZEWSKA, M. (1952). On uniform convergence in some classes of functions. *Fund. Math.* **39** 38–52.

[5] URBANIK, K. (1968). A representation of self-decomposable distributions. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **16** 209–214.

[6] URBANIK, K. (1972). Limit laws for sequences of normed sums satisfying some stability conditions. *Multivariate Analysis, III* (*Proc. Third Internat. Symp. on Multivariate Analysis*, Wright State Univ.) 225-237. Academic Press, New York.

[7] WIDDER, D. V. (1971). *An Introduction to Transform Theory*, Ser. Pure and Appl. Math. **42**, Academic Press, New York.

DEPARTMENT OF MATHEMATICS
NORTHEASTERN ILLINOIS UNIVERSITY
CHICAGO, ILLINOIS 60625

DEPARTMENT OF MATHEMATICS
WAYNE STATE UNIVERSITY
DETROIT, MICHIGAN 48202