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CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY THE RELEVATION TRANSFORM

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Abstract

A characterization of the exponential distribution was obtained by Grosswald et al. (1980) using the relevation transform introduced by Krakowski (1973). Here we obtain an improved version of the result in Grosswald et al. (1980).

CONVOLUTION; LIFE DISTRIBUTIONS

1. Introduction

The *convolution* of two distribution functions F and G is given by

$$(1.0) \qquad (F * G)(x) = \int_{-\infty}^{\infty} F(x-u)G(du), \qquad -\infty < x < \infty.$$

Suppose the supports of F and G are contained in $[0, \infty)$. Then

$$(1.1) \qquad (F * G)(x) = \int_{0}^{x} F(x-u)G(du), \qquad x \geqq 0.$$

It is the distribution of the time to failure of the second of two components when the second component with life distribution G is placed in service on the failure of the first component with life distribution F, assuming that the replacement component is new on installation. However, suppose that the failed component (with life distribution F) is replaced by another one of same age (with life distribution G). The survival function of the time to system failure (i.e failure of both components) is called the *relevation* of the survival function $\bar{F}(t) = 1 - F(t)$ with $\bar{G}(t) = 1 - G(t)$. It is denoted by $(\bar{F} \# \bar{G})(t)$ and was introduced by Krakowski (1973). Reliability applications of relevation transform are given in Baxter (1982). Note that

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$$
\begin{aligned}
(\bar{F} \# \bar{G})(t) &= P(\text{system survives beyond time } t) \\
&= P(\text{first component with survival function } \bar{F} \text{ survives beyond time } t) \\
&\quad + P(\text{first component with survival function } \bar{F} \text{ fails some time before } t \text{ and the second component with survival function } \bar{G} \text{ survives beyond time } t \text{ given that it has survived up to the time to failure of the first component}) \\
&= \bar{F}(t) + \int_0^t P[\text{the second component survives beyond time } t \text{ given that it has survived beyond time } u \text{ when the first component failed}]\, dF(u) \\
&= \bar{F}(t) + \int_0^t \frac{P(\text{life of second component} > t)}{P(\text{life of second component} > u)}\, dF(u) \\
&= \bar{F}(t) + \int_0^t \frac{\bar{G}(t)}{\bar{G}(u)}\, dF(u) \\
&= \bar{F}(t) - \int_0^t \frac{\bar{G}(t)}{\bar{G}(u)}\, d\bar{F}(u).
\end{aligned}
\tag{1.2}
$$

Even though this derivation is known, we give it here for completeness. Note that $(\bar{F} \# \bar{G})(t) \neq (\bar{G} \# \bar{F})(t)$, unlike $(F * G)(t) = (G * F)(t)$. It is easy to check that

$$
(\overline{F * G})(t) = \bar{F}(t) - \int_0^t \bar{G}(t-u)\, d\bar{F}(u). \tag{1.3}
$$

It can be seen from (1.2) and (1.3) that

$$
(\bar{F} \# \bar{G})(t) = (\overline{F * G})(t), \qquad t \geqq 0 \tag{1.4}
$$

iff

$$
\int_0^t \frac{\bar{G}(t)}{\bar{G}(u)}\, d\bar{F}(u) = \int_0^t \bar{G}(t-u)\, d\bar{F}(u), \qquad t \geqq 0. \tag{1.5}
$$

Grosswald et al. (1980) proved that (1.5) holds for all $\bar{F}(\cdot)$ if and only if $\bar{G}$ is an exponential survival function among the class of all $\bar{G}$ which can be expressed in the form of power series. They conjectured that the result should be true if $\bar{G}$ has a continuous derivative but need not have power series expansion. Here we give a proof of this conjecture under even weaker assumptions.

2. Preliminaries

We first state and prove a couple of results useful later in our discussion.

Proposition 2.1. Let $I = [0, c)$ or $[0, \infty)$. Suppose $h : I \to \mathbb{R}$ is continuous, $h(0) = 0$, $h'_+(0) = \alpha$, and for any $x \in I$, there exists $0 < \xi < x$ such that

$$
h(x) = h(\xi) + h(x - \xi). \tag{2.1}
$$

Then $h(x) = \alpha x$ for all $x \in I$.

Proof. Let $x \in I$, $x \neq 0$ and let $h(x)/x = \beta$. We first claim that there exists $0 < \xi' < x$ such that $h(\xi')/\xi' = \beta$. Indeed we let ξ be as in (2.1). Then

$$\frac{h(x) - h(\xi)}{x - \xi} = \frac{h(x - \xi)}{x - \xi}.$$

Suppose without loss of generality that $h(x - \xi) \geqq \beta(x - \xi)$. The equation (2.1) implies that

$$\frac{h(x) - h(\xi)}{x - \xi} \geqq \beta = \frac{h(x)}{x}.$$

A direct calculation shows that $\xi h(x) \geqq xh(\xi)$, i.e. $h(\xi) \leqq \beta\xi$. Now apply the intermediate value theorem to $h(t) - \beta t$ on the interval determined by ξ and $x - \xi$. There exists ξ' such that $h(\xi') - \beta\xi' = 0$ and the claim follows.

Next we show that $\beta = \alpha$. Let

$$y_0 = \inf\left\{y \in I : y \neq 0, \frac{h(y)}{y} = \beta\right\}.$$

Then $y_0 = 0$ (for, if $y_0 > 0$, then by the continuity of h, $(h(y_0)/y_0) = \beta$, and applying the claim made above to y_0, we can find $0 < \xi' < y_0$ such that $h(\xi')/\xi' = \beta$. This contradicts the fact that y_0 is the infimum). Hence there exists a sequence $x_n \to 0$ such that $h(x_n)/x_n \to \beta$. It follows that

$$\beta = \lim_{n \to \infty} \frac{h(x_n)}{x_n} = h'_+(0) = \alpha.$$

From this, we conclude that $h(x) = \alpha x$.

Proposition 2.2. Let $g : [a, b] \to \mathbb{R}$ be continuous, $l : [a, b] \to \mathbb{R}$ be increasing, and suppose the set of points of increase D of l is *not* contained in $\{a, b\}$. Then there exists $a < \xi < b$ such that

$$\int_a^b g(t)dl(t) = g(\xi)(l(b) - l(a)). \tag{2.2}$$

Remark. By the mean value theorem of integration (Apostal (1957), p. 213) there exists $a \leqq \xi \leqq b$ such that the above hold. We are showing that ξ can actually be chosen to be different from a or b.

Proof. Let $\alpha = \min_{x \in [a,b]} g(x) = g(x_1)$, $\beta = \max_{x \in [a,b]} g(x) = g(x_2)$ and $G(x) = \int_a^x g(t)dl(t)$. Without loss of generality, assume that $x_1 < x_2$. Then

$$\alpha(l(b) - l(a)) \leqq G(b) \leqq \beta(l(b) - l(a)).$$

If $G(b) = \alpha(l(b) - l(a))$, then D is contained in $\{x : g(x) = \alpha\}$. By assumption, D contains points other than a and b. We can then choose $\xi \neq a, b$ such that $g(\xi) = \alpha$, and hence

$$g(\xi)(l(b)-l(a))=\int_a^b g(t)dl(t).$$

A similar argument holds for the case $G(b)=\beta(l(b)-l(a))$. Finally if

$$\alpha(l(b)-l(a))<G(b)<\beta(l(b)-l(a)),$$

then, by the continuity of g, we can find x_1', x_2' (in the neighbourhood of x_1, x_2 respectively) such that $a<x_1'<x_2'<b$, and

$$g(x_1')(l(b)-l(a))\leqq G(b)\leqq g(x_2')(l(b)-l(a)).$$

The intermediate value theorem applied to g in $[x_1', x_2']$ implies that there exists $\xi\neq a,b$ (actually, $x_1'<\xi<x_2'$) such that

$$G(\xi)=\int_a^b g(t)dl(t).$$

3. Main theorem

We now state and prove the main theorem characterizing the exponential distribution.

Theorem 3.1. Suppose $\bar{G}$ and $\bar{F}$ are continuous survival functions and $\bar{G}'_+(0)$ exists. Further suppose that for any $x>0$, $\bar{F}$ has a point of increase in $(0,x)$. If $\bar{G}$ satisfies

$$\int_0^x \bar{G}(x-t)d\bar{F}(t)=\int_0^x \frac{\bar{G}(x)}{\bar{G}(t)}d\bar{F}(t),\quad \text{for all } x \text{ where } \bar{G}(x)\neq 0, \tag{3.1}$$

then $\bar{G}$ is exponential, that is, $\bar{G}(x)=e^{-\alpha x}$, $x\geqq 0$ for some $\alpha\geqq 0$.

Proof. Let $c=\sup\{x:\bar{G}(x)>0\}$. Let h be the non-negative function such that $h(0)=0$ and $e^{-h(x)}=\bar{G}(x)$, $x\in I=[0,c)$. Then $h'_+(0)$ exists by hypothesis and

$$\int_0^x \{e^{-h(x-t)}-e^{-h(x)+h(t)}\}d\bar{F}(t)=0,\qquad x\in I$$

by (3.1). Note that $g(t)=e^{-h(x-t)}-e^{-h(x)+h(t)}$, $0\leqq t\leqq x$ and $l(t)=\bar{F}(t)$ satisfy the conditions in Proposition 2.2. Hence there exists $0<\xi<x$ such that $g(\xi)=0$, or equivalently $h(x)-h(x-\xi)-h(\xi)=0$. Since this equation holds for all x, $x\in I$, by Proposition 2.1, $h(x)=\alpha x$ where $\alpha=h'_+(0)\geqq 0$. Hence $\bar{G}(x)=e^{-\alpha x}$, $x\in I$. Since by assumption $\bar{G}$ is continuous, I must be equal to $[0,\infty)$.

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