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CHARACTERIZATION OF THE FIRST OPERATING PERIOD OF A TWO-UNIT STANDBY REDUNDANT SYSTEM WITH THREE STATES OF UNITS

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This paper is closely connected with the paper [1]. Let us consider the same two-unit cold-standby redundant system with three states of units – good (I), degraded (II), and failed (III). This paper is devoted to the problems which arise only provided that the units of the redundant system can be in more than two states (in the operating and failed states). The following characteristics concerning the first operating period of the system (at the starting instant both units are in state I) are studied: the whole time of operation of units in state I (or II), the whole time of repairs of units of the type $II \to I$ (or $III \to I$) and the number of finished repairs of units of the type $II \to I$ (or $III \to I$).

We suppose that the condition (2.3) from [1] is fulfilled, i.e. that a failure of the system comes with probability 1. We use the same notation as in [1]. Moreover, let $\mathscr{P}_J(i)$ express the fact that the starting state of the random process $J(t)$ is e_i, $J \in \{Y, Z\}$, $i \in \{P; S; S_0; S_I; S_{II}; L; L_0; L_I; L_{II}\}$, where the processes $Y(t)$ and $Z(t)$ and the states e_{S_0}, e_{S_I}, $e_{S_{II}}$, e_{L_0}, e_{L_I} and $e_{L_{II}}$ will be determined in Section 1.

1. SOME CHARACTERISTICS OF THE BEHAVIOUR OF THE SYSTEM DURING ITS FIRST OPERATING PERIOD

We shall deal with random variables "the whole time of operation of units in state I, and II, respectively, during the first operating period of the system". Let us construct a random process $Y(t)$ with six states e_P, e_S, e_{S_0}, e_L, e_{L_0}, e_R, which changes its state at the same moments as the process $X(t)$ defined in Section 2 of [1] and, moreover, at the moments when a unit deteriorates from I to II and the other one is being repaired. Let t_0 be such a moment. We say that the process $Y(t)$ enters at t_0 the states e_P, e_S, e_L and e_R under the same conditions as the process $X(t)$ and the states:

e_{S_0} – if at t_0 a unit deteriorates from I to II and the other one is being repaired from the state II;

e_{L_0} – if at t_0 a unit deteriorates from I to II and the other one is being repaired from the state III.

Changes of states of $Y(t)$ having positive probability are illustrated in Figure 1.

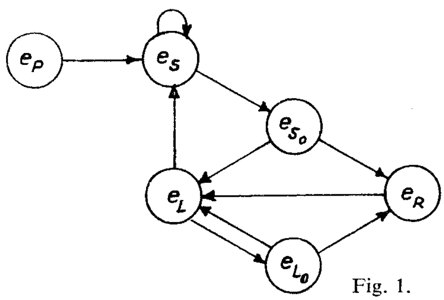

Fig. 1.

Let us denote the random variables "the whole time of operation of units in state $I(II)$ during the first operating period of the system under the conditions $\mathscr{P}_Y(P)$, $\mathscr{P}_Y(S)$, $\mathscr{P}_Y(S_0)$, $\mathscr{P}_Y(L)$ and $\mathscr{P}_Y(L_0)$", respectively, by the symbols ${}_1\mathscr{P}$, ${}_1\mathscr{S}$, ${}_1\mathscr{S}_0$, ${}_1\mathscr{L}$ and ${}_1\mathscr{L}_0({}_2\mathscr{P}, {}_2\mathscr{S}, {}_2\mathscr{S}_0, {}_2\mathscr{L}$ and ${}_2\mathscr{L}_0)$. Obviously

$$(1.1) \qquad {}_1\mathscr{P} = \mathscr{A} + {}_1\mathscr{S} .$$

Let the starting state of the system be e_{S_0} and let the first state-transition of the process $Y(t)$ lead from e_{S_0} to e_R. Then till the moment of this change of state of $Y(t)$ a unit is operating in state II and the other one is being repaired from state II. A failure of the first unit occurs sooner than the repair of the other one is finished. So from the beginning (in state e_{S_0}) to the first system failure still the same unit has been operating in state II and therefore

$$(1.2) \qquad {}_1\mathscr{S}_0 = 0 , \quad \text{if the first state-transition of } Y(t) \text{ is } e_{S_0} \to e_R .$$

On the other hand, if the first state-transition of the process $Y(t)$ leads from e_{S_0} to e_L then till the moment of its realization a unit is operating in state II and the other one is being repaired from state II. In this case, however, the repair is finished sooner than a failure of the first unit occurs. At the moment t_1 of this failure the process $Y(t)$ enters the state e_L. From the beginning (in state e_{S_0}) to t_1 still the same unit has been operating in state II and therefore the whole time of operation of units in state I during the first operating period in the cases that either the starting state of $Y(t)$ is e_L or the starting state of $Y(t)$ is e_{S_0} and its first state-transition leads to e_L, is the same, i.e.

$$(1.3) \qquad {}_1\mathscr{S}_0 = {}_1\mathscr{L} \quad \text{if the first state-transition of } Y(t) \text{ is } e_{S_0} \to e_L .$$

In the similar way one can also obtain relations for the random variable ${}_1\mathscr{L}_0$:

(1.4) $\qquad {}_1\mathscr{L}_0 = 0$ if the first state-transition of $Y(t)$ is $e_{L_0} \to e_R$,

(1.5) $\qquad {}_1\mathscr{L}_0 = {}_1\mathscr{L}$ if the first state-transition of $Y(t)$ is $e_{L_0} \to e_L$.

The relations (1.2) to (1.5) imply that

$$(1.6) \qquad {}_1\mathscr{S} = \begin{cases} {}_1\mathscr{T}_{SS} + {}_1\mathscr{S} & \text{if } \mathscr{A} \geqq \mathscr{M}, \\ {}_1\mathscr{T}_{SS_0L} + {}_1\mathscr{L} & \text{if } \mathscr{A} < \mathscr{M} \leqq \mathscr{A} + \mathscr{B}, \\ {}_1\mathscr{T}_{SS_0R} & \text{if } \mathscr{A} + \mathscr{B} < \mathscr{M}, \end{cases}$$

$$(1.7) \qquad {}_1\mathscr{L} = \begin{cases} {}_1\mathscr{T}_{LS} + {}_1\mathscr{S} & \text{if } \mathscr{A} \geqq \mathscr{N}, \\ {}_1\mathscr{T}_{LL_0L} + {}_1\mathscr{L} & \text{if } \mathscr{A} < \mathscr{N} \leqq \mathscr{A} + \mathscr{B}, \\ {}_1\mathscr{T}_{LL_0R} & \text{if } \mathscr{A} + \mathscr{B} < \mathscr{N}, \end{cases}$$

where ${}_1\mathscr{T}_{ij}$ or ${}_1\mathscr{T}_{ijk}$ are random variables "sojourn time of $Y(t)$ in state e_i under the condition that after this time the process $Y(t)$ enters state e_j or that the subsequent two states of $Y(t)$ will be the states e_j and e_k", on the right hand sides are sums of independent random variables, and the meaning of symbols $\mathscr{A}$, $\mathscr{B}$, $\mathscr{M}$ and $\mathscr{N}$ is as follows: $\mathscr{M}$ ($\mathscr{N}$) is time of the repair which was started at the moment when the system was activated in state $e_S(e_L)$; $\mathscr{A}$ and $\mathscr{B}$ are times of work in state I and II of the unit which started to work at the same moment. Random variables ${}_1\mathscr{T}_{ij}$ and ${}_1\mathscr{T}_{ijk}$ have distributions:

$$\mathsf{P}({}_1\mathscr{T}_{SS} \leqq x) = \mathsf{P}(\mathscr{A} \leqq x / \mathscr{A} \geqq \mathscr{M}),$$
$$\mathsf{P}({}_1\mathscr{T}_{LS} \leqq x) = \mathsf{P}(\mathscr{A} \leqq x / \mathscr{A} \geqq \mathscr{N}),$$
$$\mathsf{P}({}_1\mathscr{T}_{SS_0L} \leqq x) = \mathsf{P}(\mathscr{A} \leqq x / \mathscr{A} < \mathscr{M} \leqq \mathscr{A} + \mathscr{B}),$$
$$\mathsf{P}({}_1\mathscr{T}_{LL_0L} \leqq x) = \mathsf{P}(\mathscr{A} \leqq x / \mathscr{A} < \mathscr{N} \leqq \mathscr{A} + \mathscr{B}),$$
$$\mathsf{P}({}_1\mathscr{T}_{SS_0R} \leqq x) = \mathsf{P}(\mathscr{A} \leqq x / \mathscr{A} + \mathscr{B} < \mathscr{M}),$$
$$\mathsf{P}({}_1\mathscr{T}_{LL_0R} \leqq x) = \mathsf{P}(\mathscr{A} \leqq x / \mathscr{A} + \mathscr{B} < \mathscr{N}).$$

Relations of random variables ${}_2\mathscr{P}$, ${}_2\mathscr{S}$, ${}_2\mathscr{S}_0$, ${}_2\mathscr{L}$ and ${}_2\mathscr{L}_0$ can be obtained in the similar way. Following two theorems can be proved by calculation of distributions of ${}_1\mathscr{T}_{ij}$, ${}_1\mathscr{T}_{ijk}$ and of similar variables with subscript 2 and by passing to Laplace Stieltjes transforms.

Theorem 1. *Let $\hat{\pi}(t)$ be Laplace Stieltjes transform of distribution function of random variable ${}_1\mathscr{P}$.* Then

$$(1.8) \qquad \hat{\pi}(t) = \left[\alpha \cdot \frac{(\alpha - \hat{\delta}) \cdot (\hat{\varphi} - \varepsilon - 1) + (\alpha - \hat{\varphi}) \cdot (\gamma - \hat{\delta})}{(1 - \gamma) \cdot (\hat{\varphi} - \varepsilon - 1) + \varepsilon \cdot (\hat{\delta} - \gamma)}\right]_t,$$

where α, γ and ε have been determined in Section 1 of [1] *and $\hat{\delta}$ and $\hat{\varphi}$ respectively are Laplace Stieltjes transforms of functions*

$$(1.9) \qquad \hat{D}(x) = \int_{-\infty}^{x+0} \left(\int_{-\infty}^{\infty} M(y + z) \, \mathrm{d}B(z) \right) \mathrm{d}A(y),$$

$$(1.10) \qquad \hat{F}(x) = \int_{-\infty}^{x+0} \left(\int_{-\infty}^{\infty} N(y + z) \, \mathrm{d}B(z) \right) \mathrm{d}A(y).$$

Theorem 2. *Let $\tilde{\pi}(t)$ be the Laplace Stieltjes transform of the distribution function of the random variable ${}_2\mathscr{P}$. Then*

$$(1.11) \qquad \tilde{\pi}(t) = \left[\frac{(\beta - \tilde{\varphi}) \, . \, (\tilde{\delta} - c \, . \, \beta) + (\beta - \tilde{\delta}) \, . \, (1 - \tilde{\varphi} + e \, . \, \beta)}{(1 - c)(1 - \tilde{\varphi} + e \, . \, \beta) - e \, . \, (\tilde{\delta} - c \, . \, \beta)} \right]_t,$$

where c, e and β have been determined in Section 1 of [1] *and $\tilde{\delta}$ and $\tilde{\varphi}$, respectively, are the Laplace Stieltjes transforms of the functions*

$$(1.12) \qquad \tilde{D}(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x+0} M(y + z) \, \mathrm{d}B(z) \right) \mathrm{d}A(y),$$

$$(1.13) \qquad \tilde{F}(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x+0} N(y + z) \, \mathrm{d}B(z) \right) \mathrm{d}A(y).$$

For the sake of a study of the whole time of repairs of units during the first operating period of the system let us define a random process $Z(t)$ with eight states e_P, e_S, e_{S_I}, $e_{S_{II}}$, e_L, e_{L_I}, $e_{L_{II}}$, e_R, which changes its state at the same moments as the process $X(t)$ defined in Section 2 of [1] and, moreover, at the moments when a repair of a unit is finished and the other one is operating. We say that the process $Z(t)$ enters

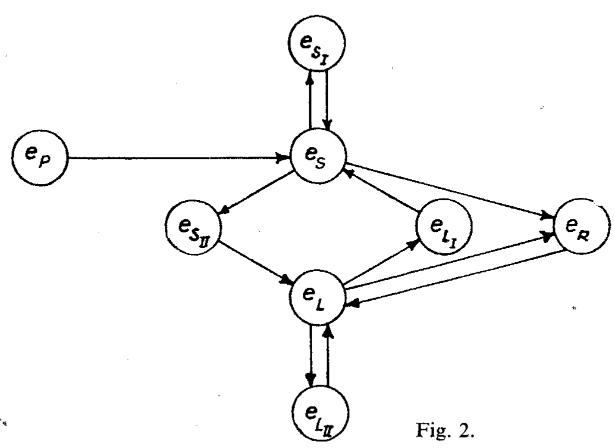

Fig. 2.

at t_0 the states e_P, e_S, e_L and e_R under the same conditions as the process $X(t)$ and the states:

e_{S_i} – if at t_0 a repair of a unit of the type $II \to I$ is finished and the other one is operating in state i, $i = I, II$;

e_{L_i} – if at t_0 a repair of a unit of the type $III \to I$ is finished and the other one is operating in state i, $i = I, II$.

The state-transitions of $Z(t)$ having positive probability are illustrated in Figure 2.

Let us denote the random variables „the whole time of repairs of units (the whole time of repairs of units of the types $II \to I$ and $III \to I$, respectively) during the first operating period of the system under the conditions $\mathscr{P}_Z(P)$, $\mathscr{P}_Z(S)$ and $\mathscr{P}_Z(L)$", respectively, by the symbols $\mathscr{P}^{(R)}$, $\mathscr{S}^{(R)}$ and $\mathscr{L}^{(R)}$ (${}_2\mathscr{P}^{(R)}$, ${}_2\mathscr{S}^{(R)}$, ${}_2\mathscr{L}^{(R)}$ and ${}_3\mathscr{P}^{(R)}$, ${}_3\mathscr{S}^{(R)}$, ${}_3\mathscr{L}^{(R)}$). Obviously

$$\mathscr{P}^{(R)} = \mathscr{S}^{(R)}, \tag{1.14}$$

$$ {}_2\mathscr{P}^{(R)} = {}_2\mathscr{S}^{(R)}, \tag{1.15}$$

$$ {}_3\mathscr{P}^{(R)} = {}_3\mathscr{S}^{(R)}. \tag{1.16}$$

It can be shown that the whole time of repairs of units (and similarly for individual types of repairs $II \to I$ or $III \to I$) is the same under each of the following three conditions: $\mathscr{P}_Z(S_I)$, $\mathscr{P}_Z(L_I)$ and $\mathscr{P}_Z(S)$. The same is true also under the conditions $\mathscr{P}_Z(S_{II})$, $\mathscr{P}_Z(L_{II})$ and $\mathscr{P}_Z(L)$. The reason is the fact that from the states e_{S_I} or e_{L_I} ($e_{S_{II}}$ or $e_{L_{II}}$) the process $Z(t)$ can enter only the state e_S (e_L) – see Figure 2 – and before this state-transition no unit is being repaired. Thus the process $X(t)$ with only four states e_P, e_S, e_L and e_R describes the behaviour of our system sufficiently as concerns the problem of the whole time of repairs of units. This implies that relations analogous to the relations (3.2) and (3.3) from [1] are fulfilled if we only replace the variables $\mathscr{S}$, $\mathscr{L}$ and $\mathscr{T}_{ij}$ by the variables $\mathscr{S}^{(R)}$, $\mathscr{L}^{(R)}$, $\mathscr{T}_{ij}^{(R)}$ or ${}_2\mathscr{S}^{(R)}$, ${}_2\mathscr{L}^{(R)}$, ${}_2\mathscr{T}_{ij}^{(R)}$ or ${}_3\mathscr{S}^{(R)}$, ${}_3\mathscr{L}^{(R)}$, ${}_3\mathscr{T}_{ij}^{(R)}$, where $\mathscr{T}_{ij}^{(R)}$ (${}_2\mathscr{T}_{ij}^{(R)}$ or ${}_3\mathscr{T}_{ij}^{(R)}$) expresses the time of repair (time of repair of the type $II \to I$ or $III \to I$) carried out while the process $X(t)$ is being in a state e_i under the condition that after that time $X(t)$ enters the state e_j, $i \in \{S; L\}$, $j \in \{S; L; R\}$. The random variables $\mathscr{T}_{ij}^{(R)}$, ${}_2\mathscr{T}_{ij}^{(R)}$ and ${}_3\mathscr{T}_{ij}^{(R)}$ have the distributions:

$$\mathsf{P}(\mathscr{T}_{SS}^{(R)} \leqq x) = \mathsf{P}({}_2\mathscr{T}_{SS}^{(R)} \leqq x) = \mathsf{P}(\mathscr{M} \leqq x / \mathscr{A} \geqq \mathscr{M}),$$

$$\mathsf{P}(\mathscr{T}_{LS}^{(R)} \leqq x) = \mathsf{P}({}_3\mathscr{T}_{LS}^{(R)} \leqq x) = \mathsf{P}(\mathscr{N} \leqq x / \mathscr{A} \geqq \mathscr{N}),$$

$$\mathsf{P}(\mathscr{T}_{SL}^{(R)} \leqq x) = \mathsf{P}({}_2\mathscr{T}_{SL}^{(R)} \leqq x) = \mathsf{P}(\mathscr{M} \leqq x / \mathscr{A} < \mathscr{M} \leqq \mathscr{A} + \mathscr{B}),$$

$$\mathsf{P}(\mathscr{T}_{LL}^{(R)} \leqq x) = \mathsf{P}({}_3\mathscr{T}_{LL}^{(R)} \leqq x) = \mathsf{P}(\mathscr{N} \leqq x / A < \mathscr{N} \leqq \mathscr{A} + \mathscr{B}),$$

$$\mathsf{P}(\mathscr{T}_{SR}^{(R)} \leqq x) = \mathsf{P}({}_2\mathscr{T}_{SR}^{(R)} \leqq x) = \mathsf{P}(\mathscr{M} \leqq x / \mathscr{A} + \mathscr{B} < \mathscr{M}),$$

$$\mathsf{P}(\mathscr{T}_{LR}^{(R)} \leqq x) = \mathsf{P}({}_3\mathscr{T}_{LR}^{(R)} \leqq x) = \mathsf{P}(\mathscr{N} \leqq x / \mathscr{A} + \mathscr{B} < \mathscr{N}),$$

$$\mathsf{P}({}_2\mathscr{T}_{Lj}^{(R)} = 0) = \mathsf{P}({}_3\mathscr{T}_{Sj}^{(R)} = 0) = 1 \quad \text{for} \quad j \in \{S; L; R\}.$$

After calculating these distributions and passing to the Laplace Stieltjes transforms one can obtain the following theorem.

Theorem 3. *Let $\pi^{(R)}(t)$, $\hat{\pi}^{(R)}(t)$ and $\tilde{\pi}^{(R)}(t)$ be the Laplace Stieltjes transforms of the distribution functions of the random variables $\mathscr{P}^{(R)}$, ${}_2\mathscr{P}^{(R)}$ and ${}_3\mathscr{P}^{(R)}$, respectively. Then*

$$(1.17)\quad \pi^{(R)}(t) = \left[\frac{(\alpha\beta - \delta)(1 + \varphi^{(R)} - \varepsilon^{(R)}) - (\alpha\beta - \varphi)(\delta^{(R)} - \gamma^{(R)})}{(1 - \mu + \gamma^{(R)}) . (1 + \varphi^{(R)} - \varepsilon^{(R)}) + (\nu - \varepsilon^{(R)}) . (\delta^{(R)} - \gamma^{(R)})}\right]_t ,$$

$$(1.18)\quad \hat{\pi}^{(R)}(t) = \left[\frac{(\alpha\beta - \delta)(1 + e - f) - (1 - f) . (\delta^{(R)} - \gamma^{(R)})}{(1 - \mu + \gamma^{(R)}) . (1 + e - f) + e . (\delta^{(R)} - \gamma^{(R)})}\right]_t ,$$

$$(1.19)\quad \tilde{\pi}^{(R)}(t) = \left[\frac{(1 - d)(1 + \varphi^{(R)} - \varepsilon^{(R)}) - (\alpha\beta - \varphi) . (c - d)}{(1 - c)(1 + \varphi^{(R)} - \varepsilon^{(R)}) + (\nu - \varepsilon^{(R)})(c - d)}\right]_t ,$$

where c, d, e, f, α, β, δ and φ have been determined in Section 1 of [1], μ and ν are the Laplace Stieltjes transforms of the function M and N defined at the same place and $\gamma^{(R)}$, $\delta^{(R)}$, $\varepsilon^{(R)}$ and $\varphi^{(R)}$ are respectively the Laplace Stieltjes transforms of the functions

$$(1.20)\qquad C^{(R)}(x) = \int_{-\infty}^{x+0} A(y - 0)\, \mathrm{d}M(y) ,$$

$$(1.21)\qquad D^{(R)}(x) = \int_{-\infty}^{x+0} (A * B)(y - 0)\, \mathrm{d}M(y) ,$$

$$(1.22)\qquad E^{(R)}(x) = \int_{-\infty}^{x+0} A(y - 0)\, \mathrm{d}N(y) ,$$

$$(1.23)\qquad F^{(R)}(x) = \int_{-\infty}^{x+0} (A * B)(y - 0)\, \mathrm{d}N(y) .$$

Now we shall study the problem of the number of repairs of units finished before the first failure of the system. Let us denote the probability that before the first system failure n repairs of units (n repairs of units of the type $II \to I$ and $III \to I$, respectively) were finished under the conditions $\mathscr{P}_X(P)$ and $\mathscr{P}_X(L)$, respectively, by the symbols x_n and y_n($x_n^{(2)}$ and $y_n^{(2)}$ or $x_n^{(3)}$ and $y_u^{(3)}$). It is obvious that the probabilities just mentioned are the same under the conditions $\mathscr{P}_X(P)$ and $\mathscr{P}_X(S)$. One can easily find that

$$(1.24)\qquad x_0 = x_0^{(2)} = 1 - d ,$$

$$(1.25)\qquad y_0 = y_0^{(3)} = 1 - f ,$$

$$(1.26)\qquad x_0^{(3)} = \sum_{k=0}^{\infty} [\mathsf{P}(\mathscr{A} \geqq \mathscr{M})]^k .$$

$$. [\mathsf{P}(\mathscr{A} + \mathscr{B} < \mathscr{M}) + \mathsf{P}(\mathscr{A} < \mathscr{M} \leqq \mathscr{A} + \mathscr{B}) . \mathsf{P}(\mathscr{A} + \mathscr{B} < \mathscr{N})] =$$

$$= 1 - \frac{(d - c) . f}{1 - c},$$

(1.27) $$y_0^{(2)} = \sum_{k=0}^{\infty} [\mathsf{P}(\mathscr{A} < \mathscr{N} \leqq \mathscr{A} + \mathscr{B})]^k .$$

$$[\mathsf{P}(\mathscr{A} + \mathscr{B} < \mathscr{N}) + \mathsf{P}(\mathscr{A} \geqq \mathscr{N}) . \mathsf{P}(\mathrm{A} + \mathscr{B} < \mathscr{M})] =$$

$$= 1 - \frac{de}{1 + e - f}.$$

Theorem 4. *The generating function $\xi(s)$ of the sequence $\{x_n\}_{n=0}^{\infty}$ has the form*

(1.28) $$\xi(s) = \frac{(1 - d) + s[(1 - c)(1 - f) - (1 - d)(1 - e)]}{(1 - cs) . (1 - fs) + es . (1 - ds)},$$

where the numbers c, d, e and f have been determined in Section 1 of [1].

Proof. We know from [1] that the chain X_n embedded into the process $X(t)$ is markovian. Hence the elements of the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ fulfil for each natural n the relations

(1.29) $$x_n = c . x_{n-1} + (\mathrm{d} - c) . y_{n-1},$$

(1.30) $$y_n = e . x_{n-1} + (f - e) . y_{n-1} .$$

If we multiply both sides of equations (1.29) and (1.30) by s^n and add each of them over all natural n we obtain

(1.31) $$\xi(s) - x_0 = cs . \xi(s) + (d - c) s . \eta(s),$$

(1.32) $$\eta(s) - y_0 = es . \xi(s) + (f - e) s . \eta(s),$$

where $\eta(s)$ is the generating function corresponding to the sequence $\{y_n\}_{n=0}^{\infty}$. The system of equations (1.31) and (1.32) has obviously the solution (1.28) – it is only necessary to substitute for x_0 and y_0 the values given in (1.24) and (1.25), respectively.

The following two theorems can be easily proved by using the Markov property of the chain X_n.

Theorem 5. *The elements of the sequence $\{x_n^{(2)}\}_{n=1}^{\infty}$ have the form*

(1.33) $$x_n^{(2)} = \hat{r} . \hat{q}^{n-1},$$

where

(1.34) $$\hat{q} = c + \frac{(d - c) . e}{1 + e - f},$$

$$\hat{r} = d(1 - c) - \frac{(d - c) \,.\, de}{1 + e - f} \,. \tag{1.35}$$

Theorem 6. *The elements of the sequence* $\{x_n^{(3)}\}_{n=1}^{\infty}$ *have the form*

$$x_n^{(3)} = \tilde{r} \,.\, \tilde{q}^{n-1} \,, \tag{1.36}$$

where

$$\tilde{q} = f - e + \frac{(d - c)\, e}{1 - c} \,, \tag{1.37}$$

$$\tilde{r} = \frac{(d - c) \,.\, f}{(1 - c)^2} \,.\, \left[(1 - c)(1 - f) + e(1 - d)\right] . \tag{1.38}$$

The result of Theorem 4 can be used for finding the probability (let us denote it by w) that the first system failure occurs during a repair of the unit which first started to operate at the moment when the system was activated under the condition $\mathscr{P}_X(P)$. This probability equals the probability that an even number of repairs (of both types $II \to I$ and $III \to I$ in sum) was finished before the first failure of the system. On the other hand, the following relations are true:

$$\xi(-1) = \sum_{n=0}^{\infty} x_{2n} - \sum_{n=0}^{\infty} x_{2n+1} = w - (1 - w) = 2w - 1 \,. \tag{1.39}$$

Thus

$$w = \frac{1 + \xi(-1)}{2} = \frac{1 + c - d - e + f}{(1 + c) \,.\, (1 + f) - e \,.\, (1 + d)} \,. \tag{1.40}$$

2. MATHEMATICAL EXPECTATIONS

In this section we shall derive mathematical expectations of the random variables in which we were interested in the preceding section. We shall deal with the random variables ${}_1\mathscr{P}$, ${}_2\mathscr{P}$, $\mathscr{P}^{(R)}$, ${}_2\mathscr{P}^{(R)}$, ${}_3\mathscr{P}^{(R)}$ and with the random variables $\mathscr{X}$, $\mathscr{X}^{(2)}$, and $\mathscr{X}^{(3)}$, expressing respectively the number of repairs of units, the number of repairs of units of the type $II \to I$ and the number of repairs of units of the type $III \to I$ finished before the first system failure under the condition $\mathscr{P}_X(P)$.

Let us have a random variable $\mathscr{R}$ with a distribution function R. Let $\mathscr{R}$ be non-negative with probability 1 and let $\varrho_{LST}(t)$ and $\varrho_{GF}(t)$ (if $\mathscr{R}$ is a discrete distribution) be respectively the Laplace Stieltjes transform and the generating function corresponding to the distribution R. If $\mathscr{R}$ has a finite mathematical expectation we know that

$$\mathsf{E}\mathscr{R} = \int_{-\infty}^{\infty} x \, \mathrm{d}R(x) = \int_0^{\infty} [1 - R(x)] \mathrm{d}x = - \varrho'_{LST}(0+) = \varrho'_{GF}(1-) \,. \tag{2.1}$$

In this section we shall suppose that the distribution functions A, B, M and N determined in Section 1 of [1] have finite mathematical expectations. Then

$$0 \leqq \int_{-\infty}^{\infty} x \, \mathrm{d}C(x) = \int_{-\infty}^{\infty} x \, M(x) \, \mathrm{d}A(x) \leqq \int_{-\infty}^{\infty} x \, \mathrm{d}A(x) < \infty$$

and

$$0 \leqq \int_{-\infty}^{\infty} x \, \mathrm{d}D(x) = \int_{-\infty}^{\infty} x \, M(x) \, \mathrm{d}(A * B)(x) \leqq$$

$$\leqq \int_{-\infty}^{\infty} x \, \mathrm{d}(A * B)(x) = \int_{-\infty}^{\infty} x \, \mathrm{d}A(x) + \int_{-\infty}^{\infty} x \, \mathrm{d}B(x) < \infty .$$

Similar relations are true also for the functions E, F, $\hat{D}$, $\hat{F}$, $\tilde{D}$, $\tilde{F}$, $C^{(R)}$, $D^{(R)}$, $E^{(R)}$ and $F^{(R)}$. This implies that each of the Laplace Stieltjes transforms α, β, μ, ν, γ, δ, ε, φ, $\hat{\delta}$, $\hat{\varphi}$, $\tilde{\delta}$, $\tilde{\varphi}$, $\gamma^{(R)}$, $\delta^{(R)}$, $\varepsilon^{(R)}$ and $\varphi^{(R)}$ have finite derivatives at the point 0 fiom the right.

Theorem 7. *Mathematical expectations of the random variables* ${}_1\mathscr{P}$ *and* ${}_2\mathscr{P}$ *have the expressions*

$$\mathsf{E}\,{}_1\mathscr{P} = \mathsf{E}\mathscr{A} + \frac{(1 - c + d + e - f) \, . \, \mathsf{E}\mathscr{A}}{(1 - c)(1 - f) + e(1 - d)}, \tag{2.2}$$

$$\mathsf{E}\,{}_2\mathscr{P} = \mathsf{E}\mathscr{B} + \frac{(d - c) \, . \, \mathsf{E}\mathscr{B}}{(1 - c)(1 - f) + e(1 - d)}, \tag{2.3}$$

where the random variables $\mathscr{A}$ *and* $\mathscr{B}$ *and the numbers* c, d, e *and* f *have been determined in Section 1 of* [1].

Proof. The proof uses the forms of the corresponding Laplace Stieltjes transforms given in Theorems 1 and 2 and the fact that

$$d = \lim_{x \to \infty} \hat{D}(x) = \lim_{x \to \infty} \tilde{D}(x), \tag{2.4}$$

$$f = \lim_{x \to \infty} \hat{F}(x) = \lim_{x \to \infty} \tilde{F}(x), \tag{2.5}$$

where the functions $\hat{D}$, $\hat{F}$, $\tilde{D}$ and $\tilde{F}$ have been determined by the relations (1.9), (1.10), (1.12) and (1.13), respectively.

Theorem 8. *The mathematical expectations of the random variables* $\mathscr{P}^{(R)}$, ${}_2\mathscr{P}^{(R)}$ *and* ${}_3\mathscr{P}^{(R)}$ *have the expressions*

$$\mathsf{E}\mathscr{P}^{(R)} = \tag{2.6}$$

$$= \frac{(1 + e - f)(\mathsf{E}\mathscr{A} + \mathsf{E}\mathscr{B} + \mathsf{E}\mathscr{M} - \mathsf{E}\mathscr{Z}_M) + (d - c)(\mathsf{E}\mathscr{A} + \mathsf{E}\mathscr{B} + \mathsf{E}\mathscr{N} - \mathsf{E}\mathscr{Z}_N)}{(1 - c)(1 - f) + e(1 - d)},$$

(2.7) $$\mathsf{E}\,{}_2\mathscr{P}^{(R)} = \frac{(1 + e - f)(\mathsf{E}\mathscr{A} + \mathsf{E}\mathscr{B} + \mathsf{E}\mathscr{M} - \mathsf{E}\mathscr{Z}_M)}{(1 - c)(1 - f) + e(1 - d)},$$

(2.8) $$\mathsf{E}\,{}_3\mathscr{P}^{(R)} = \frac{(d - c) . (\mathsf{E}\mathscr{A} + \mathsf{E}\mathscr{B} + \mathsf{E}\mathscr{N} - \mathsf{E}\mathscr{Z}_N)}{(1 - c) . (1 - f) + e(1 - d)},$$

where the numbers c, d, e and f and the random variables $\mathscr{A}, \mathscr{B}, \mathscr{M}, \mathscr{N}, \mathscr{Z}_M$ and $\mathscr{Z}_N$ have been determined in Section 1 of [1].

Proof. The corresponding Laplace Stieltjes transforms are given in Theorem 3. It is necessary to find that

(2.9) $$\mathsf{P}(\mathscr{Z}_M \leqq x) = D(x) + D^{(R)}(x),$$

(2.10) $$\mathsf{P}(\mathscr{Z}_N \leqq x) = F(x) + F^{(R)}(x),$$

so that

(2.11) $$\lim_{x \to \infty} D^{(R)}(x) = 1 - d,$$

(2.12) $$\lim_{x \to \infty} F^{(R)}(x) = 1 - f$$

and similarly

(2.13) $$\lim_{x \to \infty} C^{(R)}(x) = \lim_{x \to \infty} [A(x) . M(x) - C(x)] = 1 - c,$$

(2.14) $$\lim_{x \to \infty} E^{(R)}(x) = \lim_{x \to \infty} [A(x) . N(x) - E(x)] = 1 - e,$$

where the symbols $\mathscr{Z}_M$, $\mathscr{Z}_N$, $A, C, D, E, F, M, N, c, d, e$ and f have been determined in Section 1 of [1] and the symbols $C^{(R)}$, $D^{(R)}$, $E^{(R)}$ and $F^{(R)}$ by the relations (1.20) to (1.23).

Theorem 9. *The mathematical expectations of the random variables $\mathscr{X}$, $\mathscr{X}^{(2)}$ and $\mathscr{X}^{(3)}$ have the expressions*

(2.15) $$\mathsf{E}\mathscr{X} = \frac{d + de - cf}{(1 - c) . (1 - f) + e(1 - d)},$$

(2.16) $$\mathsf{E}\mathscr{X}^{(2)} = \frac{d(1 + e - f)}{(1 - c)(1 - f) + e(1 - d)},$$

(2.17) $$\mathsf{E}\mathscr{X}^{(3)} = \frac{(d - c) f}{(1 - c)(1 - f) + e(1 - d)},$$

where the numbers c, d, e and f have been determined in Section 1 of [1].

Proof. The existence of the mathematical expectations in question is guaranteed by the assumption that the relation (2.3) from [1] is fulfilled. The generating function

$\xi(s)$ corresponding to the random variable $\mathscr{X}$ is given in Theorem 4. The probabilities $\mathsf{P}(\mathscr{X}^{(2)} = n) = x_n^{(2)}$ and $\mathsf{P}(\mathscr{X}^{(3)} = n) = x_n^{(3)}$ are known for all $n \in \mathrm{N} \cup \{0\}$ from the relations (1.24), (1.25) and from Theorems 5 and 6. The mathematical expectations of the random variables $\mathscr{X}^{(2)}$ and $\mathscr{X}^{(3)}$ are obtained from the formulae

$$\mathsf{E}\mathscr{X}^{(i)} = \sum_{n=1}^{\infty} n \, . \, x_n^{(i)} \quad \text{for} \quad i = 2, 3 \, . \tag{2.18}$$

In this paper we have derived distributions, Laplace Stieltjes transforms, generating functions and mathematical expectations of some random variables characterizing the behaviour of the system in question during its single operating period. In all the cases it has been supposed that the condition $\mathscr{P}_X(P)$ is fulfilled. In this way all the random variables considered in Theorems of this paper deal only with the first operating period of the system (only this one can start with both new units). All the other operating periods of the system start in the state e_L (see Section 2 of [1]). It is, however, necessary to say that all characteristics of the random variables mentioned above under the condition $\mathscr{P}_X(L)$ can be found in a similar way.

Reference

[1] *A. Lešanovský:* Analysis of a Two-Unit Standby Redundant System with Three States of Units. Apl. mat. 27 (1982), 192–208.

Souhrn

CHARAKTERIZACE PRVNÍHO PRACOVNÍHO OBDOBÍ SYSTÉMU S NEZATÍŽENOU ZÁLOHOU SLOŽENÉHO ZE DVOU PRVKŮ, KTERÉ MOHOU BÝT VE TŘECH STAVECH

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V článku je uvažován jistý systém s nezatíženou zálohou složený ze dvou prvků a jednoho zařízení pro jejich opravy. Prvky mohou být ve třech stavech: bezvadném (I), zhoršeném (II) a poruchovém (III). Předpokládáme, že možné jsou pouze následující změny stavu prvků: $I \to II$, $II \to III$, $II \to I$, $III \to I$. Pozornost je věnována chování tohoto systému do jeho první poruchy. V článku je odvozena řada takových jeho charakteristik, které jsou speciální pro systémy s více než dvěma základními stavy prvků (bezvadným a poruchovým) – např. rozložení a střední hodnota doby, po kterou v systému pracují prvky ve stavu I, resp. II, celkové doby oprav prvků typu $II \to I$, resp. $III \to I$, a počtu dokončených oprav prvků typu $II \to I$, resp. $III \to I$.

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