Characterization of the ordered weighted averaging operators

Janos FODOR
Eötvös Lorand University, Budapest, Hungary
Jean-Luc MARICHAL and Marc ROUBENS
Institute of Mathematics, University of Liège, Belgium

*

Abstract

This paper deals with the characterization of two classes of monotonic and neutral (MN) aggregation operators . The first class corresponds to (MN) aggregators which are stable for the same positive linear transformations and present the ordered linkage property. The second class deals with (MN)-idempotent aggregators which are stable for positive linear transformations with same unit, independent zeroes and ordered values. These two classes correspond to the weighted ordered averaging operator (OWA) introduced by Yager in 1988. It is also shown that the OWA aggregator can be expressed as a Choquet integral.

Keywords : aggregation functions; interval scale; invariance; ordered weighted averaging operator.

1 Introduction

The ordered weighted averaging aggregation operator (OWA) was proposed by Yager in 1988 [19]. Since its introduction, it has been applied to many fields as neural networks (Yager [17, 21]), data base systems (Yager [18, 22]), fuzzy logic controlers (Yager [20, 23]) and group decision making (Yager [19], Cutello and Montero [5]). The OWA operator can also be used in decision making under uncertainty to modelize the anticipated utility (Quiggin [10], Segal [14]).

Its structural properties (Skala [15]) and its links with fuzzy integrals (Grabisch [7]) were also investigated.

Synthesizing judgments is an important part of multiple criteria decision making methods. The typical situation concerns individuals which form quantitative judgments about a measure. In order to obtain a consensus of these jugdments, classical operators

*Address correspondence to M. Roubens, Institute of Mathematics, University of Liège, Avenue des Tilleuls, 15, Bât. D1, B-4000 Liège, Belgium

are proposed : arithmetic means, geometric means, root-power means, quasi arithmetic means, fuzzy integrals and among them the OWA aggregators.

We study the OWA operator under several natural properties like monotonicity, neutrality, and its stability in the sense of the theory of measurement (Roberts [13]).

We first characterize the OWA operator under the properties of monotonicity, neutrality, stability for the same interval scales and some kind of associativity called "ordered linkage".

We then reexamine the characterization of OWA operators under the properties of monotonicity, neutrality, idempotency and stability for positive linear transformations with the same unit, independent zeroes and ordered values.

We also prove that an OWA operator can be expressed under the form of a Choquet integral.

2 Basic definitions

We consider a vector $(x_1, \ldots, x_m) \in R^m$, $m > 1$, and we are willing to substitute to that vector a single value $M^{(m)}(x_1, \ldots, x_m) \in R$ using the aggregation operator (aggregator) M.

The aggregator presents the property of

- *monotonicity* (non negative responsiveness) if

$$x_i' > x_i \text{ implies } M(x_1, \ldots, x_i', \ldots, x_m) \geq M(x_1, \ldots, x_i, \ldots, x_m)$$

- *neutrality* (symmetry, commutativity, anonimity) if

$$M(x_1, \ldots, x_m) = M(x_{i_1}, \ldots, x_{i_m}) \text{ for all } (x_1, \ldots, x_m) \in R^m,$$

  when $(i_1, \ldots, i_m) = \sigma(1, \ldots, m)$, where σ represents a permutation operation.

- *idempotency* (agreement, unanimity, identity, reflexivity) if

$$M(x, \ldots, x) = x \quad , \quad \text{for all } x \in R$$

- *associativity* if

$$\begin{aligned} M(M(x_1, x_2), x_3) &= M(x_1, M(x_2, x_3)) = M(x_1, x_2, x_2) \\ M(M(x_1, \ldots, x_{m-1}), x_m) &= M(x_1, M(x_2, \ldots, x_m)) \end{aligned}$$

  for all $(x_1, \ldots, x_m) \in R^m$.

- *decomposability* (Kolmogorov [9], Nagumo [12]) if

$$M^{(m)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) = M^{(m)}(x, \ldots, x, x_{k+1}, \ldots, x_m)$$

  when $x = M^{(k)}(x_1, \ldots, x_k)$, for all $(x_1, \ldots, x_m) \in R^m$.

- *linkage* if

$$M^{(m+1)}(M^{(m)}(x_1,\dots,x_m), M^{(m)}(x_2,\dots,x_{m+1}),\dots,M^{(m)}(x_{m+1},\dots,x_{2m}))$$
$$= M^{(m)}(M^{(m+1)}(x_1,\dots,x_{m+1}), M^{(m+1)}(x_2,\dots,x_{m+2}),\dots,M^{(m+1)}(x_m,\dots,x_{2m}))$$

for all $(x_1,\dots,x_{2m}) \in R^{2m}$.

- *ordered linkage* if $M^{(m)}$ presents the property of linkage for the ordered values $x_{(1)} \le x_{(2)} \le \cdots \le x_{(2m)}$.

- *stability for the same positive linear transformations* (SPL) if

$$M^{(m)}(rx_1+t,\dots,rx_m+t) = rM^{(m)}(x_1,\dots,x_m)+t,$$

for all $(x_1,\dots,x_m) \in R^m$, all $r>0$, all $t \in R$.

- *stability for positive linear transformations with same unit and independent zeroes* (SPLU)

$$M^{(m)}(rx_1+t_1,\dots,rx_m+t_m) = rM^{(m)}(x_1,\dots,x_m)+T^{(m)}(t_1,\dots,t_m)$$

for all $(x_1,\dots,x_m) \in R^m$, all $r>0$, all $(t_1,\dots,t_m) \in R^m$.

3 Ordered weighted averaging aggregation operators (OWA)

An OWA aggregator $M^{(m)}$ associated to the m non negative weights $(\omega_1^{(m)},\dots,\omega_m^{(m)})$ such that $\sum_{k=1}^m \omega_k^{(m)} = 1$ corresponds to

$$M^{(m)}(x_{(1)},\dots,x_{(m)}) = \sum_{i=1}^m \omega_i^{(m)} x_{(i)}, \quad x_{(1)} \le \cdots \le x_{(i)} \le \cdots \le x_{(m)}.$$

$\omega_1^{(m)}$ is linked to the lowest value $x_{(1)}$, …, $\omega_m^{(m)}$ is linked to the greatest value $x_{(m)}$.

This class of operators includes

- $\min(x_1,\dots,x_m)$ if $\omega_1^{(m)}=1$.
- $\max(x_1,\dots,x_m)$ if $\omega_m^{(m)}=1$.
- any order statistics $x_{(k)}$ if $\omega_k^{(m)}=1$, $k=1,\dots,m$.
- the arithmetic mean if $\omega_1^{(m)}=\cdots=\omega_m^{(m)}=\frac{1}{m}$.
- the median $(x_{(m/2)}+x_{(m/2)+1})/2$ if $\omega_{(m/2)}^{(m)}=\omega_{(m/2)+1}^{(m)}=\frac{1}{2}$ and m is even.
- the median $x_{(m+1)/2}$ if $\omega_{(m+1)/2}^{(m)}=1$ and m is odd.

- the arithmetic mean excluding the two extremes if $\omega_1^{(m)} = \omega_m^{(m)} = 0$ and $\omega_i^{(m)} = \frac{1}{m-2}$, $i \neq 1, m$.

Well known and easy to prove properties of the OWA aggregators are the following (see Yager [19], Cutello and Montero [5]) :

Any OWA aggregator is neutral, monotonic, idempotent and compensative ($\min \leq M^{(m)} \leq \max$) and presents stability for the same positive linear transformations.

However the OWA operator is generally not associative nor decomposable.

Proposition 1 *The OWA aggregator presents the ordered linkage property'*

Proof. Obviously

$$\begin{aligned} M^{(m)}(x_{(i)}, \ldots, x_{(m-1+i)}) &\leq M^{(m)}(x_{(i+1)}, \ldots, x_{(m+i)}), \quad i = 1, \ldots, m \\ M^{(m+1)}(x_{(j)}, \ldots, x_{(m+j)}) &\leq M^{(m+1)}(x_{(j+1)}, \ldots, x_{(m+j+1)}), \quad j = 1, \ldots, m-1 \end{aligned}$$

because $M^{(m)}$ and $M^{(m+1)}$ are OWA aggregators.

Let us first consider

$$\begin{aligned} & M^{(m+1)}(M^{(m)}(x_{(1)}, \ldots, x_{(m)}), \ldots, M^{(m)}(x_{(i)}, \ldots, x_{(m+i-1)}), \ldots, M^{(m)}(x_{(m+1)}, \ldots, x_{(2m)}) \\ &= \sum_{i=1}^{m+1} \omega_i^{(m+1)} M^{(m)}(x_{(i)}, \ldots, x_{(m+i-1)}) \\ &= \sum_{i=1}^{m+1} \omega_i^{(m+1)} \sum_{j=1}^{m} \omega_j^{(m)} x_{(i+j)} \qquad (1) \end{aligned}$$

Let us now consider

$$\begin{aligned} & M^{(m)}(M^{(m+1)}(x_{(1)}, \ldots, x_{(m+1)}), \ldots, M^{(m+1)}(x_{(j)}, \ldots, x_{(m+j-1)}), \ldots, M^{(m+1)}(x_{(m)}, \ldots, x_{(2m)})) \\ &= \sum_{j=1}^{m} \omega_j^{(m)} M^{(m+1)}(x_{(j)}, \ldots, x_{(m+j-1)}) \\ &= \sum_{j=1}^{m} \omega_j^{(m)} \sum_{i=1}^{m+1} \omega_i^{(m+1)} x_{(j+i)} \qquad (2) \end{aligned}$$

It is clear that equation (1) is identical to equation (2).

■

Proposition 2 *Any OWA aggregator $M^{(m)}$ can be expressed in an equivalent way as a Choquet integral'*

Proof. Let us consider an OWA aggregator $M^{(m)}(x_1, \ldots, x_m)$. We first introduce

$$e_i^{(m)} = M^{(m)}(\underbrace{0, \ldots, 0}_{(i)\text{ zeroes}}, \underbrace{1, \ldots, 1}_{(m-i)\text{ ones}}), \quad i = 0, \ldots, m \qquad (3)$$

We obviously have that ($M^{(m)}$ being monotonic and idempotent)

$$0 = e_m^{(m)} \leq e_i^{(m)} \leq e_0^{(m)} = 1 \quad , \quad i = 1, \ldots, m \tag{4}$$

$$e_{i-1}^{(m)} - e_i^{(m)} = \omega_i^{(m)} \quad , \quad i = 1, \ldots, m \tag{5}$$

Equation (5) indicates the equivalence between the knowledge of ($\omega_i^{(m)}$, $\sum \omega_i^{(m)} = 1$, $i = 1, \ldots, m$) and ($e_i^{(m)}$, $i = 1, \ldots, m-1$).

Finally,

$$\begin{aligned} M^{(m)}(x_1, \ldots, x_m) &= \sum_{i=1}^{m} \omega_i^{(m)} x_{(i)} = \sum_{i=1}^{m} (e_{i-1}^{(m)} - e_i^{(m)}) x_{(i)} && (6) \\ &= \sum_{i=1}^{m} e_{i-1}^{(m)} (x_{(i)} - x_{(i-1)}) && (7) \end{aligned}$$

if $x_{(0)} = 0$, by definition.

Let us now consider the subsets $A_j = \{x_{(j)}, \ldots, x_{(m)}\}$, $j = 1, \ldots, m$, $A_{m+1} = \phi$, and a fuzzy measure μ defined on these subsets :

$$\mu(A_j) = e_{j-1}^{(m)}.$$

We have

$$\mu(A_1) = \mu(x_{(1)}, \ldots, x_{(m)}) = e_0^{(m)} = 1$$

$$A_j \supset A_{j+1} \text{ and } \mu(A_j) \geq \mu(A_{j+1}) \ , \quad j = 1, \ldots, m$$

$$\mu(A_{m+1}) = \mu(\phi) = e_m^{(m)} = 0$$

and (6) can be rewritten as

$$M^{(m)}(x_1, \ldots, x_m) = \sum_{i=1}^{m} \mu(A_i)(x_{(i)} - x_{(i-1)})$$

which corresponds to the definition of a Choquet integral (see Choquet [4], Sugeno and Murofushi [15], Grabisch and Sugeno [8]).

4 Characterization of the OWA aggregator

A foundational paper of Aczél, Roberts and Rosenbaum [3] shows that the general solution of a functional equation related to the stability for the same positive linear transformations (case ♯5) :

$$M^{(m)}(rx_1 + t, \ldots, rx_m + t) = rM^{(m)}(x_1, \ldots, x_m) + t, \quad r > 0, \tag{8}$$

where M is a mapping from $R^m \to R$ given by

$$M^{(m)}(x_1, \ldots, x_m) = S(x) f\left(\frac{x_1 - A(x)}{S(x)}, \ldots, \frac{x_m - A(x)}{S(x)}\right) + aA(x) + b \text{ if } S(x) \neq 0$$

$$\text{or } ax+b \text{ if } S(x)=0 \; (\Leftrightarrow x_1 = x_2 = \cdots = x_m = x)$$

if $S^2(x) = \sum_i (x_i - A(x))^2$ and $A(x)$ represents the arithmetic mean; f is an arbitrary function from R^m to R.

It is also true that the weighted mean corresponds to monotonic and idempotent aggregators which satisfy the (SPLU)-property (see [3], case ♯ 9).

From results obtained by Marichal and Roubens [11], we know that neutral, continuous, stable for the same positive linear transformations and associative (resp. decomposable) operators are characterized by the min or max operators (resp. min or max or $A(x)$).

Weaker property than associativity or decomposability is needed to be able to characterize the OWA operators which include min, max and the arithmetic means. This intermediate property is related to the ordered linkage property.

Proposition 3 *Associativity with neutrality or decomposability with neutrality and idempotency imply the linkage property' The reciprocal is not true'*

Proof. The fact that associativity and neutrality imply linkage is trivial.

Let us now consider decomposability. We know that (see Kolmogorov [9], Fodor and Roubens [6])

$$M^{(m)}(x_1,\ldots,x_m) = M^{(m.n)}(\underbrace{x_1,\ldots,x_1}_{n \text{ times}},\ldots,\underbrace{x_m,\ldots,x_m}_{n \text{ times}}) = M^{(m.n)}(n.x_1,\ldots,n.x_m)$$

when $M^{(m)}$ is decomposable, neutral and idempotent. Let us now prove that the linkage property is satisfied.

$$\begin{aligned}
&M^{(m+1)}(M^{(m)}(x_1,\ldots,x_m), M^{(m)}(x_2,\ldots,x_{m+1}),\ldots,M^{(m)}(x_{m+1},\ldots,x_{2m})) \\
&= M^{(m(m+1))}(m.M^{(m)}(x_1,\ldots,x_m),\ldots,m.M^{(m)}(x_{m+1},\ldots,x_{2m})), \quad \text{(decomposability)} \\
&= M^{(m(m+1))}(x_1, 2.x_2,\ldots,m.x_m,m.x_{m+1},\ldots,2.x_{2m-1},x_{2m}), \quad \text{(decomposability and neutrality)} \\
&= M^{(m(m+1))}((m+1).M^{(m+1)}(x_1,\ldots,x_{m+1}),\ldots,(m+1).M^{(m+1)}(x_m,\ldots,x_{2m})), \quad \text{(decomposability)} \\
&= M^{(m)}(M^{(m+1)}(x_1,\ldots,x_{m+1}),\ldots,M^{(m+1)}(x_m,\ldots,x_{2m})), \quad \text{(decomposability).}
\end{aligned}$$

The fact that the converse is not true in general can be observed when considering the arithmetic mean (which presents linkage but is not associative) or the product (which presents linkage but is not decomposable).

■

We will now characterize the OWA operators.

Proposition 4 *The class of the ordered weighted averaging aggregators corresponds to the operators which satisfy the properties of neutrality' monotonicity' stability for the same positive linear transformations and ordered linkage'*

Proof. We already know that OWA aggregators satisfy the properties mentioned in the Proposition.

We will now prove that under the assumptions of monotonicity, neutrality, stability for the same positive linear transformations (SPL) and ordered linkage, an aggregator $M^{(m)}(x_1, \ldots, x_m)$ can be written under the form given in Equation (6).

(SPL) implies obviously idempotency.

Consider first $m = 2$.

$$\begin{aligned} M^{(2)}(x_1, x_2) &= M^{(2)}(x_{(1)}, x_{(2)}) = x_{(1)} + (x_{(2)} - x_{(1)})M^{(2)}(0, 1) \quad \text{(neutrality and SPL)} \\ &= x_{(1)}(e_0^{(2)} - e_1^{(2)}) + x_{(2)}(e_1^{(2)} - e_2^{(2)}) \qquad \text{(see (3) and idempotency).} \end{aligned}$$

Due to monotonicity $(e_0^{(2)} - e_1^{(2)}) \geq 0$ and $(e_1^{(2)} - e_2^{(2)}) \geq 0$.

Consider $m = 3$.

$$\begin{aligned} M^{(3)}(x_1, x_2, x_3) &= M^{(3)}(x_{(1)}, x_{(2)}, x_{(3)}) && \text{(neutrality)} \\ &= x_{(1)} + (x_{(3)} - x_{(1)})M^{(3)}\left(0, \frac{x_{(2)} - x_{(1)}}{x_{(3)} - x_{(1)}}, 1\right) && \text{(SPL)} \end{aligned}$$

We pose

$$M^{(2)}(0, 1) = e_1^{(2)} = \frac{x_{(2)} - x_{(1)}}{x_{(3)} - x_{(1)}} \;, \quad (M(0, 0) \leq M(0, 1) \leq M(1, 1))$$

and we apply ordered linkage property to the following boolean case :

$$M^{(3)}(e_2^{(2)}, e_1^{(2)}, e_0^{(2)}) = M^{(2)}(e_2^{(3)}, e_1^{(3)})$$

$$\begin{aligned} & M^{(3)}(x_1, x_2, x_3) \\ &= x_{(1)} + (x_{(3)} - x_{(1)})M^{(3)}(M^{(2)}(0, 0), M^{(2)}(0, 1), M^{(2)}(1, 1)) \\ &= x_{(1)} + (x_{(3)} - x_{(1)})M^{(2)}(e_2^{(3)}, e_1^{(3)}) \\ &= x_{(1)} + (x_{(3)} - x_{(1)}) \left[e_2^{(3)}(e_0^{(2)} - e_1^{(2)}) + e_1^{(3)}(e_1^{(2)} - e_2^{(2)})\right] \text{ (from case } m = 2) \\ &= x_{(1)} + (x_{(3)} - x_{(1)}) \left[e_2^{(3)}\left(1 - \frac{x_{(2)} - x_{(1)}}{x_{(3)} - x_{(1)}}\right) + e_1^{(3)}\left(\frac{x_{(2)} - x_{(1)}}{x_{(3)} - x_{(1)}}\right)\right] \\ &= x_{(1)}(e_0^{(3)} - e_1^{(3)}) + x_{(2)}(e_1^{(3)} - e_2^{(3)}) + x_{(3)}(e_2^{(3)} - e_3^{(3)}) \end{aligned}$$

In an iterative way, let us consider the case $m > 3$.

$$\begin{aligned} M^{(m)}(x_1, \ldots, x_m) &= M^{(m)}(x_{(1)}, \ldots, x_{(m)}) && \text{(neutrality)} \\ &= x_{(1)} + (x_{(m)} - x_{(1)})M^{(m)}\left(0, \ldots, \frac{x_{(i)} - x_{(1)}}{x_{(m)} - x_{(1)}}, \ldots, 1\right) && \text{(SPL).} \end{aligned}$$

We consider

$$e_{m-i}^{(m-1)} = \frac{x_{(i)} - x_{(1)}}{x_{(m)} - x_{(1)}} \left(e_{m-i}^{(m-1)} \leq e_{m-i-1}^{(m-1)}\right)$$

$$M^{(m)}(x_1,\ldots,x_m) = x_{(1)} + (x_{(m)} - x_{(1)})M^{(m)}\left(e_{m-1}^{(m-1)},\ldots,e_{m-i}^{(m-1)},\ldots,e_0^{(m-1)}\right).$$

Due to the ordered linkage property, applied to the boolean case

$$\begin{aligned}
M^{(m)}(x_1,\ldots,x_m) &= x_{(1)} + (x_{(m)} - x_{(1)})M^{(m-1)}\left(e_{m-1}^{(m)},\ldots,e_1^{(m)}\right) \\
&= x_{(1)} + (x_{(m)} - x_{(1)})\sum_{i=1}^{m-1}\left(e_{i-1}^{(m-1)} - e_i^{(m-1)}\right)e_{m-i}^{(m)} \quad \text{(by iteration)} \\
&= x_{(1)} + \sum_{i=1}^{m-1}(x_{(m-i+1)} - x_{(m-i)})e_{m-i}^{(m)} \\
&= \sum_{i=1}^{m} x_{(i)}\left(e_{i-1}^{(m)} - e_i^{(m)}\right).
\end{aligned}$$

■

Another characterization of OWA operators corresponds to the following proposition.

Proposition 5 *The class of ordered weighted averaging operators corresponds to the aggregators which satisfy the properties of neutrality' monotonicity' idempotency and stability for positive linear transformations with the same unit' independent zeroes and ordered values'*

Proof. The conditions introduced in the proposition are trivially fulfilled by an OWA operator. To prove the converse, we start from the functional equation :

$$M(rx_{(1)} + t_{(1)},\ldots,rx_{(m)} + t_{(m)}) = rM(x_{(1)},\ldots,x_{(m)}) + T(t_{(1)},\ldots,t_{(m)}) \tag{9}$$

where $x_{(1)} \leq \cdots \leq x_{(m)}$ and $t_{(1)} \leq \cdots \leq t_{(m)}$.

Due to idempotency, it is obvious that :

$$M(0,\ldots,0) = 0$$

and

$$M(t_{(1)},\ldots,t_{(m)}) = T(t_{(1)},\ldots,t_{(m)}).$$

For solving Equation (9) with $M = T$, consider the case $r = 1$:

$$M(x_{(1)} + t_{(1)},\ldots,x_{(m)} + t_{(m)}) = M(x_{(1)},\ldots,x_{(m)}) + M(t_{(1)},\ldots,t_{(m)}) \tag{10}$$

which expresses the ordered additivity of M.

Choosing particular values $x_{(1)} = \cdots = x_{(m-1)} = t_{(1)} = \cdots = t_{(m-1)} = 0$, $x = x_{(m)} \geq 0$, $t = t_{(m)} \geq 0$, Equation (10) implies

$$M(0,\ldots,0,x+t) = M(0,\ldots,0,x) + M(0,\ldots,0,t)\ , \quad x \geq 0,\ t \geq 0.$$

By introducing the function

$$f_1(x) = M(0,\ldots,0,x)\ \ , \quad x \geq 0$$

one of the four basic Cauchy equations follows from f_1 :

$$f_1(x+t) = f_1(x) + f_1(t) \quad , \quad x \geq 0,\ t \geq 0.$$

Due to monotonicity, the only solution is given by

$$f_1(x) = c_1 x \quad , \quad c_1 \geq 0$$

(see Aczél [1], Theorem 1, p. 34).

In a similar way, we can introduce $f_2, \ldots, f_m$ such that :

$$f_i(x) = M(\underbrace{0, \ldots, 0}_{(m-i)\ times}, \underbrace{x, \ldots, x}_{i\ times}) \ , \quad i = 2, \ldots, m.$$

We obtain the same Cauchy equation for each f_i, $i = 2, \ldots, m$ and

$$f_i(x) = c_i x \quad , \quad c_i \geq 0, \ \ i = 1, \ldots, m.$$

Idempotency implies $c_m = 1$ and monotonicity induces

$$c_1 \leq c_2 \leq \cdots \leq c_m = 1.$$

The value of $M(x_{(1)}, \ldots, x_{(m)}) = M(x_1, \ldots, x_m)$ (neutrality) is obtained now in a recursive way using Equation (10).

$$\begin{aligned} M(0, \ldots, 0, x_{(m-1)}, x_{(m)}) &= M(0, \ldots, 0, x_{(m-1)}, x_{(m-1)}) + M(0, \ldots, 0, x_{(m)} - x_{(m-1)}) \\ &= c_2 x_{(m-1)} + c_1(x_{(m)} - x_{(m-1)}) \\ &= (c_2 - c_1) x_{(m-1)} + c_1 x_{(m)} \end{aligned}$$

$$M(x_{(1)}, \ldots, x_{(m)}) = (c_m - c_{m-1}) x_{(1)} + (c_{m-1} - c_{m-2}) x_{(2)} + \cdots + c_1 x_{(m)}.$$

Denoting $\omega_i^{(m)} = c_{m-i+1} - c_{m-i}$, $i = 1, \ldots, m-1$

$$\omega_m^{(m)} = c_1,$$

we obtain that $M(x_{(1)}, \ldots, x_{(m)}) = \sum_i \omega_i^{(m)} x_{(i)}$, $\omega_i^{(m)} \geq 0$, $\sum_i \omega_i^{(m)} = 1$, for $0 \leq x_{(1)} \leq \cdots \leq x_{(m)}$.

We now consider any m-uple $x_{(1)} \leq \cdots \leq x_{(m)}$. Suppose that the first k values are negative, the others being non negative. In that case, it is always possible to consider :

$$(y_{(1)} = -x_1) \leq (y_{(2)} = -x_2) \leq \cdots \leq (y_{(k)} = -x_k) \leq (y_{(k+1)} = x_{k+1}) \leq \cdots \leq (y_{(m)} = x_m).$$

From Equation (9), it comes that :

$$M(y_{(1)}, \ldots, y_{(k)}, y_{(k+1)}, \ldots, y_{(m)}) = M(0, x_1 - x_2, \ldots, x_1 - x_k, x_1 + x_{k+1}, \ldots, x_1 + x_m) - x_1.$$

Using the preceding result for non negative reals, we obtain that

$$M(y_{(1)}, \ldots, y_{(m)}) = \sum_{u=1}^{m} \omega_i^{(m)} y_{(i)}.$$

■

5 Decomposable quasi-OWA aggregators

The quasi-arithmetic mean was first considered and characterized by Kolmogorov [9] and Nagumo [12]. It corresponds to the aggregator

$$M^{(m)}(x_1, \ldots, x_m) = f^{-1}\left[\frac{1}{m}\sum_i f(x_i)\right]$$

where f is a continuous strictly monotonic function.

It is natural to consider the quasi-OWA operators

$$M^{(m)}(x_1, \ldots, x_m) = f^{-1}\left[\sum_i \omega_i^{(m)} f(x_{(i)})\right]. \tag{11}$$

These aggregators have still to be characterized but we can prove the following Proposition.

Proposition 6 *Any decomposable quasi'OWA operator corresponds to the* min *or* max *or quasi'arithmetic mean'*

Proof. Min, max and quasi-arithmetic mean operators are obviously decomposable quasi-OWA operators. Let us now consider Relation (11) where M is supposed to be decomposable; we will prove that

$$F(x_1, \ldots, x_m) = f\left[M(\{f^{-1}(x_1), \ldots, f^{-1}(x_m)\}\right] = \sum_i \omega_i^{(m)} x_{(i)}$$

is also decomposable.

Indeed,

$$\begin{aligned}
&F(\underbrace{F(x_1, \ldots, x_k), \ldots F(x_1, \ldots, x_k)}_{k\ times}, x_{k+1}, \ldots, x_m) \\
&= f\Big[M\{M(f^{-1}(x_1), \ldots, f^{-1}(x_k)), \ldots, M(f^{-1}(x_1), \ldots, \\
&\qquad\qquad f^{-1}(x_k)), f^{-1}(x_{k+1}), \ldots, f^{-1}(x_m)\}\Big] \\
&= f\left[M(f^{-1}(x_1), \ldots, f^{-1}(x_m))\right] \quad \text{by decomposability of } M) \\
&= F(x_1, \ldots, x_m) \quad \text{(by definition).}
\end{aligned}$$

F is decomposable but also continuous, neutral, monotonic and presents stability for the same positive linear transformations (obvious properties of OWA). From a result obtained by Marichal and Roubens [11], we know that an operator having such properties corresponds to min, max or $A(x)$ and

$$(\omega_1^{(m)}, \ldots, \omega_m^{(m)}) = (1, 0, \ldots, 0), (0, \ldots, 0, 1) \text{ or } \left(\frac{1}{m}, \ldots, \frac{1}{m}\right).$$

Finally, decomposable M defined by (11) corresponds to min, max and the quasi-arithmetic mean.

■

References

[1] J. Aczél, *Lectures on Functional Equations and Their Applications*, (Academic Press, New York) (1966).

[2] J. Aczél and F.S. Roberts, On the possible merging functions, *Math' Social Sciences*, **17** (1989), 205–243.

[3] J. Aczél, F.S. Roberts and Z. Rosenbaum, On the scientific laws without dimensional constants, *Journal of Math' Analysis and Appl'*, **119** (1986), 389–416.

[4] G. Choquet, Theory of capacities, *Annales de l'Institut Fourier*, **5**, (1953-1954) 131-295.

[5] V. Cutello and J. Montero, Hierarchies of aggregation operators, Working paper (1993), unpublished.

[6] J. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Aid*, to be published by Kluwer.

[7] M. Grabisch, On the use of fuzzy integral as a fuzzy connective, Proceedings of the Second IEEE International Conference on Fuzzy Systems, San Francisco (1993) 213-218.

[8] M. Grabisch and M. Sugeno, Multi-attribute classification using fuzzy integral, Proceedings of the IEEE International Conference on Fuzzy Systems, San Diego (1992) 47-54.

[9] A.N. Kolmogoroff, Sur la notion de la moyenne, *Accad' Naz' Lincei Mem' Cl' Sci' Fis' Mat' Natur' Sez''* **12** (1930) 388-391.

[10] J. Quiggin, A theory of anticipated utility, *Journal of Economic Behavior and Organization*, **3** (1982) 323–343.

[11] J.L. Marichal and M. Roubens, Characterization of some stable aggregation functions, in : Proc. Intern. Conf. on Industrial Engineering and Production Management, Mons, Belgium (1993) 187–196.

[12] M. Nagumo, Über eine Klasse der Mittelwerte, *Japanese Journal of Mathematics*, **6** (1930) 71-79.

[13] F.S. Roberts, *Measurement Theory with Applications to Decision'making' Utility and the Social Sciences'* (Addison-Wesley Pub., Reading) (1979).

[14] U. Segal, Order indifference and rank-dependent probabilities, *Journal of Math' Economics*, 22 (1993) 373–397.

[15] H.J. Skala, Concerning ordered weighted averaging aggregation operators, Statistical Papers **32** (1991), 35–44.

[16] M. Sugeno and T. Murofushi, Choquet integral as an integral form for a general class of fuzzy measures, *Proceedings of the 'nd IFSA Congress*, Tokyo (1987) 408-411.

[17] R.R. Yager, On the aggregation of processing units in neural networks, Proc. 1st IEEE Int. Conference on Neural Networks, San Diego, Vol. II (1987) 927–933.

[18] R.R. Yager, A note on weighted queries in information retrieval systems, *J' Amer' Soc' Information Sciences*, **28** (1987) 23–24.

[19] R.R. Yager, On Ordered Weighted Averaging Aggregation Operators in Multicriteria Decision making, *IEEE Trans' on Systems' Man and Cybernetics*, **18** (1988), 183–190.

[20] R. R. Yager, Connectives and quantifiers in fuzzy sets, *Fuzzy Sets and Systems* **40** (1991) 39–75.

[21] R.R. Yager, OWA Neurons : A new class of fuzzy neurons, *Proc' Int' Joint on Neural Networks*, vol. 1, Baltimore (1992) 226–231.

[22] R.R. Yager, Fuzzy Quotient Operator, *Proc' Fourth Int' Conf' on Information Pro' cessing and Management of Uncertainty*, Palma de Majorca (1992) 317–322.

[23] R.R. Yager and D.P. Filev, Fuzzy logic controllers with flexible structures. *Proc' Second Int' Conf' on Fuzzy Sets and Neural Networks*, Tizuka (1992) 317–320.