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Characterization of the Unit Tangent Sphere Bundle with g-Natural Metric and Almost Contact B-metric Structure

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Abstract. We consider unit tangent sphere bundle of a Riemannian manifold (M, g) as a $(2n + 1)$-dimensional manifold and we equip it with pseudo-Riemannian g-natural almost contact B-metric structure. Then, by computing coefficients of the structure tensor F, we completely characterize the unit tangent sphere bundle equipped to this structure, with respect to the relevant classification of almost contact B-metric structures, and determine a class such that the unit tangent sphere bundle with mentioned structure belongs to it. Also, we find some curvature conditions such that the mentioned structure satisfies each of eleven basic classes.

Key Words: Almost contact structure, B-metrics, g-natural metric, Sphere bundle, Structure tensor
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Introduction

As a classical research field in Riemannian geometry, the motif of *lifted metrics* on tangent and unit tangent sphere bundle of a Riemannian manifold is widely considered by many geometrists till now. The brilliant mathematician, Sasaki, made significant contributions to this field and since then, his works have been a great inspiration for geometrists to introduce various types of lifted metrics on tangent and unit tangent sphere bundle of Riemannian manifolds such as g-natural metrics. Abbassi et al., have studied some properties of g-natural metrics on the unit tangent sphere bundle induced from g-natural metrics on the tangent bundle ([1, 2, 3, 4, 5]). Furthermore, in this context, Kowalski ([16, 17, 18, 19, 20]), Boeckx ([7, 8, 9, 10, 11, 12]), and Calvaruso ([14]) have published some worthy papers.

The notion of the almost contact structure is another concept of classical research field in differential geometry of manifolds, first given by Sasaki in [27]. In [15], authors equipped an almost contact manifold to B-metric, as a natural extension of an almost complex manifold to the odd dimensional case, and they introduced a classification of almost contact B-metric manifold with respect to the covariant derivative of the fundamental tensor of type $(1,1)$. This classification named the relevant classification, includes eleven basic classes. By taking seriously the idea of classification with respect to structure tensors, Manev characterized a wide scope of Riemannian manifolds and obtained valuable results (see [21, 25, 22, 23, 24, 26]).

In this paper, we consider pseudo-Riemannian g-natural almost contact structure on the unit tangent sphere bundle T_1M with B-metric, and we characterize this structure with respect to the relevant classification of almost contact manifold with B-metric introduced in [15]. The work is organized in the following way. We begin in the section 1 from the study on the concept of g-natural metrics on the tangent and unit tangent sphere bundle of a Riemannian manifold (M,g). We proceed to the section 2 to describe and study the behavior of the structure tensor F on the unit tangent sphere bundle. The first two sections have been organized to contain all the necessary geometric machinery for the rest of the paper. The last section contains our main results: the characterization of the unit tangent sphere bundle with respect to the relevant classification of almost contact B-metric manifolds.

1 g-natural metric on sphere bundle

This section contains some necessary information on g-natural metrics on the tangent and unit tangent sphere bundle.

1.1 g-natural metrics on the tangent bundle

Let (M,g) be an $(n+1)$-dimensional Riemannian manifold, and we denote by ∇ its Levi-Civita connection. The tangent space $TM_{(x,u)}$ of the tangent bundle TM at a point (x,u) splits as

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)},$$

where $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical spaces with respect to ∇. Actually, for a vector $X \in M_x$, there exists a unique vector $X^h \in \mathcal{H}_{(x,u)}$ (called the *horizontal lift* of X to $(x,u) \in TM$), such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. Also, the *vertical lift* of a vector $X \in M_x$ is a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(\mathrm{d}f) = Xf$, for all functions f on M. It should be noted that we consider 1-forms $\mathrm{d}f$ on

M as functions on TM (i.e., $(\mathrm{d}f)(x,u)=uf$). The map $X\to X^h$ is an isomorphism between the vector spaces M_x and $\mathcal{H}_{(x,u)}$. Similarly, the map $X\to X^v$ is an isomorphism between M_x and $\mathcal{V}_{(x,u)}$. Each tangent vector $Z\in(TM)_{(x,u)}$ can be written in the form $Z=X^h+Y^v$, where $X,Y\in M_x$, are uniquely determined vectors. The geodesic flow vector field on TM is uniquely determined by $u^h_{(x,u)}=u^i(\frac{\partial}{\partial x^i})^h_{(x,u)}$, for any point $x\in M$ and $u\in TM_x$, with respect to the local coordinates $\{\frac{\partial}{\partial x^i}\}$ on M.

One can find a comprehensive description of the class g-natural metrics on the tangent bundle of a Riemannian manifold (M,g) in [2]. Especially, we express the following characterization.

Proposition 1 ([2]) *Let (M,g) be a Riemannian manifold and G be the g-natural metric on TM. Then there are six smooth functions $\alpha_i,\beta_i:\mathbb{R}^+\to\mathbb{R}$, $i=1,2,3$, such that for every $u,X,Y\in M_x$, we have*

$$\begin{cases} G_{(x,u)}(X^h,Y^h)=(\alpha_1+\alpha_3)(r^2)g(X,Y)+(\beta_1+\beta_3)(r^2)g(X,u)g(Y,u),\\ G_{(x,u)}(X^h,Y^v)=G_{(x,u)}(X^v,Y^h)=\alpha_2(r^2)g(X,Y)+\beta_2(r^2)g(X,u)g(Y,u),\\ G_{(x,u)}(X^v,Y^v)=\alpha_1(r^2)g(X,Y)+\beta_1(r^2)g(X,u)g(Y,u),\end{cases}$$

where $r^2=g(u,u)$.

There are some well-known examples of Riemannian metrics on the tangent bundle obtained from Riemannian g-natural metrics. In particular, the Sasaki metric is a special case of Riemannian g-natural metrics with

$$\alpha_1(t)=1,\qquad \alpha_2(t)=\alpha_3(t)=\beta_1(t)=\beta_2(t)=\beta_3(t)=0.$$

1.2 g-natural metric on the unit sphere bundle

The unit tangent sphere bundle over a Riemannian manifold (M,g), is the hyperspace

$$T_1M=\{(x,u)\in TM\mid g_x(u,u)=1\},$$

in TM. The tangent space of T_1M, at a point $(x,u)\in T_1M$, is given by

$$(T_1M)_{(x,u)}=\{X^h+Y^v|X\in M_x,Y\in\{u\}^\perp\subset M_x\}.$$

A (pseudo)-Riemannian g-natural metric on T_1M, is any metric $\tilde{G}$, induced on T_1M by a g-natural metric G on TM. Using [13], we know that $\tilde{G}$ is completely determined by the values of four real constants, namely

$$a=\alpha_1(1),\qquad b=\alpha_2(1),\qquad c=\alpha_3(1),\qquad d=(\beta_1+\beta_3)(1).$$

Let (M,g) be an $(n+1)$-dimensional Riemannian manifold. We start from an orthogonal basis $\{e_0=u,e_1,\dots,e_n\}$ on $x\in M$. We define $\tilde{\delta}_0=e^h_0=u^h$, and $\tilde{\delta}_i = e_i^h$, and $\tilde{\partial}_i^T = e_i^v$, for $i = 1, \dots, n$. The metric $\tilde{G}$ on T_1M is completely determined by

$$\begin{cases} \tilde{G}_{(x,u)}(\tilde{\delta}_i, \tilde{\delta}_j) = (a+c)g_x(\partial_i, \partial_j) + dg_x(\partial_i, u)g_x(\partial_j, u), \\ \tilde{G}_{(x,u)}(\tilde{\delta}_i, \tilde{\partial}_j^T) = bg_x(\partial_i, \partial_j), \\ \tilde{G}_{(x,u)}(\tilde{\partial}_i^T, \tilde{\partial}_j^T) = ag_x(\partial_i, \partial_j), \end{cases}$$

at any point $(x, u) \in T_1M$, for all $\partial_i, \partial_j \in M_x$, with ∂_j orthogonal to u (see [13]). Obviously, we have $\tilde{G}(\tilde{\delta}_i, \tilde{\delta}_j) = \tilde{G}(\tilde{\delta}_i, \tilde{\partial}_j^T) = \tilde{G}(\tilde{\partial}_i^T, \tilde{\partial}_j^T) = 0$, when $i \neq j$. Therefore, the matrix of $\tilde{G}$ with respect to the basis $\{\tilde{\delta}_0, \tilde{\delta}_1, \tilde{\partial}_1^T, \dots, \tilde{\delta}_n, \tilde{\partial}_n^T\}$ at a point (x, u) is block diagonal:

$$\begin{pmatrix} a+c+dr^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & a+c & b & \dots & 0 & 0 \\ 0 & b & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & a+c & b \\ 0 & \dots & \dots & \dots & b & a \end{pmatrix},$$

the determinant of $\tilde{G}$ is $(a+c+dr^2)\alpha^n$, and $\tilde{G}$ has eigenvalues $a+c+dr^2$ for once and $2a+c\pm\sqrt{c^2+4b^2}$ (n-times for each of them), where $\alpha = a(a+c)-b^2$.

Further details, including a comprehensive discussion on the signature of these metrics and those conditions leading these metrics to Riemannian or pseudo-Riemannian metrics can be found in [13]. With the purpose of constructing a B-metric structure with an associated g-natural metric on the unit tangent sphere bundle T_1M, it requires to

$$a + c + d > 0, \qquad \alpha = a(a + c) - b^2 < 0.$$

By a simple calculation and using the Schmidt's orthogonalization process, it is easy to check that whenever $\phi \neq 0$, the vector field on TM defined by

$$N^G_{(x,u)} = \frac{1}{\sqrt{|(a+c+d)\phi|}}[-bu^h + (a+c+d)u^v],$$

for all $(x, u) \in TM$, is normal to T_1M and is unitary at any point of T_1M, where $\phi = a(a + c + d) - b^2$. Now, we define the *tangential lift* X^{t_G} with respect to G of a vector $X \in M_x$ to $(x, u) \in T_1M$ as the tangential projection of the vertical lift of X to (x, u) with respect to N^G that is,

$$X^{t_G} = X^v - G_{(x,u)}(X^v, N^G_{(x,u)})N^G_{(x,u)} = X^v - \sqrt{\frac{|\phi|}{|a+c+d|}}g_x(X, u)N^G_{(x,u)}.$$

If $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$. Note that if $b = 0$, then X^{t_G} coincides with the classical tangential lift X^t defined for the case of the

Sasaki metric. In the general case,

$$X^{t_G} = X^t + \frac{b}{a+c+d} g(X,u)u^h.$$

The tangent space $(T_1M)_{(x,u)}$ of T_1M at (x,u) is spanned by vectors of the form X^h and Y^{t_G} as follows,

$$(T_1M)_{(x,u)} = \{X^h + Y^{t_G} | X \in M_x, Y \in \{u\}^\perp \subset M_x\}, \tag{1}$$

where $X, Y \in M_x$. Using this fact, the pseudo-Riemannian metric $\tilde{G}$ on T_1M, induced from G, is completely determined by the following identities.

$$\begin{cases} \tilde{G}(X^h, Y^h) = (a+c)g_x(X,Y) + dg_x(X,u)g_x(Y,u), \\ \tilde{G}(X^h, Y^{t_G}) = bg_x(X,Y), \\ \tilde{G}(X^{t_G}, Y^{t_G}) = ag_x(X,Y), \end{cases}$$

where $X, Y \in M_x$. It should be noted that by the above equations, horizontal and vertical lifts are orthogonal with respect to $\tilde{G}$ if and only if $b = 0$.

1.3 Almost contact pseudo-Riemannian g-natural metric structure on sphere bundle

In this part, we consider unit tangent sphere bundle of a Riemannian manifold as an odd dimensional manifold and equip it with an almost contact pseudo-Riemannian g-natural metric structure with B-metric.

Definition 1 ([6]) *A $(2n+1)$-dimensional manifold M has an almost contact structure if it admits a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying*

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi\xi = 0, \qquad \eta \circ \varphi = 0.$$

Then, (1) yields that the tangent space of T_1M at (x,u) can be written as

$$(T_1M)_{(x,u)} = span(\xi) \oplus \{X^h | X \perp u\} \oplus \{Y^{t_G} | Y \perp u\}.$$

Notice that a Riemannian manifold (M,g) with an almost contact structure (φ, ξ, η), satisfies B-metric condition, whenever

$$g(\varphi x, \varphi y) = -g(x,y) + \eta(x)\eta(y).$$

At first, we equip the unit tangent sphere bundle with an almost contact structure with B-metric $(\varphi, \xi, \eta, \tilde{G})$ and a basis $\{\tilde{\delta}_i, \tilde{\partial}_i^T, \xi\}$, such that $\tilde{\delta}_i, \tilde{\partial}_i^T \perp \xi$ with respect to $\tilde{G}$, where $\xi = \rho u^h$. The following equations define an almost contact structure on the unit tangent sphere bundle T_1M,

$$\eta(\tilde{\delta}_i) = \eta(\tilde{\partial}_i^T) = 0, \quad \eta(\xi) = 1, \tag{2}$$

$$\varphi(\tilde{\delta}_i) = \tilde{\partial}_i^T, \quad \varphi(\tilde{\partial}_i^T) = -\tilde{\delta}_i, \quad \varphi(\xi) = 0. \tag{3}$$

The adapted pseudo-Riemannian g-natural metric on the unit tangent sphere bundle T_1M with almost contact B-metric structure is of the following form

$$\begin{cases} \tilde{G}(\tilde{\delta}_i, \tilde{\delta}_j) = (a+c)g(\partial_i, \partial_j) + dg(\partial_i, u)g(\partial_j, u), \\ \tilde{G}(\tilde{\delta}_i, \tilde{\partial}_j^T) = 0, \\ \tilde{G}(\tilde{\partial}_i^T, \tilde{\partial}_j^T) = ag(\partial_i, \partial_j). \end{cases}$$

Also, pseudo-Riemannian metric $\tilde{G}$ is a B-metric because we have

$$\tilde{G}(\varphi\tilde{\delta}_i, \varphi\tilde{\delta}_j) = -\tilde{G}(\tilde{\delta}_i, \tilde{\delta}_j), \qquad \tilde{G}(\varphi\tilde{\partial}_i^T, \varphi\tilde{\partial}_j^T) = -\tilde{G}(\tilde{\partial}_i^T, \tilde{\partial}_j^T).$$

These equations give us $a + c = -a$. Moreover, we have the following equations

$$\begin{aligned} &a_i^s(y,v)u^j g_{sj}(y) = 0, && a_i^j(y,v)a_t^s(y,v)g_{js}(y,v) = \delta_{it}, \\ &\tilde{\delta}_i = a_i^s\delta_s, && \tilde{\partial}_i^T = a_i^s\partial_s^T, \\ &\tilde{G}(\tilde{\delta}_i, \tilde{\delta}_j) = -a\delta_{ij}, && \tilde{G}(\tilde{\partial}_i^T, \tilde{\partial}_j^T) = a\delta_{ij}, \end{aligned}$$

where a_i^t are functions on M and δ_{ij} is the Kronecker symbol.

Lemma 1 ([6]) *The Lie brackets of frame $\{\tilde{\delta}_i, \tilde{\partial}_i^T, \xi\}$ of T_1M satisfy the following relations,*

$$\begin{aligned} &[\tilde{\delta}_i, \tilde{\delta}_j] = [a_i^r\delta_r, a_j^s\delta_s] = a_i^r\delta_r(a_j^s)\delta_s - a_j^s\delta_s(a_i^r)\delta_r + a_i^r a_j^s u^l \mathrm{R}_{srl}{}^k\partial_k^T, \\ &[\tilde{\delta}_i, \tilde{\partial}_j^T] = [a_i^r\delta_r, a_j^s\partial_s^T] = a_i^r\delta_r(a_j^s)\partial_s^T - a_j^s\partial_s^T(a_i^r)\delta_r + a_i^r a_j^s\Gamma_{rs}^p\partial_p^T, \\ &[\tilde{\partial}_i^T, \tilde{\partial}_j^T] = [a_i^r\partial_r^T, a_j^s\partial_s^T] = a_i^r\partial_r^T(a_j^s)\partial_s^T - a_j^s\partial_s^T(a_i^r)\partial_r^T, \\ &[\tilde{\delta}_i, \xi] = [a_i^r\delta_r, \rho u^o\delta_o] = a_i^r\delta_r(\rho u^o)\delta_o - \rho u^o\delta_o(a_i^r)\delta_r + a_i^r\rho u^o u^l \mathrm{R}_{orl}{}^k\partial_k^T, \\ &[\tilde{\partial}_i^T, \xi] = [a_i^r\partial_r^T, \rho u^o\delta_o] = a_i^r\partial_r^T(\rho u^o)\delta_o - \rho u^o\delta_o(a_i^r)\partial_r^T - a_i^r\rho u^o\Gamma_{or}^p\partial_p^T, \end{aligned}$$

where R *denotes the Riemannian curvature tensor on M with $g_{pk}\mathrm{R}_{jil}{}^k = \mathrm{R}_{jilp}$ and Γ_{ji}^k denote the Christoffel symbols on M defined by $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k$, and $\nabla_u\partial_i = \Gamma_{oi}^k\partial_k$, and $\xi = \rho u^h = \tilde{\delta}_0 = \rho u^o(\partial_o)^h$.*

2 Structure tensor F

The covariant derivatives of (φ, ξ, η) with respect to the Levi-Civita connection ∇ play a fundamental role in the differential geometry on the almost contact manifolds. The structural tensor F of the type $(0,3)$ on $(T_1M, \varphi, \xi, \eta, \tilde{G})$ is defined by the following way.

Definition 2 *The structure tensor F is defined by*

$$F(x,y,z) = \tilde{G}((\tilde{\nabla}_x\varphi)y, z), \tag{4}$$

and has the following properties:

$$F(x,y,z) = F(x,z,y) = F(x,\varphi y,\varphi z) + \eta(y)F(x,\xi,z) + \eta(z)F(x,y,\xi), \tag{5}$$

$$\tilde{G}(\tilde{\nabla}_x\xi, y) = F(x,\varphi y,\xi). \tag{6}$$

The following 1*-forms are associated with F,*

$$\theta(z) = \tilde{G}^{ij}F(\tilde{e}_i,\tilde{e}_j,z), \qquad \theta^*(z) = \tilde{G}^{ij}F(\tilde{e}_i,\varphi\tilde{e}_j,z), \qquad \omega(z) = F(\xi,\xi,z), \tag{7}$$

where $\tilde{G}^{ij}$ are the coefficients of the inverse matrix of pseudo-Riemannian metric $\tilde{G}$ with respect to the basis $\{\tilde{e}_i,\xi\}$, $(i = 1,2,\ldots,2n)$ at any point $p \in M$ and $\tilde{\nabla}$ denotes the Levi-Civita connection on T_1M.

Proposition 2 *The* ***essential*** *coefficients (which may not be zero) of structure tensors F are of following forms*

$$\begin{aligned}
F(\tilde{\delta}_i,\tilde{\delta}_j,\tilde{\delta}_t) &= \frac{a}{2}\{\mathrm{R}(\partial_i,\partial_t,u,\partial_j) - \mathrm{R}(\partial_j,\partial_i,u,\partial_t)\},\\
F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\xi) &= -\frac{a\rho}{2}\{\mathrm{R}(\partial_j,u,u,\partial_i)\},\\
F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\tilde{\delta}_t) &= \frac{a}{2}\{\mathrm{R}(\partial_t,\partial_j,u,\partial_i)\},\\
F(\xi,\tilde{\delta}_i,\tilde{\delta}_j) &= \frac{a\rho}{2}\{\mathrm{R}(u,\partial_j,u,\partial_i) - \mathrm{R}(\partial_i,u,u,\partial_j)\},\\
F(\tilde{\delta}_i,\tilde{\delta}_j,\xi) &= \frac{a\rho}{2}\{\mathrm{R}(\partial_i,u,u,\partial_j)\},
\end{aligned}$$

where R *denotes the Riemannian curvature tensor on M.*

Proof. First, we compute $F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\tilde{\partial}_t^T)$, as a zero coefficient. Taking into account the definition 2 we get

$$F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\tilde{\partial}_t^T) = -\tilde{G}(\tilde{\nabla}_{\tilde{\partial}_i^T}\tilde{\delta}_j,\tilde{\partial}_t^T) + \tilde{G}(\tilde{\nabla}_{\tilde{\partial}_i^T}\tilde{\partial}_j^T,\tilde{\delta}_t). \tag{8}$$

Using Koszul formula we obtain

$$\begin{aligned}
\tilde{G}(\tilde{\nabla}_{\tilde{\partial}_i^T}\tilde{\delta}_j,\tilde{\partial}_t^T) &= \frac{1}{2}\{\tilde{G}([\tilde{\partial}_i^T,\tilde{\delta}_j],\tilde{\partial}_t^T) + \tilde{G}([\tilde{\partial}_t^T,\tilde{\partial}_i^T],\tilde{\delta}_j) \qquad (9)\\
&\quad - \tilde{G}([\tilde{\delta}_j,\tilde{\partial}_t^T],\tilde{\partial}_i^T) + \tilde{\delta}_j\tilde{G}(\tilde{\partial}_i^T,\tilde{\partial}_t^T)\}\\
&= \frac{1}{2}\{-a_j^s a_t^m\delta_s(a_i^r)ag_{rm} - a_i^r a_t^m a_j^s\Gamma_{rs}^p ag_{pm} - a_i^r a_j^s\delta_s(a_t^m)(ag_{mr})\\
&\quad - a_i^r a_j^s a_t^m\Gamma_{sm}^p ag_{pr} + a_j^s\delta_s(a_i^r a_t^m ag_{rm})\},
\end{aligned}$$

and also,

$$\begin{aligned}\tilde{G}(\tilde{\nabla}_{\tilde{\partial}_i^T}\tilde{\partial}_j^T,\tilde{\delta}_t) &= \frac{1}{2}\{\tilde{G}([\tilde{\partial}_i^T,\tilde{\partial}_j^T],\tilde{\delta}_t)+\tilde{G}([\tilde{\delta}_t,\tilde{\partial}_i^T],\tilde{\partial}_j^T) \\ &- \tilde{G}([\tilde{\partial}_j^T,\tilde{\delta}_t],\tilde{\partial}_i^T)-\tilde{\delta}_t\tilde{G}(\tilde{\partial}_i^T,\tilde{\partial}_j^T)\} \\ &= \frac{1}{2}\{a_t^m a_j^s\delta_m(a_i^r)ag_{rs}+a_i^r a_j^s a_t^m\Gamma_{mr}^p ag_{ps}+a_i^r a_t^m\delta_m(a_j^s)a(g_{sr}) \\ &+ a_i^r a_j^s a_t^m\Gamma_{sm}^p ag_{pr}-a_t^m\delta_m(a_i^r a_j^s ag_{rs})\}.\end{aligned} \tag{10}$$

Substituting (9) and (10) into (8), we have

$$\begin{aligned}F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\tilde{\partial}_t^T) &= \frac{1}{2}\{a_j^s a_t^m\delta_s(a_i^r)ag_{rm}+a_i^r a_t^m a_j^s\Gamma_{rs}^p ag_{pm}+a_i^r a_j^s\delta_s(a_t^m)(ag_{mr}) \\ &+ a_i^r a_j^s a_t^m\Gamma_{sm}^p ag_{pr}+a_j^s a_i^r\delta_s(a_t^m)ag_{rm}+a_j^s a_t^m\delta_s(a_i^r)ag_{rm} \\ &+ a_j^s a_i^r a_t^m\delta_s(ag_{rm})+a_t^m a_j^s\delta_m(a_i^r)ag_{rs}+a_i^r a_j^s a_t^m\Gamma_{mr}^p ag_{ps} \\ &+ a_i^r a_t^m\delta_m(a_j^s)a(g_{sr})+a_i^r a_j^s a_t^m\Gamma_{sm}^p ag_{pr}-a_t^m\delta_m(a_i^r a_j^s ag_{rs})\} \\ &= \frac{1}{2}\{a_j^s a_t^m\delta_s(a_i^r)ag_{rm}+a_i^r a_t^m a_j^s\Gamma_{rs}^p ag_{pm}+a_i^r a_j^s\delta_s(a_t^m)(ag_{mr}) \\ &+ a_i^r a_j^s a_t^m\Gamma_{sm}^p ag_{pr}-a_j^s a_i^r\delta_s(a_t^m)ag_{rm}-a_j^s a_t^m\delta_s(a_i^r)ag_{rm} \\ &- a_j^s a_i^r a_t^m\delta_s(ag_{rm})+a_t^m a_j^s\delta_m(a_i^r)ag_{rs}+a_i^r a_j^s a_t^m\Gamma_{mr}^p ag_{ps} \\ &+ a_i^r a_t^m\delta_m(a_j^s)a(g_{sr})+a_i^r a_j^s a_t^m\Gamma_{sm}^p ag_{pr}-a_t^m a_i^r a_j^s\delta_m(ag_{rs}) \\ &- a_t^m a_i^r\delta_m(a_j^s)ag_{rs}-a_t^m a_j^s\delta_m(a_i^r)ag_{rs}\} \\ &= \frac{1}{2}\{a_i^r a_t^m a_j^s\Gamma_{rs}^p ag_{pm}+a_i^r a_j^s a_t^m\Gamma_{sm}^p ag_{pr}-a_j^s a_i^r a_t^m\delta_s(ag_{rm}) \\ &+ a_i^r a_j^s a_t^m\Gamma_{mr}^p ag_{ps}+a_i^r a_j^s a_t^m\Gamma_{sm}^p ag_{pr}-a_t^m a_i^r a_j^s\delta_m(ag_{rs})\} \\ &= \frac{1}{2}\{-\nabla_s g_{rm}-\nabla_m g_{rs}\}=0.\end{aligned}$$

Now we compute $F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\xi)$. Taking into account the definition 2 and using Koszul formula we have

$$\begin{aligned}F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\xi) &= \tilde{G}(\tilde{\nabla}_{\tilde{\partial}_i^T}\xi,\tilde{\delta}_j) \\ &= \frac{1}{2}\{\tilde{G}([\tilde{\partial}_i^T,\xi],\tilde{\delta}_j)+\tilde{G}([\tilde{\delta}_j,\tilde{\partial}_i^T],\xi)-\tilde{G}([\xi,\tilde{\delta}_j],\tilde{\partial}_i^T)\} \\ &= \frac{1}{2}\{-a_i^r a_j^s\partial_r^T(\rho u^o)ag_{os}+a_i^r\rho u^o\partial_r^T(a_j^s)ag_{os} \\ &- a_i^r a_j^s\rho u^o u^l \mathrm{R}_{sol}{}^k ag_{kr}\}=\frac{1}{2}\{-a_i^r a_j^s\rho u^o u^l\mathrm{R}_{sol}{}^k ag_{kr}\}.\end{aligned}$$

So we have $F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\xi)=-\frac{a\rho}{2}\{\mathrm{R}(\partial_j,u,u,\partial_i)\}$. Similarly, it can be shown that our assertion is valid for other essential coefficients of the structure tensor F. $\square$

Lemma 2 *The associated* 1*-forms with respect to structure tensor* F *on* $(T_1M, \varphi, \xi, \eta, \tilde{G})$ *are of the following forms*

$$
\begin{aligned}
&\theta(\tilde{\delta}_i) = -\mathrm{Ric}(\partial_i, u), && \theta(\tilde{\partial}_i^T) = 0, && \theta(\xi) = -\mathrm{Ric}(u, u),\\
&\theta^*(\tilde{\delta}_i) = 0, && \theta^*(\tilde{\partial}_i^T) = -\mathrm{Ric}(\partial_i, u), && \theta^*(\xi) = 0,\\
&\omega(\tilde{\delta}_i) = 0, && \omega(\tilde{\partial}_i^T) = 0, && \omega(\xi) = 0,
\end{aligned}
$$

where Ric *denotes the Ricci curvature tensor on* M.

Proof. Using Proposition 2, the Bianchi identity and (7) we have

$$
\begin{aligned}
\theta(\tilde{\delta}_t) &= \tilde{G}^{bc}F(e_b, e_c, \tilde{\delta}_t) = \frac{-1}{a}g^{bc}F(\tilde{\delta}_b, \tilde{\delta}_c, \tilde{\delta}_t) + \frac{1}{a}g^{bc}F(\tilde{\partial}_b^T, \tilde{\partial}_c^T, \tilde{\delta}_t)\\
&= \frac{-1}{a}g^{bc}\{\frac{-a}{2}\mathrm{R}(\partial_c, u, \partial_b, \partial_t) + \frac{a}{2}\mathrm{R}(\partial_b, \partial_c, u, \partial_t)\}\\
&+ \frac{1}{a}g^{bc}\{\frac{a}{2}\mathrm{R}(\partial_b, u, \partial_c, \partial_t)\}\\
&= \frac{1}{a}g^{bc}\{\frac{a}{2}\mathrm{R}(\partial_c, u, \partial_b, \partial_t) - \frac{a}{2}\mathrm{R}(\partial_b, \partial_c, u, \partial_t) + \frac{a}{2}\mathrm{R}(\partial_b, u, \partial_c, \partial_t)\}\\
&= \frac{1}{2}g^{bc}\{2R(\partial_c, u, \partial_b, \partial_t)\} = g^{bc}\mathrm{R}(\partial_c, u, \partial_b, \partial_t) = -g^{bc}\mathrm{R}(\partial_b, \partial_t, u, \partial_c)\\
&= -\mathrm{Ric}(\partial_t, u),
\end{aligned}
$$

and also,

$$
\begin{aligned}
\theta^*(\xi) &= \tilde{G}^{bc}F(e_b, \varphi e_c, \xi) = -\frac{1}{a}g^{bc}F(\tilde{\delta}_b, \varphi\tilde{\delta}_c, \xi) + \frac{1}{a}g^{bc}F(\tilde{\partial}_b^T, \varphi\tilde{\partial}_c^T, \xi)\\
&= -\frac{1}{a}g^{bc}F(\tilde{\delta}_b, \tilde{\partial}_c^T, \xi) - \frac{1}{a}g^{bc}F(\tilde{\partial}_b^T, \tilde{\delta}_c, \xi) = 0.
\end{aligned}
$$

Similarly, we can prove the other equations. □

3 Classification of the unit tangent sphere bundle

A classification of the almost contact B-metric manifold with respect to the structure tensor F is given in [15]. This classification includes eleven basic classes $\mathcal{F}_1, \ldots, \mathcal{F}_{11}$. Their intersection is the special class $\mathcal{F}_0$ determined by the condition $F(x, y, z) = 0$. Hence, $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with ∇-parallel structures, i.e., $\nabla\varphi = \nabla\xi = \nabla\eta = \nabla\tilde{G} = 0$.

This classification is determined by the following relations.

$$\begin{aligned}
&\mathcal{F}_1 : F(x,y,z) = \frac{1}{2n}\{\tilde{G}(x,\varphi y)\theta(\varphi z) + \tilde{G}(\varphi x,\varphi y)\theta(\varphi^2 z) \\
&\qquad + \tilde{G}(x,\varphi z)\theta(\varphi y) + \tilde{G}(\varphi x,\varphi z)\theta(\varphi^2 y)\};\\
&\mathcal{F}_2 : F(\xi,y,z) = F(x,\xi,z) = 0, \quad \underset{x,y,z}{\mathfrak{S}} F(x,y,\varphi z) = 0, \quad \theta = 0;\\
&\mathcal{F}_3 : F(\xi,y,z) = F(x,\xi,z) = 0, \quad \underset{x,y,z}{\mathfrak{S}} F(x,y,z) = 0;\\
&\mathcal{F}_4 : F(x,y,z) = -\frac{\theta(\xi)}{2n}\{\tilde{G}(\varphi x,\varphi y)\eta(z) + \tilde{G}(\varphi x,\varphi z)\eta(y)\};\\
&\mathcal{F}_5 : F(x,y,z) = -\frac{\theta^*(\xi)}{2n}\{\tilde{G}(x,\varphi y)\eta(z) + \tilde{G}(x,\varphi z)\eta(y)\};\\
&\mathcal{F}_{6/7} : \begin{cases} F(x,y,z) = F(x,y,\xi)\eta(z) + F(x,z,\xi)\eta(y),\\ F(x,y,\xi) = \pm F(y,x,\xi) = -F(\varphi x,\varphi y,\xi), \ \theta = \theta^* = 0;\end{cases}\\
&\mathcal{F}_{8/9} : \begin{cases} F(x,y,z) = F(x,y,\xi)\eta(z) + F(x,z,\xi)\eta(y),\\ F(x,y,\xi) = \pm F(y,x,\xi) = +F(\varphi x,\varphi y,\xi);\end{cases}\\
&\mathcal{F}_{10} : F(x,y,z) = F(\xi,\varphi y,\varphi z)\eta(x);\\
&\mathcal{F}_{11} : F(x,y,z) = \eta(x)\{\eta(y)\omega(z) + \eta(z)\omega(y)\};
\end{aligned}$$

where $\mathfrak{S}$ is a notation for the cyclic sum by three arguments.

Now we characterize the five essential coefficients of the structure tensor F on T_1M as follows.

Proposition 3 *The essential coefficients of F on $(T_1M,\varphi,\xi,\eta,\tilde{G})$ mentioned in the Proposition 2, satisfy the following relations.*

i) $F(\tilde{\delta}_i,\tilde{\delta}_j,\tilde{\delta}_t) \in \mathcal{F}_3$;

ii) $F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\tilde{\delta}_t) \in \mathcal{F}_2$;

iii) $F(\tilde{\delta}_i,\tilde{\delta}_j,\xi)$, $F(\tilde{\partial}_i^T,\tilde{\partial}_j^T,\xi) \in \mathcal{F}_6$;

iv) $F(\xi,\tilde{\delta}_i,\tilde{\delta}_j) \in \mathcal{F}_{10}$.

Proof. We only prove the first item. Using Proposition 2 and some direct calculations we have $\underset{i,j,t}{\mathfrak{S}} F(\tilde{\delta}_i,\tilde{\delta}_j,\tilde{\delta}_t) = 0$, therefore, $F(\tilde{\delta}_i,\tilde{\delta}_j,\tilde{\delta}_t)$ belongs to $\mathcal{F}_3$. $\square$

Remark 1 ([15]) *Let (M,φ,η,ξ,g) be an almost contact manifold with B-metric. Using the decomposition of the space $\mathcal{F}$, we define the corresponding subclasses of the class of almost contact manifolds with B-metric with respect to the covariant derivative of the structure tensor field φ. An almost contact manifold with B-metric is said to be in the class $\mathcal{F}_i$ $(i = 1,\dots,11)$, if the structure tensor F belongs to the class $\mathcal{F}_i$. In a similar way we define the classes $\mathcal{F}_i \oplus \mathcal{F}_j$, etc. It is clear that 2^{11} classes of almost contact manifolds with B-metric are possible.*

Here, according to Proposition 3 and Remark 1 we express main theorem of the present paper.

Theorem 1 (Characterization of $(T_1M, \varphi, \xi, \eta, \tilde{G})$**)** *Let M be an $(n+1)$-dimensional Riemannian manifold and we denote by $(T_1M, \varphi, \xi, \eta, \tilde{G})$ its unit tangent sphere bundle equipped with pseudo-Riemannian g-natural almost contact B-metric structure. Then T_1M with mentioned structure belongs to the class $\mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_6 \oplus \mathcal{F}_{10}$.*

Proof. According to Proposition 3, the essential coefficients (and obviously the zero coefficients) of the structure tensor F belong to the class $\mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_6 \oplus \mathcal{F}_{10}$. Therefore, $(T_1M, \varphi, \xi, \eta, \tilde{G})$ belongs to the class $\mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_6 \oplus \mathcal{F}_{10}$. □

Now, we focus on the essential coefficients of the structure tensor F, in order to find some curvature conditions such that the essential coefficient satisfy some basic classes.

Proposition 4 *The essential coefficients of the structure tensor F on the unit tangent sphere bundle $(T_1M, \varphi, \xi, \eta, \tilde{G})$ mentioned in the Proposition 2, satisfy the following relations.*

i) $F(\tilde{\delta}_i, \tilde{\delta}_j, \tilde{\delta}_t) \in \mathcal{F}_1 \Longleftrightarrow$

$$\mathrm{R}(\partial_i, \partial_t, u, \partial_j) - \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = \frac{1}{an}\{g_{ij}\mathrm{Ric}(u, \partial_t) - g_{it}\mathrm{Ric}(u, \partial_j)\};$$

ii) $F(\tilde{\delta}_i, \tilde{\delta}_j, \tilde{\delta}_t) \in \mathcal{F}_2 \Longleftrightarrow \mathrm{R}(\partial_i, u, u, \partial_j) = 0, \quad \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0;$

iii) $F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \tilde{\delta}_t) \in \mathcal{F}_1 \Longleftrightarrow$

$$\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = \frac{-1}{an}\{-g_{ij}\mathrm{Ric}(u, \partial_t) + g_{it}\mathrm{Ric}(u, \partial_j)\};$$

iv) $F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \tilde{\delta}_t) \in \mathcal{F}_3 \Longleftrightarrow \mathrm{R}(\partial_t, \partial_j, u, \partial_i) + \mathrm{R}(\partial_t, \partial_i, u, \partial_j) = 0;$

v) $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi) \in \mathcal{F}_4 \Longleftrightarrow \mathrm{R}(\partial_j, u, u, \partial_i) = \frac{1}{an} g_{ij}\mathrm{Ric}(u, u);$

vi) $F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \xi) \in \mathcal{F}_4 \Longleftrightarrow \mathrm{R}(\partial_j, u, u, \partial_i) = \frac{1}{an} g_{ij}\mathrm{Ric}(u, u).$

Proof. Direct computation yields the truthfulness of our assertion. □

3.1 Characterization of the unit tangent sphere bundle $(T_1M, \varphi, \xi, \eta, \tilde{G})$ under some curvature conditions

In this part, we focus on each of the eleven basic classes separately, in order to find some conditions such that structure tensors F satisfy each of classes under those curvature conditions.

Theorem 2 *Let M be an $(n+1)$-dimensional Riemannian manifold and we denote by $(T_1M, \varphi, \xi, \eta, \tilde{G})$ its unit tangent sphere bundle equipped with pseudo-Riemannian g-natural almost contact B-metric structure. Then,*

i) $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_1 \Longleftrightarrow$

$$\mathrm{R}(\partial_i, \partial_t, u, \partial_j) - \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = \frac{1}{an}\{g_{ij}\mathrm{Ric}(u, \partial_t) - g_{it}\mathrm{Ric}(u, \partial_j)\},$$
$$\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = \frac{1}{an}\{g_{ij}\mathrm{Ric}(u, \partial_t) - g_{it}\mathrm{Ric}(u, \partial_j)\},\ \mathrm{R}(\partial_i, u, u, \partial_j) = 0;$$

ii) $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_2 \Longleftrightarrow \mathrm{R}(\partial_i, u, u, \partial_j) = 0, \quad \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0;$

iii) $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_3 \Longleftrightarrow$

$$\mathrm{R}(\partial_t, \partial_j, u, \partial_i) + \mathrm{R}(\partial_t, \partial_i, u, \partial_j) = 0, \quad \mathrm{R}(\partial_i, u, u, \partial_j) = 0;$$

iv) $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_4, \mathcal{F}_5 \Longleftrightarrow \mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0,\ \mathrm{R}(\partial_j, u, u, \partial_i) = 0;$

v) $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_6, \mathcal{F}_7 \Longleftrightarrow \mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0,\ \mathrm{R}(\partial_j, u, u, \partial_i) = 0;$

vi) $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_8, \mathcal{F}_9 \Longleftrightarrow \mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0,\ \mathrm{R}(\partial_j, u, u, \partial_i) = 0;$

vii) $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_{10}, \mathcal{F}_{11} \Longleftrightarrow$

$$\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0, \qquad \mathrm{R}(\partial_j, u, u, \partial_i) = 0.$$

Proof. We use Proposition 3 and Proposition 4 in the proof as follows.

i) It is obvious that all zero coefficients of the structure tensor F satisfy $\mathcal{F}_1$, hence, we concentrate on the five essential coefficients. Using Proposition 4, it is clear that the structure tensors $F(\tilde{\delta}_i, \tilde{\delta}_j, \tilde{\delta}_t)$ and $F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \tilde{\delta}_t)$, belong to the class $\mathcal{F}_1$ if and only if $\mathrm{R}(\partial_i, \partial_t, u, \partial_j) - \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = \frac{1}{an}\{g_{ij}\mathrm{Ric}(u, \partial_t) - g_{it}\mathrm{Ric}(u, \partial_j)\}$ and $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = \frac{-1}{an}\{-g_{ij}\mathrm{Ric}(u, \partial_t) + g_{it}\mathrm{Ric}(u, \partial_j)\}$. Also, direct calculations show that $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi)$, and $F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \xi)$, and $F(\xi, \tilde{\delta}_i, \tilde{\delta}_j)$ belong to $\mathcal{F}_1$ if and only if $\mathrm{R}(\partial_i, u, u, \partial_j)$ vanishes. Therefore, our assertion is valid.

ii) First, we consider the essential coefficients of the structure tensors F. A glimpse into Proposition 3 shows that one of the five essential coefficients of the structure tensor F belongs to $\mathcal{F}_2$. Moreover, using Proposition 4 we have $F(\tilde{\delta}_i, \tilde{\delta}_j, \tilde{\delta}_t)$ belongs to $\mathcal{F}_2$ if and only if $\mathrm{R}(\partial_i, u, u, \partial_j) = \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0$. Also, the structure tensors $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi)$, and $F(\tilde{\partial}^T_i, \tilde{\partial}^T_j, \xi)$, and $F(\xi, \tilde{\delta}_i, \tilde{\delta}_j)$ belong to $\mathcal{F}_2$ if and only if $\mathrm{R}(\partial_i, u, u, \partial_j) = 0$. Now we focus on zero coefficients. We consider $F(\tilde{\delta}_i, \tilde{\delta}_j, \tilde{\partial}^T_t)$ as a zero coefficient and a computation shows that $\mathfrak{S}F(\tilde{\delta}_i, \tilde{\delta}_j, \varphi\tilde{\partial}^T_t)$ vanishes if and only if $\mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0$. All the other coefficients of the structure tensor F belong to $\mathcal{F}_2$ without any conditions. Therefore, T_1M belongs to $\mathcal{F}_2$ if and only if $\mathrm{R}(\partial_i, u, u, \partial_j) = \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0$.

iii) Using Proposition 3 we know that one of the five essential coefficients of F belongs to $\mathcal{F}_3$. Thus, we concentrate on other essential coefficients. Taking advantage of Proposition 4, it can be shown that $F(\tilde{\partial}^T_i, \tilde{\partial}^T_j, \tilde{\delta}_t)$ belongs to $\mathcal{F}_3$ if and only if $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) + \mathrm{R}(\partial_t, \partial_i, u, \partial_j) = 0$. Also, the structure tensors $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi)$, and $F(\tilde{\partial}^T_i, \tilde{\partial}^T_j, \xi)$, and $F(\xi, \tilde{\delta}_i, \tilde{\delta}_j)$ belong to $\mathcal{F}_3$ if and only if $\mathrm{R}(\partial_i, u, u, \partial_j)$ vanishes. Obviously, all zero coefficients of the structure tensor F belong to $\mathcal{F}_3$, so the result holds.

iv) It is clear that all zero coefficients of the structure tensors F belong to $\mathcal{F}_4$, hence, we concentrate on essential coefficients. Direct computations show that $F(\tilde{\delta}_i, \tilde{\delta}_j, \tilde{\delta}_t)$ and $F(\tilde{\partial}^T_i, \tilde{\partial}^T_j, \tilde{\delta}_t)$ belong to $\mathcal{F}_4$ if and only if both equations $\mathrm{R}(\partial_i, \partial_t, u, \partial_j) - \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0$ and $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0$ hold. Also, using Proposition 4, the structure tensors $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi)$ and $F(\tilde{\partial}^T_i, \tilde{\partial}^T_j, \xi)$ belong to $\mathcal{F}_4$ if and only if $\mathrm{R}(\partial_j, u, u, \partial_i) = \frac{1}{an} g_{ij}\mathrm{Ric}(u, u)$. Moreover, the structure tensor $F(\xi, \tilde{\delta}_i, \tilde{\delta}_j)$ belongs to $\mathcal{F}_4$ if and only if $\mathrm{R}(\partial_j, u, u, \partial_i) = 0$. Notice that these equations are equivalent to $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0$ and $\mathrm{R}(\partial_j, u, u, \partial_i) = 0$ which prove our claim for the class $\mathcal{F}_4$.

For the class $\mathcal{F}_5$, the fact that $\theta^*(\xi) = 0$ implies that essential coefficients of the structure tensors F belong to $\mathcal{F}_5$ if and only if they vanish i.e. $\mathrm{R}(\partial_i, \partial_t, u, \partial_j) - \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0$, and $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0$, and $\mathrm{R}(\partial_j, u, u, \partial_i) = 0$. These equations are equivalent to $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0$ and $\mathrm{R}(\partial_j, u, u, \partial_i) = 0$. So the result holds.

v) Proposition 3 shows that $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi)$ and $F(\tilde{\partial}^T_i, \tilde{\partial}^T_j, \xi)$ belong to $\mathcal{F}_6$ and also, it can be checked that $F(\tilde{\delta}_i, \tilde{\delta}_j, \tilde{\delta}_t)$, and $F(\tilde{\partial}^T_i, \tilde{\partial}^T_j, \tilde{\delta}_t)$, and $F(\xi, \tilde{\delta}_i, \tilde{\delta}_j)$ belong to $\mathcal{F}_6$ if and only if $\mathrm{R}(\partial_i, \partial_t, u, \partial_j) - \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0$, and $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0$, and $\mathrm{R}(\partial_j, u, u, \partial_i) = 0$. These equations are equivalent to $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0$ and $\mathrm{R}(\partial_j, u, u, \partial_i) = 0$. By repeating this procedure, we obtain same results for the class $\mathcal{F}_7$. Therefore, our claim holds.

vi) The proof of this item is completely similar to the previous one.

vii) Taking into account Proposition 3, one of the five essential coefficients of F belongs to $\mathcal{F}_{10}$. Using Proposition 2 and some standard calculations, it is easy to see that the structure tensors $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi)$, and $F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \xi)$, and $F(\tilde{\delta}_i, \tilde{\delta}_j, \tilde{\delta}_t)$, and $F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \tilde{\delta}_t)$ belong to $\mathcal{F}_{10}$ if and only if $\mathrm{R}(\partial_i, \partial_t, u, \partial_j) - \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0$, and $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0$, and $\mathrm{R}(\partial_j, u, u, \partial_i) = 0$. These equations are equivalent to $\mathrm{R}(\partial_t, \partial_j, u, \partial_i) = 0$ and $\mathrm{R}(\partial_j, u, u, \partial_i) = 0$. So the result holds. For the class $\mathcal{F}_{11}$ we can use the same method.

□

Remark 2 *According to Theorem* 3.10 *in [15], the decomposition* $\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_{11}$ *is orthogonal and hence, all mutual intersections of these classes are reduced to the class* $\mathcal{F}_0$*, i.e.,* $\mathcal{F}_i \cap \mathcal{F}_j = \mathcal{F}_0$ $(i = 1, \ldots, 11)$.

Corollary 1 *Theorem 2 shows that* $(T_1M, \varphi, \xi, \eta, \tilde{G})$ *belongs to nine of the eleven basic classes under one condition. More precisely, if* $\mathrm{R}(\partial_i, u, u, \partial_j) = \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0$ *then* $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_2 \cap \mathcal{F}_4 \cap \ldots \cap \mathcal{F}_{11}$*. Taking into account Remark 2, if* $\mathrm{R}(\partial_i, u, u, \partial_j) = \mathrm{R}(\partial_j, \partial_i, u, \partial_t) = 0$ *then* $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_0$.

Corollary 2 *Items five and six of Proposition 4 imply that if* $\mathrm{R}(\partial_j, u, u, \partial_i) = \frac{1}{an} g_{ij} \mathrm{Ric}(u, u)$*, then* $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi), F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \xi) \in \mathcal{F}_4$*. Also, using Theorem 3 it is obvious that* $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi), F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \xi) \in \mathcal{F}_6$*. Therefore, if* $\mathrm{R}(\partial_j, u, u, \partial_i) = \frac{1}{an} g_{ij} \mathrm{Ric}(u, u)$*, then Remark 2 implies that* $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi), F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \xi) \in \mathcal{F}_4 \cap \mathcal{F}_6 = \mathcal{F}_0$*. In other words, if* $\mathrm{R}(\partial_j, u, u, \partial_i) = \frac{1}{an} g_{ij} \mathrm{Ric}(u, u)$*, then essential coefficients* $F(\tilde{\delta}_i, \tilde{\delta}_j, \xi), F(\tilde{\partial}_i^T, \tilde{\partial}_j^T, \xi)$ *must be zero and then Proposition 2 implies that three other essential coefficients of* F *must be zero too. So, if we have* $\mathrm{R}(\partial_j, u, u, \partial_i) = \frac{1}{an} g_{ij} \mathrm{Ric}(u, u)$*, then all five essential coefficients of the structure tensor* F *must be zero and consequently if* $\mathrm{R}(\partial_j, u, u, \partial_i) = \frac{1}{an} g_{ij} \mathrm{Ric}(u, u)$*, then* $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_0$.

Corollary 3 *Let* M *be a* ***flat*** $(n+1)$*-dimensional Riemannian manifold and* T_1M *its unit tangent sphere bundle. Using Proposition 2, all essential coefficients of* F *are zero and hence,* $(T_1M, \varphi, \xi, \eta, \tilde{G})$ *belongs to the intersection of eleven basic classes, i.e.,* $(T_1M, \varphi, \xi, \eta, \tilde{G}) \in \mathcal{F}_0$.

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