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CHARACTERIZATION OF TRACES  
OF THE WEIGHTED SOBOLEV SPACE  $W^{1,p}(\Omega, d_M^\varepsilon)$  ON  $M$

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1. INTRODUCTION

In this paper we shall use the following notation. Let  $N > 0$ ,  $k \geq 0$  be integers. Let  $\varepsilon$ ,  $p$  be real numbers,  $1 < p < \infty$ . Denote by  $p'$  the conjugate exponent, i.e.  $p' = \frac{p}{p-1}$ . Let  $\Omega$  be a non-empty, open, bounded subset of  $\mathbf{R}^N$ . Let  $M$  be a closed subset of  $\partial\Omega$  and let  $d_M(x)$  be the distance function,  $d_M(x) = \text{dist}(x, M)$ . For simplicity we shall write  $d(x)$  instead of  $d_M(x)$ . For an integer  $m$ ,  $1 \leq m \leq N$ , we set  $Q_m = (0, 1)^m$ .

**Definition 1.1.** We shall write  $(\Omega, M) \in B(k, N)$  for  $1 \leq k \leq N - 1$ ,  $N \geq 2$  if and only if there exists a bilipschitz mapping

$$B: Q_N \rightarrow \Omega$$

such that  $B(\overline{Q}_k) = M$ .

By  $C^\infty(\overline{\Omega})$  we denote the set of real functions  $u$  defined on  $\overline{\Omega}$  such that the derivatives  $D^\alpha u$  can be continuously extended to  $\overline{\Omega}$  for all multiindices  $\alpha$ .

Define the weighted Sobolev space  $W^{1,p}(\Omega, d^\varepsilon)$  as the closure of  $C^\infty(\overline{\Omega})$  with respect to the norm

$$\|u\|_{W^{1,p}(\Omega, d^\varepsilon)} = \left( \int_{\Omega} |u(x)|^p d^\varepsilon(x) dx + \int_{\Omega} \sum_{i=1}^N |D_i u(x)|^p d^\varepsilon(x) dx \right)^{1/p},$$

where  $D_i u = \frac{\partial u}{\partial x_i}$  stands for the generalized derivative of the function  $u$ . The space  $H^{1,p}(\Omega, d^\varepsilon)$  is the class of all functions locally integrable on  $\Omega$ , with a finite norm

$$\|u\|_{H^{1,p}(\Omega, d^\varepsilon)} = \left( \int_{\Omega} |u(x)|^p d^{\varepsilon-p}(x) dx + \int_{\Omega} \sum_{i=1}^N |D_i u(x)|^p d^\varepsilon(x) dx \right)^{1/p}.$$

Now, let  $(\Omega, M) \in B(k, N)$ . Let  $0 < s < 1$ . Let us recall the definition of the Slobodeckij space  $W^{s,p}(M)$  as the set of all functions  $u$  defined on  $M$  with a finite norm

$$\|u|W^{s,p}(M)\| = \left( \int_M |u(x)|^p dx + \iint_{M \times M} \frac{|u(x) - u(y)|^p}{|x - y|^{k+sp}} dx dy \right)^{1/p}.$$

It is well known (see [1] and [4]) that for  $\varepsilon \leq k - N$  or  $\varepsilon > p + k - N$  the space  $W^{1,p}(\Omega, d^\varepsilon)$  is isomorphically and topologically equivalent to the space  $H^{1,p}(\Omega, d^\varepsilon)$ . The space  $H^{1,p}(\Omega, d^\varepsilon)$  has zero traces on  $M$  in the case  $\varepsilon \leq k - N$  while for  $\varepsilon > p + k - N$  the traces on  $M$  have no sense in general.

In this paper we show that in the case  $k - N < \varepsilon < p + k - N$  the class of traces on  $M$  of the space  $W^{1,p}(\Omega, d^\varepsilon)$  for  $(\Omega, M) \in B(k, N)$  is equal to the space  $W^{1-\frac{N-k+\varepsilon}{p},p}(M)$ . Section 2 contains the direct trace theorems, in Section 3 we find a corresponding extension operator.

## 2. DIRECT THEOREMS

**Lemma 2.1.** *Let  $M = \{0\}$ ,  $-N < \varepsilon < p - N$ . Then there exists a constant  $c > 0$  dependent only on  $\varepsilon, p, N$  such that for all functions  $u \in C^\infty(\overline{Q}_N)$  the inequality*

$$(2.1) \quad |u(0)| \leq c \|u|W^{1,p}(Q_N, d^\varepsilon)\|$$

holds.

*Proof.* Let  $u \in C^\infty(\overline{Q}_N)$ . For  $x \in Q_N$  we have

$$u(x) = u(0) + \int_0^1 \sum_{i=1}^N x_i D_i u(tx) dt,$$

and so

$$|u(0)| \leq |u(x)| + \int_0^1 \sum_{i=1}^N x_i |D_i u(tx)| dt.$$

Multiplying the last inequality by  $d^{\varepsilon/p}(x)$  and integrating over  $Q_N$  we get

$$(2.2) \quad |u(0)| \int_{Q_N} d^{\varepsilon/p}(x) dx \leq \int_{Q_N} |u(x)| d^{\varepsilon/p}(x) dx + \int_0^1 \int_{Q_N} \sum_{i=1}^N x_i |D_i u(tx)| d^{\varepsilon/p}(x) dx dt = I_1 + I_2.$$

Since  $\varepsilon/p > -N$ , we have  $c_1 = \int_{Q_N} d^{\varepsilon/p}(x) dx < \infty$ .

The Hölder inequality yields

$$(2.3) \quad I_1 \leq \left( \int_{Q_N} dx \right)^{1/p'} \left( \int_{Q_N} |u(x)|^p d^\varepsilon(x) dx \right)^{1/p} \leq \|u\| W^{1,p}(Q_N, d^\varepsilon).$$

Let us estimate the integral  $I_2$ . By the assumptions there exists  $\alpha$ , such that  $\varepsilon + N - 1 < \alpha p < p - 1$ . We use the substitution  $tx = y$  and the Hölder inequality and obtain

$$\begin{aligned} I_2 &\leq \int_0^1 \int_{(0,t)^N} \sum_{i=1}^N |D_i u(y)| d^{\varepsilon/p}(y) t^{-N-\varepsilon/p} dy dt \\ &\leq \left( \int_0^1 t^{-\alpha p'} dt \right)^{1/p'} \left( \int_0^1 \left[ \int_{(0,t)^N} \sum_{i=1}^N |D_i u(y)| d^{\varepsilon/p}(y) t^{\alpha-N-\varepsilon/p} dy \right]^p dt \right)^{1/p}. \end{aligned}$$

Since  $-\alpha p' > -1$ , we have  $(\int_0^1 t^{-\alpha p'} dt)^{1/p'} = c_2 < \infty$ .

Using again the Hölder inequality we obtain

$$(2.4) \quad \begin{aligned} I_2 &\leq \left( \int_0^1 t^{p\alpha-N-\varepsilon} dt \right)^{1/p} \left( \int_{(0,1)^N} \sum_{i=1}^N |D_i u(y)|^p d^\varepsilon(y) dy \right)^{1/p} \\ &\leq c_3 \|u\| W^{1,p}(Q_N, d^\varepsilon), \end{aligned}$$

where  $c_3 = c_2 (\int_0^1 t^{p\alpha-N-\varepsilon} dt)^{1/p} < \infty$ , because  $p\alpha - N - \varepsilon > -1$ . Now (2.1) follows from (2.2), (2.3) and (2.4).  $\square$

**Lemma 2.2.** *Let  $1 \leq k \leq N - 1$ ,  $k - N < \varepsilon < p + k - N$  and let  $M = [0, 1]^k$ . Then there exists a unique bounded linear operator*

$$T: W^{1,p}(Q_N, d^\varepsilon) \rightarrow L^p(M)$$

satisfying

$$Tu(x) = u(x_1, x_2, \dots, x_k, 0, \dots, 0), \quad x \in Q_N,$$

for every function  $u \in C^\infty(\overline{Q_N})$ .

**Proof.** Let  $u \in C^\infty(\overline{Q_N})$ . Fix  $x_1, \dots, x_k \in (0, 1)$  and define a function  $v$  by

$$v(x_{k+1}, \dots, x_N) = u(x), \quad (x_{k+1}, \dots, x_N) \in (0, 1)^{N-k}.$$

Obviously,  $v(x_{k+1}, \dots, x_N) \in W^{1,p}(Q_{N-k}, d^\varepsilon)$ .

By Lemma 2.1 there exists a constant  $c$  dependent only on  $\varepsilon, p, N, k$  such that

$$|u(x_1, x_2, \dots, x_k, 0, \dots, 0)|^p \leq c \left( \int_{Q_{N-k}} |u(x)|^p d^\varepsilon(x) dx_{k+1} \dots dx_N + \int_{Q_{N-k}} \sum_{i=k+1}^N |D_i u(x)|^p d^\varepsilon(x) dx_{k+1} \dots dx_N \right).$$

Integrating over  $x_1, \dots, x_k$  we obtain

$$\|u(x_1, \dots, x_k, 0, \dots, 0)\|_{L^p(M)}^p \leq c \|u\|_{W^{1,p}(Q_N, d^\varepsilon)}^p.$$

The operator  $T$  is now the unique bounded linear extension of the mapping  $u \mapsto u(x_1, x_2, \dots, x_k, 0, \dots, 0)$ .  $\square$

Using a similar argument we can prove a more general assertion:

**Theorem 2.3.** *Let  $1 \leq k \leq N-1$ ,  $k-N < \varepsilon < p+k-N$  and  $(\Omega, M) \in B(k, N)$ . Then there exists a unique bounded linear operator*

$$T: W^{1,p}(\Omega, d^\varepsilon) \rightarrow L^p(M)$$

such that

$$Tu = u|_M$$

for all  $u \in C^\infty(\bar{\Omega})$ .

Now we shall prove that the trace operator is a bounded mapping from  $W^{1,p}(\Omega, d^\varepsilon)$  into  $W^{1-\frac{N-k+\varepsilon}{p}, p}(M)$  if  $(\Omega, M) \in B(k, N)$  and  $k-N < \varepsilon < p+k-N$ .

**The case  $k = N-1$ .**

**Lemma 2.4** (see [2]). *Let  $-1 < \varepsilon < p-1$  and let  $a$  be a real number such that  $0 < a < 1$ . Then there exists a constant  $c > 0$  independent of  $a$ , such that for all functions  $u \in C^\infty([0, a])$  and  $v \in C^\infty([a, 1])$  the inequalities*

$$\int_0^a \left| \frac{1}{a-x} \int_x^a u(t) dt \right|^p (a-x)^\varepsilon dx \leq c \int_0^a |u(x)|^p (a-x)^\varepsilon dx,$$

$$\int_a^1 \left| \frac{1}{x-a} \int_a^x v(t) dt \right|^p (x-a)^\varepsilon dx \leq c \int_a^1 |v(x)|^p (x-a)^\varepsilon dx$$

hold.

**Lemma 2.5.** Let  $\Omega = \{(x, y) \in \mathbf{R}^2 : x \in (0, 1), y \in (0, x)\}$ ,  $M = \{(x, x) : 0 \leq x \leq 1\}$  and let  $-1 < \varepsilon < p - 1$ . Then there exists a constant  $c > 0$  such that

$$(2.5) \quad \int_0^1 \int_0^t \frac{|u(t, t) - u(\tau, \tau)|^p}{|t - \tau|^{p-\varepsilon}} d\tau dt$$

$$\leq c \int_0^1 \int_0^t (|D_1 u(t, \tau)|^p + |D_2 u(t, \tau)|^p) (t - \tau)^\varepsilon d\tau dt$$

for all  $u \in C^\infty(\bar{\Omega})$ .

*Proof.* Using the Fubini theorem we obtain

$$\int_0^1 \int_0^t \frac{|u(t, t) - u(\tau, \tau)|^p}{|t - \tau|^{p-\varepsilon}} d\tau dt$$

$$\leq 2^{p-1} \left[ \int_0^1 \left( \int_0^t \left| \frac{1}{t - \tau} \int_\tau^t D_2 u(t, \xi) d\xi \right|^p (t - \tau)^\varepsilon d\tau \right) dt \right.$$

$$\left. + \int_0^1 \left( \int_\tau^1 \left| \frac{1}{t - \tau} \int_\tau^t D_1 u(\xi, \tau) d\xi \right|^p (t - \tau)^\varepsilon dt \right) d\tau \right]$$

$$= 2^{p-1} [I_1 + I_2].$$

According to Lemma 2.4 we have

$$I_1 \leq c \int_0^1 \int_0^t |D_2 u(t, \tau)|^p (t - \tau)^\varepsilon d\tau dt,$$

$$I_2 \leq c \int_0^1 \int_\tau^1 |D_1 u(t, \tau)|^p (t - \tau)^\varepsilon dt d\tau.$$

The inequality (2.5) follows. □

**Lemma 2.6.** Let  $N \geq 2$ ,  $-1 < \varepsilon < p - 1$ . Define  $A_i(u)$  by

$$A_i(u) = \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{(N-2)\text{-fold}} \left( \int_0^1 \int_0^1 |u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{N-1}) - u(x_1, \dots, x_{i-1}, \tau, x_{i+1}, \dots, x_{N-1})|^p \frac{d\tau dt}{|t - \tau|^{p-\varepsilon}} \right) dx_1 dx_2 \dots dx_{i-1} dx_{i+1} \dots dx_{N-1}.$$

Then there exists a positive constant  $c$  such that

$$\|u\|_{W^{1-\frac{1+\varepsilon}{p}, p}(Q_{N-1})}^p \leq c \left( \|u\|_{L^p(Q_{N-1})}^p + \sum_{i=1}^{N-1} A_i(u) \right)$$

for all functions  $u \in L^p(Q_{N-1})$  such that  $A_i(u) < \infty$  for  $i = 1, 2, \dots, N - 1$ .

**Proof.** The proof can proceed in a way similar to that of the proof of Lemma 6.8.10 in [3].  $\square$

**Lemma 2.7.** Let  $N \geq 2$ ,  $-1 < \varepsilon < p - 1$  and  $M = [0, 1]^{N-1}$ . Then there exists a constant  $c > 0$  such that

$$\|u(x_1, \dots, x_{N-1}, 0)\|_{W^{1-\frac{1+\varepsilon}{p}, p}(Q_{N-1})} \leq c \|u\|_{W^{1,p}(Q_N, d^\varepsilon)}$$

for all  $u \in C^\infty(\overline{Q}_N)$ .

**Proof.** Let  $u \in C^\infty(\overline{Q}_N)$ . Set  $v(x_1, x_2, \dots, x_{N-1}) = u(x_1, x_2, \dots, x_{N-1}, 0)$ . According to Lemma 2.6 and Lemma 2.2 we have

$$\begin{aligned} & \|u(x_1, \dots, x_{N-1}, 0)\|_{W^{1-\frac{1+\varepsilon}{p}, p}(Q_{N-1})}^p = \|v\|_{W^{1-\frac{1+\varepsilon}{p}, p}(Q_{N-1})}^p \\ & \leq c_1 \left( \|v\|_{L^p(Q_{N-1})}^p + \sum_{i=1}^{N-1} A_i(v) \right) \leq c_2 \left( \|u\|_{W^{1,p}(Q_N, d^\varepsilon)}^p + \sum_{i=1}^{N-1} A_i(v) \right). \end{aligned}$$

Obviously, it remains to estimate  $A_i(v)$ ,  $i = 1, 2, \dots, N - 1$ . Fix  $i = 1, \dots, N - 1$ . Set  $v_i(x_1, \dots, x_N) = u(x_1, \dots, x_{N-1}, x_i - x_N)$  in the domain  $0 \leq x_j \leq 1$  for

$j = 1, \dots, N - 1$  and  $0 \leq x_N \leq x_i$ . Lemma 2.5 and the direct calculation yield

$$\begin{aligned}
 A_i(v) &= 2 \int_0^1 \dots \int_0^1 \left( \int_0^1 \int_0^t |u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{N-1}, 0) \right. \\
 &\quad \left. - u(x_1, \dots, x_{i-1}, \tau, x_{i+1}, \dots, x_{N-1}, 0) \right|^p \frac{d\tau dt}{|t - \tau|^{p-\varepsilon}} \Big) \\
 &\quad dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_{N-1} \\
 &= 2 \int_0^1 \dots \int_0^1 \left( \int_0^1 \int_0^t |v_i(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{N-1}, t) \right. \\
 &\quad \left. - v_i(x_1, \dots, x_{i-1}, \tau, x_{i+1}, \dots, x_{N-1}, \tau) \right|^p \frac{d\tau dt}{|t - \tau|^{p-\varepsilon}} \Big) \\
 &\quad dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_{N-1} \\
 &\leq 2c \int_0^1 \dots \int_0^1 \left( \int_0^1 \int_0^t (|D_i v_i(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{N-1}, \tau)|^p \right. \\
 &\quad \left. + |D_N v_i(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{N-1}, \tau)|^p) (t - \tau)^\varepsilon d\tau dt \Big) \\
 &\quad dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_{N-1} \\
 &\leq 2^{p+1} c \int_0^1 \dots \int_0^1 \left( \int_0^1 \int_0^t (|D_i u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{N-1}, t - \tau)|^p \right. \\
 &\quad \left. + |D_N u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{N-1}, t - \tau)|^p) (t - \tau)^\varepsilon d\tau dt \Big) \\
 &\quad dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_{N-1}.
 \end{aligned}$$

The substitution  $s = t - \tau$  gives

$$A_i(v) \leq 2^{p+1} c \|u\|_{W^{1,p}(Q_N, d^\varepsilon)}^p, \quad i = 1, 2, \dots, N - 1,$$

which completes the proof. □

As an immediate consequence we have

**Theorem 2.8.** *Let  $N \geq 2$ ,  $-1 < \varepsilon < p - 1$  and  $(\Omega, M) \in B(N - 1, N)$ . Then there exists a unique bounded linear operator*

$$T: W^{1,p}(\Omega, d^\varepsilon) \rightarrow W^{1-\frac{1+\varepsilon}{p},p}(M)$$

such that

$$Tu = u|_M$$



for all  $u \in C^\infty(\overline{Q}_N)$ .

**The case  $1 \leq k \leq N - 2$ .**

**Lemma 2.9.** *Let  $K, L, s > 0$ . Then there exists  $c > 0$  such that*

$$\frac{c}{r^{1+s}} \leq \int_0^L \int_0^L \frac{1}{(r + |\varrho_1 - \varrho_2|)^{2+s}} d\varrho_1 d\varrho_2$$

for every  $r \in (0, K]$ .

**Proof.** Obviously,

$$(2.6) \quad \int_0^L \int_0^L \frac{d\varrho_1 d\varrho_2}{(r + |\varrho_1 - \varrho_2|)^{2+s}} = 2 \int_0^L \int_{\varrho_2}^L \frac{d\varrho_1 d\varrho_2}{(r + \varrho_1 - \varrho_2)^{2+s}} = 2I.$$

Direct calculation yields

$$I = \frac{1}{1+s} \frac{L}{r^{1+s}} + \frac{1}{s(s+1)} \left( \frac{1}{(r+L)^s} - \frac{1}{r^s} \right).$$

By the Lagrange Mean Value Theorem there exists  $\xi \in (r, r+L)$  such that

$$I = \frac{L}{1+s} \left( \frac{1}{r^{1+s}} - \frac{\xi^{s-1}}{r^s(r+L)^s} \right).$$

If  $s \leq 1$  we have

$$\begin{aligned} I &\geq \frac{L}{1+s} \left( \frac{1}{r^{1+s}} - \frac{r^{s-1}}{r^s(r+L)^s} \right) = \frac{L}{1+s} \frac{1}{r^{1+s}} \left( 1 - \left( \frac{r}{r+L} \right)^s \right) \\ &\geq \frac{L}{1+s} \left( 1 - \left( \frac{K}{K+L} \right)^s \right) \frac{1}{r^{1+s}}. \end{aligned}$$

For  $s \geq 1$  we get

$$\begin{aligned} I &\geq \frac{L}{1+s} \left( \frac{1}{r^{1+s}} - \frac{(r+L)^{s-1}}{r^s(r+L)^s} \right) = \frac{L}{1+s} \frac{1}{r^{1+s}} \left( 1 - \frac{r}{r+L} \right) \\ &\geq \frac{L}{1+s} \left( 1 - \frac{K}{K+L} \right) \frac{1}{r^{1+s}}. \end{aligned}$$

Thus, (2.6) holds with  $c = 2 \min \left( \frac{L}{1+s} \left( 1 - \left( \frac{K}{K+L} \right)^s \right), \frac{L}{1+s} \left( 1 - \frac{K}{K+L} \right) \right)$ . □

**Lemma 2.10.** Let  $N \geq 2$ ,  $k - N < \varepsilon < p + k - N$  and  $M = [0, 1]^k$ . Then there exists a unique bounded linear operator

$$T: W^{1,p}(Q_N, d^\varepsilon) \rightarrow W^{1-\frac{N-k+\varepsilon}{p},p}(M)$$

such that

$$Tu = u|_M$$

for all functions  $u \in C^\infty(\overline{Q}_N)$ .

**Proof.** Let  $u \in C^\infty(\overline{Q}_N)$ . We introduce general cylindrical coordinates  $(x', r, \varrho)$ ,  $x' = (x_1, \dots, x_k)$ ,  $\varrho = (\varrho_1, \dots, \varrho_{N-k-1})$ , by

$$\begin{aligned} x_1 &= x_1, \\ &\dots \\ x_k &= x_k, \\ x_{k+1} &= r \cos \varrho_1, \\ x_{k+2} &= r \sin \varrho_1 \cos \varrho_2, \\ &\dots \\ x_N &= r \sin \varrho_1 \dots \sin \varrho_{N-k-2} \cos \varrho_{N-k-1}. \end{aligned}$$

Let  $\Omega' = \{(x', r, \varrho) : x' \in (0, 1)^k, r \in (0, 1), \varrho \in (\frac{\pi}{6}, \frac{\pi}{3})^{N-k-1}\}$ . Obviously  $M = \{(x', r, \varrho) \in \overline{\Omega}' : r = 0\}$ . Let  $\Omega$  be the set of points on  $\mathbb{R}^N$  whose cylindrical coordinates  $(x', r, \varrho)$  belong to  $\Omega'$ . Then  $\Omega \subset Q_N$ ,  $M \subset \overline{\Omega}$ ,  $d(x) = r$  and there exist two constants  $c_1, c_2 > 0$  such that

$$(2.7) \quad c_1 r^{N-k-1} \leq J(x', r, \varrho) \leq c_2 r^{N-k-1}, \quad (x', r, \varrho) \in \Omega',$$

where  $J(x', r, \varrho)$  is the Jacobian of the mapping  $(x', r, \varrho) \mapsto x$ . Define a function  $v$  by

$$v(x', r, \varrho) = u(x).$$

It is easy to see that  $v \in C^\infty(\overline{\Omega}')$  and

$$\begin{aligned} \left| \frac{\partial v}{\partial x_i}(x', r, \varrho) \right| &= \left| \frac{\partial u}{\partial x_i}(x) \right| \quad \text{for } i = 1, 2, \dots, k, \\ \left| \frac{\partial v}{\partial r}(x', r, \varrho) \right| &\leq \sum_{j=k+1}^N \left| \frac{\partial u}{\partial x_j}(x) \right|, \\ \left| \frac{\partial v}{\partial \varrho_j}(x', r, \varrho) \right| &\leq r \sum_{i=k+1}^N \left| \frac{\partial u}{\partial x_i}(x) \right| \quad \text{for } j = 1, \dots, N - k - 1. \end{aligned}$$

These inequalities and (2.7) yield

$$\begin{aligned}
 (2.8) \quad I(v) &:= \int_{\Omega'} |v(x', r, \varrho)|^p r^{N-k-1} r^\varepsilon dx' dr d\varrho \\
 &+ \int_{\Omega'} \left( \sum_{i=1}^k \left| \frac{\partial v}{\partial x_i}(x', r, \varrho) \right|^p + \left| \frac{\partial v}{\partial r}(x', r, \varrho) \right|^p \right) r^{N-k-1} r^\varepsilon dx' dr d\varrho \\
 &+ \int_{\Omega'} \sum_{i=1}^{N-k-1} \left| \frac{\partial v}{\partial \varrho_i}(x', r, \varrho) \right|^p r^{N-k-1} r^{\varepsilon-p} dx' dr d\varrho \\
 &\leq c_3 \|u\|_{W^{1,p}(Q_N, d^\varepsilon)}^p.
 \end{aligned}$$

Set  $\varepsilon_1 = \varepsilon - N - k - 1$  and define the anisotropic weighted space  $V^{1,p}(\Omega', r^{\varepsilon_1})$  as the closure of  $C^\infty(\overline{\Omega'})$  with respect to the norm  $\| \cdot \|_V = (I(\cdot))^{1/p}$ . Obviously

$$(2.9) \quad V^{1,p}(\Omega', r^{\varepsilon_1}) \hookrightarrow W^{1,p}(\Omega', r^{\varepsilon_1}).$$

Putting  $\varepsilon = \varepsilon_1$  in Theorem 2.8 we obtain the existence of a constant  $c_4 > 0$  such that

$$\begin{aligned}
 (2.10) \quad &c_4 \|v\|_{W^{1,p}(\Omega', r^{\varepsilon_1})}^p \\
 &\geq \int_{(0,1)^k} \int_{(0,1)^k} \int_{(\frac{\pi}{6}, \frac{\pi}{3})^{N-k-1}} \int_{(\frac{\pi}{6}, \frac{\pi}{3})^{N-k-1}} \frac{|v(x', 0, \varrho) - v(y', 0, \psi)|^p}{(|x' - y'| + |\varrho - \psi|)^{N+p-2-\varepsilon_1}} d\varrho d\psi dx' dy'.
 \end{aligned}$$

Denote the right-hand side of (2.10) by  $I_1$ . Since  $v(x', 0, \varrho) = u(x_1, \dots, x_k, 0, \dots, 0) = u(x', 0)$ , we can write

$$(2.11) \quad I_1 = \int_{(0,1)^k} \int_{(0,1)^k} |u(x', 0) - u(y', 0)|^p J(|x' - y'|) dx' dy',$$

where

$$\begin{aligned}
 &J(|x' - y'|) \\
 &\geq c_5 \int_{(\frac{\pi}{6}, \frac{\pi}{3})^{N-k-1}} \int_{(\frac{\pi}{6}, \frac{\pi}{3})^{N-k-1}} \frac{d\varrho_1 \dots d\varrho_{N-k-1} d\psi_1 \dots d\psi_{N-k-1}}{(|x' - y'| + |\varrho_1 - \psi_2| + \dots + |\varrho_{N-k-1} - \psi_{N-k-1}|)^{N+p-2-\varepsilon_1}}.
 \end{aligned}$$

Using  $(N - k - 1)$ -times Lemma 2.9 we arrive at the estimate

$$(2.12) \quad J(|x' - y'|) \geq \frac{c_6}{|x' - y'|^{p+k-1-\varepsilon_1}} = \frac{c_6}{|x' - y'|^{p+2k-N-\varepsilon}}.$$

From (2.11) and (2.12) we obtain

$$I_1 \geq c_6 \int_{(0,1)^k} \int_{(0,1)^k} \frac{|u(x', 0) - u(y', 0)|^p}{|x' - y'|^{p+2k-N-\epsilon}} dx' dy'.$$

The estimates (2.8), (2.9), (2.10) and (2.12) imply the assertion of the lemma.  $\square$

Again, it is not difficult to prove the following more general statement:

**Theorem 2.11.** *Let  $N \geq 2$ ,  $k - N < \epsilon < p + k - N$  and let  $(\Omega, M) \in B(k, N)$ . Then there exists a unique bounded linear operator*

$$T: W^{1,p}(\Omega, d^\epsilon) \rightarrow W^{1-\frac{N-k+\epsilon}{p},p}(M)$$

such that

$$Tu = u|_M$$

for all  $u \in C^\infty(\bar{\Omega})$ .

### 3. INVERSE THEOREMS

Let  $N \geq 2$  and let  $k$  be fixed,  $1 \leq k \leq N - 1$ . For  $x \in (x_1, \dots, x_N) \in \mathbf{R}^N$  we shall write  $x = (x', x'')$ ,  $x' = (x_1, \dots, x_k)$ ,  $x'' = (x_{k+1}, \dots, x_N)$ . Let  $\Phi$  be a function of  $C^\infty(\mathbf{R}^k)$  such that  $\Phi(x') = 0$  for  $|x'| \geq 1$ ,  $\Phi(x') > 0$  for  $|x'| < 1$  and  $\int_{|x'| \leq 1} \Phi(x') dx' = 1$ . We define an operator  $R$  by

$$(3.1) \quad Ru(x', x'') = |x''|^{-k} \int_{|x' - t'| \leq |x''|} \Phi\left(\frac{x' - t'}{|x''|}\right) u(t') dt', \quad u \in L^1_{\text{loc}}(\mathbf{R}^k).$$

**Lemma 3.1.** *Let  $k - N < \epsilon < p + k - N$  and  $K > 0$ . Let  $Q^k = (-K, K)^k$ ,*

$$D = \{x \in \mathbf{R}^N : x = (x', x''), x' \in Q^k, |x''| \leq K - \max_{i=1, \dots, k} |x_i|, x_j \geq 0 \text{ for } k+1, \dots, N\}$$

and  $M = \bar{Q}^k$ .

Then the operator  $R$  is a bounded linear mapping of  $W^{1-\frac{N-k+\epsilon}{p},p}(Q^k)$  in  $W^{1,p}(D, d^\epsilon)$ .

**Proof.** The linearity of the operator  $R$  is obvious. Now, we shall follow the idea of the proof of Lemma 6.9.1 in [3]. Without loss of generality we may assume  $K = 1$ . From (3.1) we get

$$(3.2) \quad D_i Ru(x', x'') = \frac{1}{|x''|^{k+1}} \int_{|x' - t'| < |x''|} D_i \Phi\left(\frac{x' - t'}{|x''|}\right) u(t') dt', \quad i = 1, 2, \dots, k,$$

$$(3.3) \quad D_i Ru(x', x'') = \frac{1}{|x''|^{k+1}} \int_{|x'-t'| < |x''|} \left( -\frac{kx_i}{|x''|} \Phi\left(\frac{x'-t'}{|x''|}\right) - \sum_{j=1}^k D_j \Phi\left(\frac{x'-t'}{|x''|}\right) \frac{(x_j - t_j)x_i}{|x''|^2} \right) u(t') dt',$$

$$i = k+1, \dots, N.$$

If we take  $u \equiv 1$  in (3.1) we get  $Ru(x', x'') = 1$  and from (3.2) and (3.3) we obtain

$$0 = \frac{1}{|x''|^{k+1}} \int_{|x'-t'| \leq |x''|} D_i \Phi\left(\frac{x'-t'}{|x''|}\right) dt', \quad i = 1, 2, \dots, k,$$

$$0 = \frac{1}{|x''|^{k+1}} \int_{|x'-t'| \leq |x''|} \left( -\frac{kx_i}{|x''|} \Phi\left(\frac{x'-t'}{|x''|}\right) - \sum_{j=1}^k D_j \Phi\left(\frac{x'-t'}{|x''|}\right) \frac{(x_j - t_j)x_i}{|x''|^2} \right) dt',$$

$$i = k+1, \dots, N.$$

Thus, (3.2) and (3.3) can be rewritten in the following way:

$$D_i Ru(x', x'') = \frac{1}{|x''|^{k+1}} \int_{|x'-t'| \leq |x''|} D_i \Phi\left(\frac{x'-t'}{|x''|}\right) (u(t') - u(x')) dt',$$

$$i = 1, \dots, k,$$

$$D_i Ru(x', x'') = \frac{1}{|x''|^{k+1}} \int_{|x'-t'| \leq |x''|} \left( -\frac{kx_i}{|x''|} \Phi\left(\frac{x'-t'}{|x''|}\right) - \sum_{j=1}^k D_j \Phi\left(\frac{x'-t'}{|x''|}\right) \frac{(x_j - t_j)x_i}{|x''|^2} \right) (u(t') - u(x')) dt',$$

$$i = k+1, \dots, N.$$

We shall estimate  $Ru$  and  $D_i Ru$  in the norm of  $L^p(D, d^\epsilon)$ . Note that  $d(x) = |x''|$  for  $x \in D$ .

Let us estimate  $I_0 = \|Ru\|_{L^p(D, d^\epsilon)}^p$ :

We put  $b(x') = 1 - \max_{i=1, \dots, k} |x_i|$ . According to (3.1), we have

$$I_0 = \int_{Q^k} \int_{|x''| < b(x')} \left| \frac{1}{|x''|^k} \int_{|x'-t'| \leq |x''|} \Phi\left(\frac{x'-t'}{|x''|}\right) u(t') dt' \right|^p |x''|^\epsilon dx'' dx'.$$

Using the substitution  $s' = \frac{x' - t'}{|x''|}$ , the Hölder inequality, the spherical coordinates for  $x''$  and the Fubini theorem we obtain

$$\begin{aligned} I_0 &= \int_{Q^k} \left( \int_{|x''| < b(x')} \left| \int_{|s'| \leq 1} \Phi(s') u(x' - s'|x''|) ds' \right|^p |x''|^\epsilon dx'' \right) dx' \\ &\leq c_1 \int_{Q^k} \int_{0 < r < b(x')} \int_{|s'| \leq 1} |u(x' - s'r)|^p ds' r^{N-k-1+\epsilon} dr dx' \\ &= c_1 \int_{|s'| \leq 1} \int_0^1 \int_{(r-1, 1-r)^k} |u(x' - s'r)|^p dx' r^{N-k-1+\epsilon} dr ds'. \end{aligned}$$

Now we use the substitution  $y' = x' - s'r$ . Obviously,  $|y'| - |s'r| \leq |y' + s'r| = |x'| \leq 1 - r$ , and so  $|y'| \leq 1 - r(1 - |s'|) \leq 1$ .

Increasing the integration domain we obtain

$$I_0 \leq c_1 \int_{|s'| \leq 1} \int_0^1 \int_{|y'| \leq 1} |u(y')|^p dy' r^{N-k-1+\epsilon} dr ds' \leq c_2 \|u\|_{L^p(Q^k)}^p,$$

where  $c_2 = c_1 \int_{|s'| \leq 1} ds' \int_0^1 r^{N-k-1+\epsilon} dr$ .

To estimate  $I_i = \|D_i R u\|_{L^p(D, d^\epsilon)}^p$  for  $i = 1, 2, \dots, N$  we proceed in the same way as in the previous case. We obtain

$$\begin{aligned} I_i &\leq c_3 \int_{Q^k} \int_{0 < r < b(x')} \int_{|s'| \leq 1} \frac{|u(x' - s'r) - u(x')|^p}{r^p} r^{N-k-1+\epsilon} ds' dr dx' \\ &\leq c_3 \int_{Q^k} \int_{|t' - x'| \leq b(x')} \frac{|u(t') - u(x')|^p}{|t' - x'|^{p+2k-N-\epsilon}} \int_{|t' - x'|}^{b(x')} \frac{|t' - x'|^{p+2k-N-\epsilon}}{r^{p+2k-N-\epsilon-1}} dr dt' dx'. \end{aligned}$$

Direct calculation yields

$$\int_{|t' - x'|}^{b(x')} \frac{|t' - x'|^{p+2k-N-\epsilon}}{r^{p+2k-N-\epsilon-1}} dr \leq \frac{1}{p+2k-N-\epsilon}.$$

Since  $\{t' : |x' - t'| \leq 1 - \max_{i=1, \dots, k} |x_i|\} \subset Q^k$ , we have

$$I_i \leq c_4 \int_{Q^k} \int_{Q^k} \frac{|u(t') - u(x')|^p}{|t' - x'|^{p+2k-N-\epsilon}} dt' dx' \leq c_4 \|u\|_{W^{1-\frac{N-k-\epsilon}{p}, p}(Q^k)}^p,$$

which completes the proof. □

**Lemma 3.2.** Let  $G \subset \mathbf{R}^k$  have a Lipschitz boundary, i.e.  $G \in C^{0,1}$  in the sense of Definition 5.5.6 in [3],  $s \in \mathbf{R}$ ,  $0 < s < 1$ . Then there exists a bounded linear operator

$$S: W^{s,p}(G) \rightarrow W^{s,p}(\mathbf{R}^k)$$

such that

$$Su = u \quad \text{on} \quad \Omega.$$

**Proof.** We shall write  $x = (x', x_k)$  where  $x' = (x_1, \dots, x_{k-1})$ . Let  $Q_k^- = \{x = (x', x_k), x' \in Q_{k-1}, -1 < x_k < 0\}$ . Define an extension operator  $S_0$  in the following way:

$$(S_0 u)(x', x_k) = \begin{cases} u(x', x_k) & \text{for } x \in Q_k \\ u(x', -x_k) & \text{for } x \in Q_k^- \end{cases}$$

It is not difficult to prove that  $S_0$  is a bounded linear operator from  $W^{s,p}(Q_k)$  into  $W^{s,p}(Q_k \cup Q_k^-)$ .

Using the local description of  $\partial G$ , the corresponding partition of unity and the standard technique we obtain the assertion of the lemma.  $\square$

**Lemma 3.3.** Let  $N \geq 2$ ,  $k - N < \varepsilon < p + k - N$  and let  $M = [0, 1]^k$ . Then there exists a bounded linear operator

$$R_0: W^{1-\frac{N-k+\varepsilon}{p},p}(M) \rightarrow W^{1,p}(Q_N, d^\varepsilon)$$

such that

$$T(R_0 u) = u.$$

**Proof.** Take  $K$  sufficiently large so that  $D \supset Q_N$  where  $D$  is the domain defined in Lemma 3.1. Set  $Q^k = [-K, K]^k$ . Let  $S$  be the operator from Lemma 3.2 and  $R$  the operator from Lemma 3.1 (with  $G = Q^k$  and  $s = 1 - \frac{N-k+\varepsilon}{p}$ ). Let  $u \in W^{1-\frac{N-k+\varepsilon}{p},p}(M)$ . Obviously,

$$\|RSu\|_{W^{1,p}(Q_N, d^\varepsilon)} \leq c_1 \|u\|_{W^{1-\frac{N-k+\varepsilon}{p},p}(M)}.$$

It remains to prove that  $T(RSu) = u$ . Note that  $Su \in L^p(Q^k)$  and  $Su = u$  on  $M$ . Put

$$I(x'') = \int_M |(RSu)(x', x'') - u(x')|^p dx'.$$

Then

$$\begin{aligned} I(x'') &= \int_M \left| \int_{|s'| < 1} \Phi(s') ((Su)(x' - s'|x''|) - u(x')) \, ds' \right|^p dx' \\ &\leq c_2 \int_M \int_{|s'| < 1} |(Su)(x' - s'|x''|) - (Su)(x')|^p dx'. \end{aligned}$$

The  $p$ -mean continuity of  $Su$  yields

$$\lim_{|x''| \rightarrow 0} I(x'') = 0.$$

By virtue of the trivial imbedding

$$L^p(M) \hookrightarrow L^1(M)$$

we have

$$(3.4) \quad \lim_{|x''| \rightarrow 0} \|(RSu)(x', x'') - u(x')\|_{L^1(M)} = 0.$$

Since  $RSu \in W^{1,p}(D, d^\epsilon)$ , by Theorem 2.11 the trace of  $RSu$  on  $M$  exists. Denote this trace by  $v$ . We shall prove that

$$\lim_{|x''| \rightarrow 0} \|(RSu)(x', x'') - v(x'')\|_{L^1(M)} = 0.$$

Let  $u_n \in C^\infty(\overline{Q_N})$ ,  $u_n \rightarrow RSu$  in  $W^{1,p}(Q_N, d^\epsilon)$ . We can write

$$\begin{aligned} (3.5) \quad &\int_M |(RSu)(x', x'') - v(x')| \, dx' \leq \int_M |(RSu)(x', x'') - (RSu_n)(x', x'')| \, dx' \\ &+ \int_M |(RSu_n)(x', x'') - u_n(x', 0)| \, dx' + \int_M |u_n(x', 0) - v(x')| \, dx' \\ &= J_1(n, x'') + J_2(n, x'') + J_3(n). \end{aligned}$$

Set  $\sigma_{N-k} = \frac{1}{N-k} s_{N-k}$  where  $s_{N-k}$  is the  $(N-k-1)$ -dimensional Hausdorff measure of the  $(N-k)$ -dimensional unit sphere. For simplicity we write  $f_n$  instead of  $RSu - RSu_n$ . From the definition of the operator  $RS$  we have  $f_n(x', x'_1) = f_n(x', x'_2)$  for  $|x'_1| = |x'_2|$ . Thus, we can rewrite the integral  $J_1(u, x'')$  in the following way:

$$\begin{aligned} (3.6) \quad J_1(u, x'') &= \int_M |f_n(x', x'')| \, dx' \\ &= \frac{1}{\sigma_{N-k} |x''|^{N-k-1}} \int_M \int_{|y''|=|x''|} |f_n(x', y'')| \, dx' \, dy''. \end{aligned}$$



Put

$$M(|x''|) = \{y'' : |y''| < |x''|, y_i > 0 \text{ for } i = k + 1, \dots, N\}.$$

Since the traces of  $W^{1,1}(\Omega)$  are in  $L^1(\partial\Omega)$ , there exists a constant  $c_3 > 0$  independent of  $|x''|$  and such that

$$J_1(u, x'') \leq \frac{c_3}{\sigma_{N-k}|x''|^{N-k-1}} \int_M \int_{M(|x''|)} (|f_n(x', y'')| + \sum_{i=1}^N |D_i f_n(x', y'')|) dx' dy''.$$

We use (3.5), (3.6), the Hölder inequality and the general cylindrical coordinates to obtain

$$J_1(n, x'') \leq c_4 \int_M \int_{M(|x''|)} (|f_n(x', y'')|^p + \sum_{i=1}^N |D_i f_n(x', y'')|^p) |y''|^\epsilon dx' dy''^{1/p} \cdot \left( \int_0^{|x''|} r^{N-k-1-\frac{\epsilon r'}{p}} dr \right)^{1/p} \leq c_5 |x''|^{1-\frac{N-k+\epsilon}{p}} \|f_n\|_{W^{1,p}(Q_N, d^\epsilon)}^p.$$

Consequently,

$$(3.7) \quad J_1(n, x'') \leq c_5 |x''|^{1-\frac{N-k+\epsilon}{p}} \|RSu - RSu_n\|_{W^{1,p}(Q_N, d^\epsilon)}.$$

Note that evidently  $1 - \frac{N-k+\epsilon}{p} > 0$ .

Let  $\delta > 0$  be a fixed real number. Then for every  $n$  there exists  $r(n) > 0$  such that

$$(3.8) \quad J_2(n, x'') = \int_M |(RSu_n)(x', x'') - u_n(x', 0)| dx' < \delta/3 \quad \text{for } |x''| < r(n).$$

The proof is analogous to the estimate of  $I(x'')$ .

Finally, from the trivial imbedding

$$W^{1,p}(Q_N, d^\epsilon) \hookrightarrow W^{1,1}(Q_N)$$

and from Lemma 2.2 we have

$$\lim_{n \rightarrow \infty} J_3(n) = 0.$$

Now, fix  $n_0$  such that  $J_3(n_0) < \delta/3$ . From (3.8) we can find a corresponding number  $r(n_0)$  such that  $J_2(n_0, x'') < \delta/3$  for  $|x''| < r(n_0)$ . According to (3.7) there exists a number  $r_1 > 0$  such that

$$J_1(n_0, x'') < \delta/3 \quad \text{for} \quad |x''| < r_1.$$

Let  $r = \min(r_1, r(n_0))$ .

The estimate (3.5) implies

$$J(x'') < \delta \quad \text{for} \quad |x''| < r.$$

Therefore,

$$(3.9) \quad \lim_{|x''| \rightarrow 0} J(x'') = 0.$$

By (3.4) and (3.9) we have  $u = v$  in  $L^1(M)$ . Setting  $R_0 = RS$  we complete the proof.  $\square$

**Theorem 3.4.** *Let  $N \geq 2$ ,  $1 \leq k \leq N - 1$ ,  $\varepsilon \in \mathbf{R}$ ,  $k - N < \varepsilon < p + k - N$  and let  $(\Omega, M) \in B(k, N)$ . Then there exists a bounded linear operator*

$$R: W^{1-\frac{N-k+\varepsilon}{p}, p}(M) \rightarrow W^{1, p}(\Omega, d^\varepsilon)$$

such that

$$T(Ru) = u.$$

**Proof.** The theorem is an easy consequence of Lemma 3.3.  $\square$

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