Characterization of Weakly Efficient Points

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Abstract: Weakly efficient points of a mapping $F : S \to Y$ are characterized, where the feasible set S is given by infinitely many constraints, and Y is equipped with an arbitrary convex ordering. In the linear and in the convex case a necessary and sufficient condition is given, which needs no constraint qualification.

Zusammenfassung: Es werden schwach effiziente Punkte einer Abbildung $F : S \to Y$ charakterisiert, wobei der zulässige Bereich S durch unendlich viele Restriktionen bestimmt wird und Y mit einem beliebigen konvexen Ordnungskegel versehen ist. Im linearen und im konvexen Fall wird eine notwendige und hinreichende Bedingung angegeben, die keine Regularitätsvoraussetzung benötigt.

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1. Introduction

Recently B.Brosowski [3] has given characterization theorems for weakly efficient points (sometimes called weak Pareto-points) of a mapping $F : S \to \mathbb{R}^m$, if S, the feasible set, is described by infinitely many constraints, and if $\mathbb{R}^m$, the range space of the mapping F, is provided with its natural ordering. In what follows we extend these results to mappings into a linear topological space provided with an arbitrary convex ordering. Also instead of differentiable mappings we consider mappings which admit convex approximants. Two types of conditions are considered. The first one, which concerns inconsistency of a system of strict inequalities, reduces, if F is scalar-valued, to a certain generalization of Kolmogorov's criterion which has been described in [6 , p.163] and, for convex approximants, in [2], [1 , p.291]. The second type of conditions involves Lagrange multipliers. Here passing from linear to convex approximants is made possible through the use of Helly's theorem. In each case the corresponding results of [3] can be obtained readily by specialization. We conclude with a necessary and sufficient condition which, contrary to [3] and to similar conditions found in the literature, does not need a constraint qualification.

2. Preliminaries

Throughout this paper we make the following assumptions:

Y is a real linear topological space, Y^* its topological dual;
$P \subset Y$ is a convex cone with int $P \neq \emptyset$[1];
$P^* := \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0 \quad \forall y \in P\}$ is the polar cone of P.

P will be used as an ordering cone in Y, i.e., we define for arbitrary $y^1, y^2 \in Y$:

$$y^1 \leq_P y^2 :\iff y^1 - y^2 \in -P,$$
$$y^1 <_P y^2 :\iff y^1 - y^2 \in \text{int}(-P).$$

Given the ordering cone P in Y, a mapping $F : C \to Y$, and a subset $S \subset C$, we consider the problem

(1) $\quad$ w-eff $\{F(x), P \mid x \in S\}$.

By definition, x^o is a solution of (1) iff $x^o \in S$ and there is no $x \in S$ such that $F(x) - F(x^o) <_P 0$; x^o is then said to be a *weakly efficient point* of F over the feasible set S. Note that for $Y = \mathbb{R}^1$ and $P = \mathbb{R}^1_+$ the above problem (1) reduces to the ordinary minimization problem

$$\min \{F(x) \mid x \in S\}.$$

In what follows, we assume in particular that the feasible set S is of the form

$$S_f := \{x \in C \mid f(t,x) \leq 0 \quad \forall t \in T\},$$

i.e., we consider

(2) $\quad$ w-eff $\{F(x), P \mid x \in C, f(t,x) \leq 0 \quad \forall t \in T\}$,

where

$F : C \to Y$, $f : T \times C \to \mathbb{R}$,
T is a topological space,
C is a subset of a real linear topological space

(the latter two assumptions will be relaxed in the final section). The precise assumptions to be made on T,C,F and f will be different with each section and will be formulated as needed.

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1) "int" denotes the topological interior of a set, "cl" the closure.

For the present we give some more definitions. For a fixed element $x^o \in S_f$, we define

$$T^o := \{t \in T \mid f(t,x^o) = 0\},$$
$$S_f^o := \{x \in C \mid f(t,x) < 0 \quad \forall t \in T^o\}.$$

Problem (2) will be called weakly regular in x^o, iff $S_f^o \neq \emptyset$ implies $S_f \subset \mathrm{cl}\, S_f^o$. Moreover, (2) will be called strongly regular in x^o, iff $C_M := \{x \in C \mid f(t,x) < 0 \quad \forall t \in M,\ F(x) - F(x^o) <_P 0\} \neq \emptyset$ implies $x^o \in \mathrm{cl}\, C_M$ for all closed sets M with $T^o \subset M \subset T$.

Recall that $F : C \to Y$ is called P-convex iff C is convex and

$$F(\lambda x^1 + (1-\lambda)x^2) \leq_P \lambda F(x^1) + (1-\lambda)F(x^2)$$

for all $x^1, x^2 \in C$ and all $\lambda \in [0,1]$. Convexity is closely related to regularity, as shown in the following lemma.

Lemma 1. Let C be convex.
(i) If $f(t,\cdot)$ is convex for all $t \in T$, then (2) is weakly regular in x^o.
(ii) If $f(t,\cdot)$ is convex for all $t \in T$ and $F(\cdot)$ is P-convex, then (2) is strongly regular in x^o.

Proof. From the convexity assumptions it follows that $f(t,x^1) < 0$ and $f(t,x^2) \leq 0$ implies $f(t,x) < 0$ for all $x \in [x^1,x^2)$, and $F(x^1) - F(x^2) <_P 0$ implies $F(x) - F(x^2) <_P 0$ for all $x \in [x^1,x^2)$. q.e.d.

3. The non-convex case

In this section we want to characterize a solution x^o of (2) by the inconsistency of the system

(3) $\quad x \in C, \qquad f(t,x) < 0 \quad \forall t \in T^o, \qquad F(x) - F(x^o) <_P 0.$

We assume in this section that

T is a compact set,

$F(\cdot) : C \to Y$ and $f(\cdot,\cdot) : T \times C \to \mathbb{R}$ are continuous mappings.

Theorem 1.

(i) Let (2) be weakly regular in $x^o \in S_f$ and $S_f^o \neq \emptyset$. If (3) has no solution, then x^o solves (2).

(ii) Let (2) be strongly regular in $x^o \in S_f$. If x^o solves (2), then (3) has no solution.

Proof.

(i) Assume, for contradiction, that x^o does not solve (2). Then there exists $\xi \in S_f$ with $F(\xi) - F(x^o) <_P 0$. But weak regularity and the continuity of F imply that we can find an element $\xi^o \in S_f^o$ still satisfying $F(\xi^o) - F(x^o) <_P 0$, i.e., ξ^o is a solution of (3).

(ii) Assume, for contradiction, that ξ is a solution of (3). Then, since T^o is compact and $f(\cdot,\xi)$ is continuous, we can find a constant $K > 0$ such that

$$f(t,\xi) \leq -K < 0 \quad \forall t \in T^o.$$

Let us set $U := \{t \in T \mid f(t,\xi) < -\frac{K}{2}\}$, which implies in particular that $T^o \subset U$. The set $T \setminus U$ is again compact, and by the same argument as before we can find a constant $L > 0$ such that

$$f(t,x^o) \leq -L < 0 \quad \forall t \in T \setminus U.$$

Now ξ satisfies the inequalities

$$f(t,\xi) < 0 \quad \forall t \in \mathrm{cl}\, U, \qquad F(\xi) - F(x^o) <_P 0.$$

Since $T^o \subset \mathrm{cl}\, U \subset T$, strong regularity guarantees that in any neighbourhood W of x^o there exists an element $x_W \in W$ such that

(4) $\quad x_W \in C, \qquad f(t,x_W) < 0 \quad \forall t \in U, \qquad F(x_W) - F(x^o) <_P 0.$

In view of the continuity of $f(\cdot,\cdot)$ we can choose W in such a way that $f(t,x_W) - f(t,x^o) \leq \frac{L}{2}$ for all $t \in T \setminus U$, and hence

$$f(t,x_W) \leq -\frac{L}{2} < 0 \quad \forall t \in T \setminus U.$$

With (4), this contradicts x^o being a solution of (2). q.e.d.

4. Convex approximants

In this section we assume that, in addition to functions $F: C \to Y$ and $f: T\times C \to \mathbb{R}$ we are given functions $\Phi: C \to Y$ and $\varphi: T\times C \to \mathbb{R}$ such that the following requirements are satisfied:

$$T \text{ is a compact set; } C \text{ is a convex set;}$$
$$\Phi(\cdot) \text{ is P-convex, } \varphi(t,\cdot) \text{ is convex for all } t \in T;$$
$$\varphi(\cdot,x) \text{ is upper semicontinuous for all } x \in C;$$
$$\Phi(x^o) = F(x^o),\ \varphi(t,x^o) = f(t,x^o) \quad \forall t \in T;$$

$$(5)\quad \begin{cases} \text{for all } \xi \in C \text{ there exist Landau-functions } o_1(\cdot): [0,1] \to Y \text{ and} \\ o_2(\cdot): [0,1] \to \mathbb{R} \text{ such that for all } \lambda \in [0,1]: \\ F(x^o+\lambda(\xi-x^o)) \leq_P \Phi(x^o+\lambda(\xi-x^o)) + o_1(\lambda), \\ f(t,x^o+\lambda(\xi-x^o)) \leq \varphi(t,x^o+\lambda(\xi-x^o)) + o_2(\lambda) \quad \forall t \in T. \end{cases}$$

Here $o_1(\cdot)$ being a Landau-function means: for all neighbourhoods W of 0_Y there exists $\lambda_o > 0$ such that $\lambda \in [0,\lambda_o] \Rightarrow o_1(\lambda) \in \lambda W$. Similarly for $o_2(\cdot)$.

Let now be $x^o \in S_f$. We want to consider the analogue of (3) with f and F replaced by their convex approximants φ and Φ, i.e.,

$$(6)\qquad x \in C, \qquad \varphi(t,x) < 0 \quad \forall t \in T^o, \qquad \Phi(x) - \Phi(x^o) <_P 0.$$

As before, $T^o := \{t \in T \mid f(t,x^o) = 0\} = \{t \in T \mid \varphi(t,x^o) \geq 0\}$ is compact, since $\varphi(\cdot,x^o)$ is upper semicontinuous. No topology on C is needed in this section.

<u>Theorem 2</u>. If x^o solves (2), then (6) has no solution.

Proof. Assume that (6) has a solution ξ, i.e.,

$$\xi \in C, \qquad \varphi(t,\xi) < 0 \quad \forall t \in T^o, \qquad \Phi(\xi) - \Phi(x^o) <_P 0.$$

In view of the compactness of T and the upper semicontinuity of $\varphi(\cdot,\xi)$ there exists a constant $K > 0$ such that

$$\varphi(t,\xi) \leq -K \quad \forall t \in T^o.$$

With $U := \{t \in T \mid \varphi(t,\xi) < -\frac{K}{2}\}$ we obtain for some $L > 0$ that

$$\varphi(t,x^o) \leq -L \quad \forall t \in T \setminus U,$$

since $T \setminus U$ is compact and disjoint from T^o. Likewise

$$\varphi(t,\xi) \leq M < \infty \quad \forall t \in T \setminus U.$$

For $\lambda \in (0,1]$ let us set $x_\lambda := \lambda\xi + (1-\lambda)x^o$. Then there exists $\lambda_1 > 0$ such that for all $t \in U$ it follows

$$\begin{aligned} f(t,x_\lambda) &\le \varphi(t,x_\lambda) + o_2(\lambda) \\ &\le \lambda\varphi(t,\xi) + (1-\lambda)\varphi(t,x^o) + o_2(\lambda) \\ &\le \lambda\cdot(-\tfrac{K}{2}) + o_2(\lambda) \\ &< 0 \quad \text{if } \lambda \in (0,\lambda_1). \end{aligned}$$

Similarly there exists $\lambda_2 > 0$ such that for all $t \in T \setminus U$ we obtain

$$\begin{aligned} f(t,x_\lambda) &\le \lambda\cdot M + (1-\lambda)\cdot(-L) + o_2(\lambda) \\ &< 0 \quad \text{if } \lambda \in (0,\lambda_2). \end{aligned}$$

Furthermore, since $\Phi(\xi) - \Phi(x^o) \in \mathrm{int}(-P)$, there exists a neighbourhood W of 0_Y such that $\Phi(\xi) - \Phi(x^o) + W \subset \mathrm{int}(-P)$. Then there exists $\lambda_3 > 0$ such that $o_1(\lambda) \in \lambda\cdot W$ if $\lambda \in (0,\lambda_3)$. Consequently with

$$\begin{aligned} F(x_\lambda) - F(x^o) &\le_P \Phi(x_\lambda) - \Phi(x^o) + o_1(\lambda) \\ &\le_P \lambda(\Phi(\xi) - \Phi(x^o)) + o_1(\lambda) \end{aligned}$$

it follows for all $\lambda \in (0,\lambda_3)$ that

$$\begin{aligned} F(x_\lambda) - F(x^o) &\in \lambda(\Phi(\xi) - \Phi(x^o) + W) - P \\ &\subset \mathrm{int}(-P) - P = \mathrm{int}(-P), \end{aligned}$$

and hence

$$F(x_\lambda) - F(x^o) <_P 0.$$

Altogether we have obtained that x^o does not solve (2), a contradiction.

q.e.d.

Remark. If among the above assumptions we replace (5) by the following:

$$\begin{aligned} &\Phi(x) \le_P F(x) \quad \forall x \in C, \qquad \varphi(t,x) \le f(t,x) \quad \forall x \in C, \forall t \in T, \\ &S_\varphi^o := \{x \in C \mid \varphi(t,x) < 0 \quad \forall t \in T^o\} \neq \emptyset, \end{aligned}$$

then the converse implication of theorem 2 is also true, i.e., if $x^o \in S_f$ and if (6) has no solution, then x^o solves (2).

Indeed, if x^o does not solve (2), then there exists $\xi \in C$ such that

$$f(t,\xi) \le 0 \quad \forall t \in T, \qquad F(\xi) - F(x^o) <_P 0,$$

and hence

$$\varphi(t,\xi) \le 0 \quad \forall t \in T, \qquad \Phi(\xi) - \Phi(x^o) <_P 0.$$

Let $\xi^o \in S_\varphi^o$; then due to the convexity assumptions all $x \in [\xi^o,\xi)$ sufficiently close to ξ satisfy

$$\varphi(t,x) < 0 \quad \forall t \in T^o, \qquad \Phi(x) - \Phi(x^o) <_P 0,$$

i.e., they solve (6), a contradiction.

5. The convex case

From now on we shall utilize the polar cone P^*, defined in Section 2. The following lemma will be needed repeatedly; it is similar to [4, Thm. 5.13].

Lemma 2. Let S be any convex set and let $\Phi : S \to Y$ be P-convex. Then $x^o \in S$ solves

(7) $\quad$ $\text{w-eff}\,\{\Phi(x), P \mid x \in S\}$

if and only if there exists $y^* \in P^* \setminus \{0\}$ such that

$$\langle y^*, \Phi(x) - \Phi(x^o)\rangle \geq 0 \quad \forall x \in S.$$

Proof. It can easily be seen that $y^* \in P^* \setminus \{0\}$ implies $\langle y^*, y\rangle < 0$ for all $y \in \operatorname{int}(-P)$. Hence, if $\langle y^*, \Phi(x) - \Phi(x^o)\rangle \geq 0$ for all $x \in S$, there exists no $x \in S$ such that $\Phi(x) - \Phi(x^o) <_P 0$, i.e., x^o solves (7).
For the converse implication let x^o solve (7) and set $V := \Phi(S) - \Phi(x^o) + P$. V is a nonvoid convex set, and, since x^o solves (7) and $P + \operatorname{int} P = \operatorname{int} P$, one has $V \cap \operatorname{int}(-P) = \emptyset$. Hence it follows from the weak separation theorem that there exists $y^* \in Y^* \setminus \{0\}$ such that

$$\langle y^*, y\rangle \begin{cases} \leq 0 & \forall y \in -P, \\ \geq 0 & \forall y \in V. \end{cases}$$

The first inequality yields $y^* \in P^*$. The second inequality yields then

$$\langle y^*, \Phi(x) - \Phi(x^o)\rangle \geq \sup_{y \in -P} \langle y^*, y\rangle = 0 \quad \forall x \in S.$$ q.e.d.

If $\Phi : C \to Y$ is P-convex and $y^* \in P^*$, then the function $\langle y^*, \Phi(\cdot)\rangle$ is easily seen to be convex. We define $\Phi : C \to Y$ to be P-lower semicontinuous iff $\langle y^*, \Phi(\cdot)\rangle$ is lower semicontinuous on C for all $y^* \in P^*$.

In the remainder of this section we consider the problem

(8) $\quad$ $\text{w-eff}\,\{\Phi(x), P \mid x \in C,\ \varphi(t,x) \leq 0 \quad \forall t \in T\}$.

We assume:

- T is compact;
- C is a closed convex subset of $\mathbb{R}^n$;
- $\Phi(\cdot) : C \to Y$ is P-convex and P-lower semicontinuous;
- $\varphi(t,\cdot) : C \to \mathbb{R}$ is convex and lower semicontinuous for all $t \in T$;
- $\varphi(\cdot,x) : T \to \mathbb{R}$ is upper semicontinuous for all $x \in C$.

In analogy to our previous notation let us set

$$S_\varphi := \{x \in C \mid \varphi(t,x) \leq 0 \quad \forall t \in T\},$$

and, for a fixed element $x^o \in S_\varphi$,

$$T^o := \{t \in T \mid \varphi(t,x^o) = 0\}.$$

In the following theorem solutions of (8) are characterized by the existence of Lagrange-multipliers. The technique of proof is similar to [1, pp.90–100].

Theorem 3. Assume that

(9) $\left\{ \begin{array}{l} \text{for all finite subsets } \mathcal{T} \subset T^o \text{ there exists an element } x \in C \text{ such} \\ \text{that } \varphi(t,x) < 0 \text{ for all } t \in \mathcal{T}. \end{array} \right.$

Then $x^o \in S_\varphi$ solves (8) if and only if there exist a finite subset $\mathcal{T}^o \subset T^o$ with $|\mathcal{T}^o| \leq n$, $u_t \geq 0$ $(t \in \mathcal{T}^o)$, and $y^* \in P^* \setminus \{0\}$ such that

(10) $$0 \leq \langle y^*, \Phi(x) - \Phi(x^o) \rangle + \sum_{t \in \mathcal{T}^o} u_t \varphi(t,x) \quad \forall x \in C.$$

Proof. If (10) is satisfied with nonnegative numbers u_t $(t \in \mathcal{T}^o)$, then $\langle y^*, \Phi(x) - \Phi(x^o) \rangle \geq 0$ for all $x \in S_\varphi$. Hence, by lemma 2, x^o solves (8).
For the converse implication – in order to avoid trivial case distinctions – let us suppose that T^o contains at least n elements. Let $x^o \in S_\varphi$ solve (8). By lemma 2, there exists $y^* \in P^* \setminus \{0\}$ such that x^o is a solution of

$$\min \{\langle y^*, \Phi(x) \rangle \mid x \in C,\ \varphi(t,x) \leq 0 \quad \forall t \in T\},$$

a special case of weak efficiency with $P = \mathbb{R}^1_+$. With $C_\rho := C \cap \{x \in \mathbb{R}^n \mid \|x - x^o\| \leq \rho\}$ for some $\rho > 0$ and $\varphi(0,x) := \langle y^*, \Phi(x) - \Phi(x^o) \rangle$ it follows from theorem 2 that the system

$$x \in C_\rho, \qquad \varphi(t,x) < 0 \quad \forall t \in T^o \cup \{0\}$$

has no solution; hence, for $\varepsilon > 0$, the system

$$x \in C_\rho, \qquad \varphi(t,x) \leq -\varepsilon \quad \forall t \in T^o \cup \{0\}$$

has no solution either. Since the sets $\{x \in C_\rho \mid \varphi(t,x) \leq -\varepsilon\}$ are convex and compact for all $t \in T^o \cup \{0\}$, it follows from a theorem by Helly and König about the intersection of closed convex sets over a compact subset of $\mathbb{R}^n$ [5] that there exist $t_i \in T^o \cup \{0\}$ $(i=1,\dots,n+1)$ such that the system

$$x \in C_\rho, \qquad \varphi(t_i,x) \leq -\varepsilon. \quad (i=1,\dots,n+1)$$

has no solution. Hence $\max_{i=1,\dots,n+1} \varphi(t_i,x) \geq -\varepsilon$ $\forall x \in C_\rho$. By a result of Fan-Glicksberg-Hoffman [8, p.65] there exists $u \in \Lambda_{n+1} := \{u \in \mathbb{R}^{n+1} \mid u \geq 0,\ \Sigma_{i=1}^{n+1} u_i = 1\}$ such that

$$\Sigma_{i=1}^{n+1} u_i \varphi(t_i,x) \geq -\varepsilon \qquad \forall x \in C_\rho.$$

Hence with $K := (T^o \cup \{0\})^{n+1}$ the sets

$$F(\varepsilon) := \{(t_1,\dots,t_{n+1},u_1,\dots,u_{n+1}) \in K \times \Lambda_{n+1} \mid$$
$$\Sigma_{i=1}^{n+1} u_i \varphi(t_i,x) \geq -\varepsilon \qquad \forall x \in C_\rho\}$$

are nonempty for all $\varepsilon > 0$. This implies at the same time that any finite collection of these sets $F(\varepsilon)$ has nonempty intersection. Since the sets $F(\varepsilon)$ are closed and $K \times \Lambda_{n+1}$ is compact, the collection of all $F(\varepsilon)$ with $\varepsilon > 0$ has nonempty intersection. Now from $(\bar{t}_1,\dots,\bar{t}_{n+1},\bar{u}_1,\dots,\bar{u}_{n+1}) \in \bigcap_{\varepsilon>0} F(\varepsilon)$ follows, since $\varphi(\bar{t}_i,x^o) = 0$ and the functions $\varphi(\bar{t}_i,\cdot)$ are convex, that

$$\Sigma_{i=1}^{n+1} \bar{u}_i \varphi(\bar{t}_i,x) \geq 0 \qquad \forall x \in C.$$

Due to assumption (9) this implies that $0 \in \{\bar{t}_1,\dots,\bar{t}_{n+1}\}$, and that one of the multipliers $\bar{u}_i$ corresponding to $\bar{t}_i = 0$ must be positive. Altogether we have obtained $\bar{t}_1,\dots,\bar{t}_n \in T^o$ and $\bar{u}_o > 0$, $\bar{u}_1 \geq 0,\dots, \bar{u}_n \geq 0$ such that

$$\bar{u}_o \, \varphi(0,x) + \sum_{i=1}^{n} \bar{u}_i \varphi(\bar{t}_i,x) \geq 0 \qquad \forall x \in C.$$

Since we may normalize $(\bar{u}_o,\bar{u}_1,\dots,\bar{u}_n)$ such that $\bar{u}_o = 1$, this is the desired result.

q.e.d.

Assumption (9) is commonly termed a constraint qualification.

Let us specialize theorem 3 to the case

$$C = \mathbb{R}^n, \ \Phi(x) := Ax, \ \varphi(t,x) := \langle a(t),x\rangle + b(t),$$

where

$A : \mathbb{R}^n \to Y$ is a continuous linear mapping,

$a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$ are both continuous.

Then problem (8) becomes

(11) $\quad$ $\text{w-eff}\ \{Ax, P \mid x \in \mathbb{R}^n, \langle a(t),x\rangle + b(t) \leq 0 \quad \forall t \in T\}$,

and condition (10) becomes

$$0 \leq \langle y^*, Ax - Ax^o\rangle + \sum_{t\in\mathcal{T}^o} u_t(\langle a(t),x\rangle + b(t)) \qquad \forall x \in \mathbb{R}^n.$$

Since $\mathcal{T}^o \subset T^o$ and hence $\langle a(t),x^o\rangle + b(t) = 0 \quad \forall t \in \mathcal{T}^o$, this is equivalent to

$$0 = \langle y^*, Ax\rangle + \sum_{t\in\mathcal{T}^o} u_t \langle a(t),x\rangle \qquad \forall x \in \mathbb{R}^n, \text{ i.e.,}$$

$$0 = A^*y^* + \sum_{t\in \mathcal{T}^o} u_t a(t),$$

where $A^* : Y^* \to \mathbb{R}^n$ is the adjoint of A. Making use of Carathéodory's Theorem (that every point of the convex conical hull of a set $B \subset \mathbb{R}^n$ can be represented as a nonnegative linear combination of at most n elements of B), we can then specialize theorem 3 as follows:

Let (9) hold. Then $x^o \in S_\varphi$ solves (11) if and only if

(12) $\quad 0 \in A^*(P^* \setminus \{0\}) + \text{conv cone } \{a(t) \mid t \in T^o\}$ 2).

If $Y = \mathbb{R}^m$, $P = P^* = \mathbb{R}^m_+$, this yields theorem 6.3 in [3]. The usefulness of (12) as a necessary condition for a solution of (11) is somewhat limited, since its validity requires a constraint qualification, finite-dimensionality of the underlying space, and continuous dependence on t. In the next section we shall get rid of these limitations.

2) "conv cone" denotes the convex conical hull.

6. Elimination of the constraint qualification

In this section we want to characterize weakly efficient points in a more general setting and without need for a constraint qualification. First we consider the linear case, i.e.,

$$\text{(13)} \qquad \text{w-eff}\ \{Ax,P \mid x \in X,\ \langle a(t)^*,x\rangle + b(t) \le 0 \quad \forall\, t \in T\},$$

where we assume:

T is an arbitrary set;
X is a linear topological space; X*, its topological dual, is equipped with the weak*-topology;
$A : X \to Y$ is a continuous linear mapping with $A^* : Y^* \to X^*$ its adjoint;
$a(t)^* \in X^*$ and $b(t) \in \mathbb{R}$ for all $t \in T$.

S_φ is the set of feasible solutions of (13).

Let us recall that, for $B \subset X^*$, ξ^* is an element of the weak*-closure of B if and only if for all finite subsets $\mathfrak{X} \subset X$ and for all $\varepsilon > 0$ there exists $x^* \in B$ such that

$$|\langle \xi^*,x\rangle - \langle x^*,x\rangle| \le \varepsilon \quad \forall\, x \in \mathfrak{X}.$$

Theorem 4. $x^o \in S_\varphi$ is a solution of (13) if and only if there exists $y^* \in P^* \setminus \{0\}$ such that

$$\text{(14)} \qquad (0_{X^*},0) \in (A^*y^*,\langle y^*,-Ax^o\rangle) + \mathrm{cl}\,\Gamma,$$

where $\Gamma := \mathrm{conv\ cone}\ \{(a(t)^*,b(t)) \mid t \in T\} \subset X^* \times \mathbb{R}$.

Proof. Let x^o be a solution of (13); then, by lemma 2, there exists $y^* \in P^* \setminus \{0\}$ such that the system

$$\text{(15)} \qquad \langle a(t)^*,\xi\rangle + b(t) \le 0 \quad \forall\, t \in T, \qquad \langle y^*,A\xi - Ax^o\rangle < 0$$

has no solution. Assume now that (14) does not hold for this particular choice of y*. Then the point $(-A^*y^*,\langle y^*,Ax^o\rangle)$ is disjoint from the closed convex cone $\mathrm{cl}\,\Gamma$. By the strong separation theorem, and since X* is provided with the weak*-topology, there exists $(x,r) \in X \times \mathbb{R}$ (i.e., a continuous linear functional on $X^* \times \mathbb{R}$), such that

$$-\langle A^*y^*,x\rangle + \langle y^*,Ax^o\rangle \cdot r > 0,$$
$$\langle a(t)^*,x\rangle + b(t) \cdot r \le 0 \quad \forall\, t \in T.$$

In case $r > 0$ we obtain that

$$\langle y^*, A(r^{-1}x - x^o)\rangle < 0,$$
$$\langle a(t)^*, r^{-1}x\rangle + b(t) \le 0 \quad \forall t \in T,$$

and therefore $\xi := r^{-1}x$ is a solution of (15). In case $r \le 0$ we obtain, since $r \cdot (\langle a(t)^*, x^o\rangle + b(t)) \ge 0$, that

$$\langle y^*, A(x - rx^o)\rangle < 0,$$
$$\langle a(t)^*, x - rx^o\rangle \le 0 \quad \forall t \in T,$$

and therefore $\xi := x^o + x - rx^o$ is a solution of (15). In both cases we have obtained a contradiction. Hence (14) must be satisfied.
Assume now that (14) holds. Let $\mathfrak{X} \subset X$ be a finite subset and $\varepsilon > 0$. Then it follows from (14), since X^* is provided with the weak*-topology, that there exists $(x^*, r) \in \Gamma$ such that

$$|-\langle A^*y^*, x\rangle - \langle x^*, x\rangle| \le \frac{\varepsilon}{2} \quad \forall x \in \mathfrak{X},$$
$$|\langle y^*, Ax^o\rangle - r| \le \frac{\varepsilon}{2},$$

and hence

$$\langle y^*, Ax^o\rangle \le \langle y^*, Ax\rangle + \langle x^*, x\rangle + r + \varepsilon \quad \forall x \in \mathfrak{X}.$$

Taking into account that $(x^*, r) \in \Gamma$ has a representation

$$(x^*, r) = \sum_{t \in \mathfrak{T}} v_t (a(t)^*, b(t)), \qquad v_t \ge 0 \ (t \in \mathfrak{T})$$

with some finite subset $\mathfrak{T} \subset T$, it follows that

(16) $$\langle y^*, Ax^o\rangle \le \langle y^*, Ax\rangle + \sum_{t \in \mathfrak{T}} v_t (\langle a(t)^*, x\rangle + b(t)) + \varepsilon \quad \forall x \in \mathfrak{X}.$$

Choosing in particular $\mathfrak{X} = \{x\}$, where $x \in S_\varphi$, we obtain from (16) that

$$\langle y^*, Ax^o\rangle \le \langle y^*, Ax\rangle + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\langle y^*, Ax - Ax^o\rangle \ge 0$ for all $x \in S_\varphi$. In view of lemma 2, x^o solves (13). q.e.d.

In the proof of theorem 4 we have established that from (14) follows (16). Moreover, if $Y = \mathbb{R}^1$, $P = P^* = \mathbb{R}^1_+$, then $y^* \in P^* \setminus \{0\}$ may be normalized to $y^* = 1$. Combining these two observations we obtain the following corollary to theorem 4, where $c^* \in X^*$.

Corollary. If x^o solves

(17) $$\min \{\langle c^*, x\rangle \mid x \in X, \langle a(t)^*, x\rangle + b(t) \le 0 \quad \forall t \in T\},$$

then for all finite subsets $\mathfrak{X} \subset X$ and for all $\varepsilon > 0$ there exist a finite subset

$\mathcal{T} \subset T$ and $v_t \geq 0$ $(t \in \mathcal{T})$ such that

$$\langle c^*, x^o \rangle \leq \langle c^*, x \rangle + \sum_{t \in \mathcal{T}} v_t (\langle a(t)^*, x \rangle + b(t)) + \varepsilon \quad \forall x \in \mathcal{X}.$$

Now we consider the convex case, i.e., we consider the problem

(18) $\quad \text{w-eff } \{\Phi(x), P \mid x \in C, \varphi(t,x) \leq 0 \quad \forall t \in T\},$

where we assume:

- T is an arbitrary set;
- C is a convex set (in some real linear space);
- $\Phi(\cdot) : C \to Y$ is P-convex;
- $\varphi(t,\cdot) : C \to \mathbb{R}$ is convex for all $t \in T$.

S_φ denotes the set of feasible points of (18).

Theorem 5. $x^o \in S_\varphi$ is a solution of (18) if and only if there exists $y^* \in P^* \setminus \{0\}$ such that for all finite subsets $\mathcal{X} \subset C$ and for all $\varepsilon > 0$ there exist a finite subset $\mathcal{T} \subset T$ and $v_t \geq 0$ $(t \in \mathcal{T})$ satisfying

(19) $\quad \langle y^*, \Phi(x^o) \rangle \leq \langle y^*, \Phi(x) \rangle + \sum_{t \in \mathcal{T}} v_t \varphi(t,x) + \varepsilon \quad \forall x \in \mathcal{X}.$

Proof. If (19) holds, then $\langle y^*, \Phi(x) - \Phi(x^o) \rangle \geq 0$ for all $x \in S_\varphi$; hence it follows from lemma 2 that x^o solves (18).
Conversely, let x^o be a solution of (18). Then, by lemma 2, there exists $y^* \in P^* \setminus \{0\}$ such that

$$\langle y^*, \Phi(x) - \Phi(x^o) \rangle \geq 0 \quad \forall x \in S_\varphi.$$

With the abbreviation $\tilde{\Phi}(x) := \langle y^*, \Phi(x) \rangle$ this means that x^o solves

(20) $\quad \min \{\tilde{\Phi}(x) \mid x \in C, \varphi(t,x) \leq 0 \quad \forall t \in T\}.$

Now let $\mathcal{X}$ be an arbitrary finite subset of C. Without loss of generality we may assume that $x^o \in \mathcal{X}$. So let $\mathcal{X} := \{x^o, x^1, \dots, x^n\}$. With $u := (u_o, u_1, \dots, u_n) \in \mathbb{R}^{n+1}$ we consider the following problem:

(21) $$\min \{ \sum_{i=0}^{n} u_i \tilde{\Phi}(x^i) \mid u \in \mathbb{R}^{n+1}, \sum_{i=0}^{n} u_i \varphi(t, x^i) \leq 0 \quad \forall t \in T,$$
$$u_i \geq 0 \ (i=0,\dots,n), \ \sum_{i=0}^{n} u_i - 1 = 0 \}.$$

In view of our convexity assumptions, and since x^o solves (20), it is easily verified that $u^o := (1,0,\dots,0) \in \mathbb{R}^{n+1}$ solves (21). If we replace the equality constraint by two inequalities, (21) has the same structure as (17). Hence we may apply the corollary. For this purpose let $\mathcal{U}$ be a finite subset of $\mathbb{R}^{n+1}$ and $\varepsilon > 0$. Then it follows from the corollary that there exist a finite subset $\mathcal{T} \subset T$ and multipliers $v_t \geq 0$ $(t \in \mathcal{T})$, $\mu_i \geq 0$ $(i=0,\dots,n)$ and $\nu \in \mathbb{R}$ such that

$$\tilde{\Phi}(x^o) \leq \sum_{i=0}^{n} u_i \tilde{\Phi}(x^i) + \sum_{t\in\mathcal{T}} v_t \Big(\sum_{i=0}^{n} u_i \varphi(t,x^i)\Big) - \sum_{i=0}^{n} \mu_i u_i + \nu\Big(\sum_{i=0}^{n} u_i - 1\Big) + \varepsilon$$
$$\forall u \in \mathcal{U}.$$

Here we have used the fact that $\sum_{i=0}^{n} u_i^o \tilde{\Phi}(x^i) = \tilde{\Phi}(x^o)$. Choosing in particular for $\mathcal{U}$ the set of all unit vectors $e^i \in \mathbb{R}^{n+1}$ $(i=0,1\dots,n)$ we obtain in turn

$$\tilde{\Phi}(x^o) \leq \tilde{\Phi}(x^i) + \sum_{t\in\mathcal{T}} v_t \varphi(t,x^i) - \mu_i + \varepsilon \qquad (i=0,1,\dots,n),$$

and since $\mu_i \geq 0$ $(i=0,1,\dots,n)$, this is the desired result

$$\tilde{\Phi}(x^o) \leq \tilde{\Phi}(x) + \sum_{t\in\mathcal{T}} v_t \varphi(t,x) + \varepsilon \quad \forall x \in \mathcal{X}.$$ q.e.d.

We cannot maintain in theorem 5 that $\mathcal{T} \subset T^o := \{t \in T \mid \varphi(t,x^o) = 0\}$, but setting $x := x^o$ in the last line of the proof we obtain the bounds

$$-\varepsilon \leq \sum_{t\in\mathcal{T}} v_t \varphi(t,x^o) \leq 0.$$

If T is finite, the closure in (14) can be omitted, since Γ as a finitely generated convex cone in a separated space is then already closed. If we apply (14) in this strengthened form in the proof above, it follows that for T finite theorem 5 even holds true with $\varepsilon = 0$.
Theorems 4 and 5 improve upon previous results in [7] and [9].

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